## CHAPTER 10

## $\mathscr{D}$-MODULES ENRICHED WITH A SESQUILINEAR PAIRING


#### Abstract

Summary. Our aim in this chapter is to extend the notion of $(\nabla, \bar{\nabla})$-flat sesquilinear pairing, used for the definition of a variation of $\mathbb{C}$-Hodge structure (see Definition 4.1.3), to the case where the flat bundle is replaced with a $\mathscr{D}$-module. The case of a holonomic $\mathscr{D}$-module will of course be the most interesting for us, but it is useful to develop the notion with some generality in order to apply derived functors.


In this chapter, we keep Notation 7.0.1. However, we will only consider $\mathscr{O}_{X}$-modules and $\mathscr{D}_{X}$-modules, as coherent filtrations will not play any role here. We will use the constructions and results of Chapter 7 in this framework.

### 10.1. Introduction

One of the ingredients of a variation of polarized Hodge structure is a flat Hermitian pairing (that we have denoted by Q), which is $(-1)^{p}$-definite on $\mathcal{H}^{p, w-p}$. In this chapter, we introduce the notion of sesquilinear pairing between holonomic $\mathscr{D}_{X^{-}}$ modules. It takes values in the sheaf of distributions (in fact a smaller sheaf, but we are not interested in characterizing the image). This notion will not be used directly as in classical Hodge theory to furnish the notion of polarization. Instead, we will take up the definition of a $\mathbb{C}$-Hodge structure as a triple (see Section 2.4.c) and mimic this definition in higher dimension. Our aim is therefore to define a category of $\mathscr{D}$-triples (an object consists of a pair of $\mathscr{D}_{X}$-modules and a sesquilinear pairing between them) and to extend to this abelian category the various functors considered in Chapter 7.

### 10.2. Sesquilinear pairings for $\mathscr{D}_{X}$-modules and the category $\mathscr{D}$-Triples

10.2.a. Distributions an currents on a complex manifold. Let $\bar{X}$ denote the complex manifold conjugate to $X$, i.e., with structure sheaf $\mathscr{O}_{\bar{X}}$ defined as the sheaf of anti-holomorphic functions $\overline{\mathscr{O}_{X}}$. Correspondingly is defined the sheaf of antiholomorphic differential operators $\mathscr{D}_{\bar{X}}$. The sheaf of $C^{\infty}$ functions on $X$ is acted on
by $\mathscr{D}_{X}$ and $\mathscr{D}_{\bar{X}}$ on the left and both actions commute, i.e., $\mathscr{C}_{X}^{\infty}$ is a left $\mathscr{D}_{X} \otimes_{\mathbb{C}} \mathscr{D}_{\bar{X}^{-}}$ module. Similarly, the sheaf of distributions $\mathfrak{D b}_{X}$ is a left $\mathscr{D}_{X} \otimes_{\mathbb{C}} \mathscr{D}_{\bar{X}}$-module: by definition, on any open set $U \subset X, \mathfrak{D b}_{X}(U)$ is dual to the space $\mathscr{A}_{\mathrm{c}}^{2 n}(U)$ of $C^{\infty}$ $2 n$-forms with compact support, equipped with a suitable topology, and the presheaf defined in this way is a sheaf. On the other hand, the space of $\mathfrak{C}_{X}(U)$ of currents of degree 0 on $X$ is dual to $C_{\mathrm{c}}^{\infty}(U)$ with suitable topology. Then $\mathfrak{C}_{X}$ is the right $\mathscr{D}_{X} \otimes_{\mathbb{C}} \mathscr{D}_{\bar{X}}$-module obtained from $\mathfrak{D b}_{X}$ by the left-to-right transformation for such objects, i.e.,

$$
\mathfrak{C}_{X}=\left(\omega_{X} \otimes_{\mathbb{C}} \omega_{\bar{X}}\right) \otimes_{\left(\mathscr{O}_{X} \otimes \mathscr{O}_{\bar{X}}\right)} \mathfrak{D} \mathfrak{b}_{X}
$$

The stupid conjugation functor $\mathcal{M} \mapsto \overline{\mathcal{M}}$ transforms $\mathscr{O}_{X}$-modules (resp. $\mathscr{D}_{X}$-modules) into $\mathscr{O}_{\bar{X}}$-modules (resp. $\mathscr{D}_{\bar{X}}$-modules): let us regard $\mathscr{O}_{\bar{X}}$ as an $\mathscr{O}_{X}$-module by setting $f \cdot \bar{g}:=\bar{f} \bar{g}$, and similarly let us regard $\mathscr{D}_{\bar{X}}$ as a $\mathscr{D}_{X}$-module; for an $\mathscr{O}_{X}$-module (resp. a $\mathscr{D}_{X}$-module) $\mathcal{M}$ we then define $\overline{\mathcal{M}}$ as $\mathscr{O}_{\bar{X}} \otimes_{\mathscr{O}_{X}} \mathcal{M}$ (resp. $\left.\mathscr{D}_{\bar{X}} \otimes_{\mathscr{D}_{X}} \mathcal{M}\right)$. In other words, for a local section $m$ of $\mathcal{M}$, we denote by $\bar{m}$ the same local section, that we act on by $\bar{f} \in \mathscr{O}_{\bar{X}}$ (resp. $\mathscr{D}_{\bar{X}}$ ) with the formula $\bar{f} \cdot \bar{m}:=\overline{f m}$.

Notation 10.2.1. From now on, the notation $\mathscr{A}_{X, \bar{X}}$ will mean $\mathscr{A}_{X} \otimes_{\mathbb{C}} \mathscr{A}_{\bar{X}}(\mathscr{A}=\mathscr{O}$ or $\mathscr{D})$.

One can easily adapt Exercise A.5.5 to prove that the $C^{\infty}$-de Rham complex $\mathscr{E}_{X}^{2 n+\bullet} \otimes_{\mathscr{C}_{X}^{\infty}} \mathscr{D}_{X, \bar{X}}=\mathscr{E}_{X}^{\bullet} \otimes_{\mathscr{O}_{X, \bar{X}}} \mathscr{D}_{X, \bar{X}}[2 n]$, where the differential is obtained from the standard differential on $C^{\infty} k$-forms and the universal connection $\nabla_{X}+\bar{\nabla}_{X}$ on $\mathscr{D}_{X, \bar{X}}$, is a resolution of $\mathscr{E}_{X}^{n, n}=\mathscr{E}_{X}^{2 n}$ as a right $\mathscr{D}_{X, \bar{X}}$-module.

We denote by $\mathfrak{D b}_{X}^{n-p, n-q}=\mathscr{E}_{X}^{n-p, n-q} \otimes \mathscr{C}_{X}^{\infty} \mathfrak{D b}_{X}$ or $\mathfrak{D b}_{X, p, q}$ the sheaf of currents of degree $(p, q)$ (we also say of type $(n-p, n-q)$ ), that is, continuous linear forms on $C_{\mathrm{c}}^{\infty}$ differential forms of degree $p, q$.

The distributional de Rham complex gives then a resolution of $\mathfrak{C}_{X}$ as a right $\mathscr{D}_{X, \bar{X}^{-}}$ module:

$$
\begin{equation*}
\mathfrak{D b}_{X}^{\bullet}[2 n] \otimes_{\mathscr{O}_{X, X}} \mathscr{D}_{X, \bar{X}} \xrightarrow{\sim} \mathfrak{C}_{X} \tag{10.2.2}
\end{equation*}
$$

Let us make precise that the morphism is induced by

$$
\mathfrak{D b}_{X}^{n, n} \otimes \mathscr{D}_{X, \bar{X}}=\mathfrak{C}_{X} \otimes \mathscr{D}_{X, \bar{X}} \longrightarrow \mathfrak{C}_{X}, \quad u \otimes P \longmapsto u \cdot P
$$

10.2.b. Sesquilinear pairings. Let us start with the case of right $\mathscr{D}_{X}$-modules.

## Definition 10.2.3 (Sesquilinear pairing).

(1) A sesquilinear pairing $\mathfrak{c}$ between right $\mathscr{D}_{X}$-modules $\mathcal{M}^{\prime}, \mathcal{M}^{\prime \prime}$ is a $\mathscr{D}_{X, \bar{X}}$-linear morphism $\mathfrak{c}: \mathcal{M}^{\prime} \otimes_{\mathbb{C}} \overline{\mathcal{M}^{\prime \prime}} \rightarrow \mathfrak{C}_{X}$. When $\mathcal{M}^{\prime}=\mathcal{M}^{\prime \prime}=\mathcal{M}$, we speak of a sesquilinear pairing on $\mathcal{M}$.
(2) The adjoint of $\mathfrak{c}$ is $\mathfrak{c}^{*}: \mathcal{M}^{\prime \prime} \otimes_{\mathbb{C}} \overline{\mathcal{M}^{\prime}} \rightarrow \mathfrak{C}_{X}$ defined by $\mathfrak{c}^{*}\left(m^{\prime \prime}, \overline{m^{\prime}}\right)=\overline{\mathfrak{c}\left(m^{\prime}, \overline{m^{\prime \prime}}\right)}$, with $\left.\left\langle\overline{\mathfrak{c}\left(m^{\prime}, \overline{m^{\prime \prime}}\right.}\right), \eta\right\rangle:=\left\langle\mathfrak{c}\left(m^{\prime}, \overline{m^{\prime \prime}}\right), \bar{\eta}\right\rangle$ for any test function $\eta$. We clearly have $\mathfrak{c}^{* *}=\mathfrak{c}$.

Remark 10.2.4 (Extension to $C^{\infty}$ coefficients). Let us define a right action of $\mathscr{D}_{X, \bar{X}}$ on $\mathcal{M} \otimes_{\mathscr{O}_{X}} \mathscr{C}_{X}^{\infty}$ by setting $(m \otimes \eta) \cdot \partial_{x_{i}}=m \partial_{x_{i}} \otimes \eta-m \otimes \partial \eta / \partial x_{i}$ and $(m \otimes \eta) \cdot \partial_{\bar{x}_{i}}=$ $-m \otimes \partial \eta / \partial \bar{x}_{i}$. Then $\mathfrak{c}$ extends in a unique way as a $\mathscr{C}_{X}^{\infty}$-linear morphism

$$
\left(\mathcal{N}^{\prime} \otimes_{\mathscr{O}_{X}} \mathscr{C}_{X}^{\infty}\right) \otimes_{\mathscr{C}_{X}^{\infty}} \overline{\left(\mathcal{M}^{\prime \prime} \otimes_{\mathscr{O}_{X}} \mathscr{C}_{X}^{\infty}\right)} \longrightarrow \mathfrak{C}_{X}
$$

which satisfies, for any local section $\xi$ of $\Theta_{X}$ or $\overline{\Theta_{X}}$,

$$
\mathfrak{c}\left(\mu^{\prime}, \overline{\mu^{\prime \prime}}\right) \xi=\mathfrak{c}\left(\mu^{\prime} \xi, \overline{\mu^{\prime \prime}}\right)+\mathfrak{c}\left(\mu^{\prime}, \overline{\mu^{\prime \prime}} \xi\right)
$$

by setting

$$
\mathfrak{c}\left(m^{\prime} \otimes \eta^{\prime}, \overline{m^{\prime \prime} \otimes \eta^{\prime \prime}}\right):=\mathfrak{c}\left(m^{\prime}, \overline{m^{\prime \prime}}\right) \eta^{\prime} \overline{\eta^{\prime \prime}}
$$

Conversely, given such a pairing, one recovers the original $\mathfrak{c}$ by restricting to $\mathcal{M}^{\prime} \otimes_{\mathbb{C}} \overline{\mathcal{M}^{\prime \prime}}$.

## Remark 10.2.5 (The case of left $\mathscr{D}_{X}$-modules and side-changing)

If $\mathcal{N}^{\prime}, \mathcal{M}^{\prime \prime}$ are left $\mathscr{D}_{X}$-modules and $\mathfrak{c}=\mathfrak{c}^{\text {left }}: \mathcal{M}^{\prime} \otimes_{\mathbb{C}} \overline{\mathcal{N}^{\prime \prime}} \rightarrow \mathfrak{D b}_{X}$ is a sesquilinear pairing, that is, a left $\mathscr{D}_{X, \bar{X}}$-linear morphism, then it determines in a canonical way a sesquilinear pairing

$$
\begin{align*}
\mathfrak{c}^{\text {right }}:\left(\omega_{X} \otimes \mathcal{M}^{\prime}\right) \otimes \mathbb{C}\left(\overline{\omega_{X} \otimes \mathcal{N}^{\prime \prime}}\right) & \longrightarrow \omega_{X} \otimes \overline{\omega_{X}} \otimes \mathfrak{D b}_{X}=\mathfrak{C}_{X} \\
\left(\omega^{\prime} \otimes m^{\prime}, \overline{\omega^{\prime \prime} \otimes m^{\prime \prime}}\right) & \longmapsto \omega^{\prime} \wedge \overline{\omega^{\prime \prime}} \otimes \mathfrak{c}^{\text {left }}\left(m^{\prime}, \overline{m^{\prime \prime}}\right) . \tag{10.2.5*}
\end{align*}
$$

Conversely, from a sesquilinear pairing between right $\mathscr{D}_{X}$-modules one recovers one for left $\mathscr{D}_{X}$-modules.

The compatibility with adjunction is given by the following relation:
$(10.2 .5 * *) \quad\left(\mathfrak{c}^{\text {right }}\right)^{*}=(-1)^{n}\left(\mathfrak{c}^{*}\right)^{\text {right }}$,
since $\overline{\omega^{\prime}} \wedge \omega^{\prime \prime}=(-1)^{n} \omega^{\prime \prime} \wedge \overline{\omega^{\prime}}$.
Let us notice the following.
Lemma 10.2.6. If $\mathcal{M}^{\prime}$ and $\mathcal{N}^{\prime \prime}$ are $\mathscr{O}_{X}$-coherent (hence $\mathscr{O}_{X}$-locally free of finite rank), the pairing $\mathfrak{c}$ takes values in $C^{\infty}$ forms of maximal degree (resp. functions).

Proof. We know (see Example A.4.b) that $\mathcal{M}^{\prime}, \mathcal{M}^{\prime \prime}$ are $\mathscr{O}_{X}$-generated by their flat local sections. For such local sections $m^{\prime}, m^{\prime \prime}$, the current (resp. distribution) $\mathfrak{c}\left(m^{\prime}, \overline{m^{\prime \prime}}\right)$ is annihilated by $\partial$ and $\bar{\partial}$, hence is locally a constant. It follows that, for any local sections $m^{\prime}, m^{\prime \prime}, \mathfrak{c}\left(m^{\prime}, \overline{m^{\prime \prime}}\right)$ is real-analytic, so in particular $C^{\infty}$.
10.2.c. The category of $\mathscr{D}$-triples. The category $\mathscr{D}$ - $\operatorname{Triples}(X)$ is a prototype, without any Hodge filtration, of the category of Hodge modules. It serves as a model for it, but is also a constituent of it, as we will see in Chapter 12. It is an abelian category, and possesses the basic functors we need for studying pure Hodge modules, as a consequence of the results of the previous sections. Moreover, we will define sign rules for various functors in order to eliminate various signs which have appeared in the previous formulas, and to be compatible with the notion of Lefschetz structure (see Chapter 3) when possible.

Definition 10.2.7. The category $\mathscr{D}$-Triples $(X)$ has

- objects consisting of triples $\mathcal{T}=\left(\mathcal{M}^{\prime}, \mathcal{M}^{\prime \prime}, \mathfrak{c}\right)$, where $\mathcal{M}^{\prime}, \mathcal{M}^{\prime \prime}$ are $\mathscr{D}_{X}$-modules and $\mathfrak{c}$ is a sesquilinear pairing between them (with values in $\mathfrak{D b}_{X}$ in the left case, and in $\mathfrak{C}_{X}$ in the right case),
- morphisms $\varphi: \mathcal{T}_{1} \rightarrow \mathcal{T}_{2}$ consisting of pairs $\varphi=\left(\varphi^{\prime}, \varphi^{\prime \prime}\right)$, where $\varphi^{\prime}: \mathcal{M}_{1}^{\prime} \rightarrow \mathcal{M}_{2}^{\prime}$ and $\varphi^{\prime \prime}: \mathcal{M}_{2}^{\prime \prime} \rightarrow \mathcal{M}_{1}^{\prime \prime}$ are $\mathscr{D}_{X}$-linear, such that for all local sections $m_{1}^{\prime}$ of $\mathcal{M}_{1}^{\prime}$ and $m_{1}^{\prime \prime}$ of $\mathcal{M}_{2}^{\prime \prime}$,

$$
\begin{equation*}
\mathfrak{c}_{1}\left(m_{1}^{\prime}, \overline{\varphi^{\prime \prime}\left(m_{2}^{\prime \prime}\right)}\right)=\mathfrak{c}_{2}\left(\varphi^{\prime}\left(m_{1}^{\prime}\right), \overline{m_{2}^{\prime \prime}}\right) \tag{10.2.7*}
\end{equation*}
$$

In particular, $\mathscr{D}$-Triples $(X)$ is an abelian subcategory of $\operatorname{Mod}\left(\mathscr{D}_{X}\right) \times \operatorname{Mod}\left(\mathscr{D}_{X}\right)^{\text {op }}$.
We say that an object $\mathcal{T}$ of $\mathscr{D}$-Triples $(X)$ is coherent, resp. $\mathbb{R}$-specializable, resp. smooth, if its components $\mathcal{M}^{\prime}, \mathcal{N}^{\prime \prime}$ are $\mathscr{D}_{X}$-coherent, resp. $\mathbb{R}$-specializable, resp. $\mathscr{O}_{X}$-locally free of finite rank.

Definition 10.2.8 (Side-changing in $\mathscr{D}$ - $\operatorname{Triples}(X)$ ). Let $\mathcal{T}=\left(\mathcal{N}^{\prime}, \mathcal{N}^{\prime \prime}, \mathfrak{c}\right)$ be a left $\mathscr{D}_{X^{-}}$ triple. We set

$$
\mathcal{T}^{\text {right }}:=\left(\mathcal{N}^{/ \text {right }}, \mathcal{M}^{\prime / \text { right }}, \mathfrak{c}^{\text {right }}\right)
$$

where $\boldsymbol{c}^{\text {right }}$ is defined by $(10.2 .5 *)$. The right-to-left side changing is defined correspondingly, so that the composition of both is the identity.

Definition 10.2.9 (Adjunction). The adjoint of an object $\mathcal{T}=\left(\mathcal{M}^{\prime}, \mathcal{M}^{\prime \prime}, \mathfrak{c}\right)$ of $\mathscr{D}$-Triples $(X)$ is the object $\mathcal{T}^{*}:=\left(\mathcal{N}^{\prime \prime}, \mathcal{M}^{\prime}, \mathfrak{c}^{*}\right)$. The adjoint of a morphism $\varphi=\left(\varphi^{\prime}, \varphi^{\prime \prime}\right)$ is the morphism $\varphi^{*}:=\left(\varphi^{\prime \prime}, \varphi^{\prime}\right)$. We clearly have $\mathcal{T}^{* *}=\mathcal{T}$ and $\varphi^{* *}=\varphi$.

Remark 10.2.10 (Adjoint of a graded triple). Let $\mathcal{T}_{\bullet}=\bigoplus_{k} \mathcal{T}_{k}$ be a graded object in $\mathscr{D}$-Triples $(X)$. We write $\mathcal{T}_{k}$ as $\left(\mathcal{M}_{k}^{\prime}, \mathcal{M}_{-k}^{\prime \prime}, \mathfrak{c}_{k}\right)$. The adjoint object is then

$$
\mathcal{T}_{\bullet}^{*}=\bigoplus_{k} \mathcal{T}_{k}^{*}:=\bigoplus_{k}\left(\mathcal{T}_{-k}\right)^{*}
$$

Remark 10.2.11 (Side-changing and adjunction in $\mathscr{D}$ - $\operatorname{Triples}(X)$ )
With the previous definitions, adjunction does not commute with side-changing when $n$ is odd, because of the sign in (10.2.5**).

Definition 10.2.12 (Pre-polarization, see Definition 2.4.32). A pre-polarization of weight $w$ of an object $\mathcal{T}$ of $\mathscr{D}$-Triples $(X)$ is a morphism $\mathrm{Q}: \mathcal{T} \rightarrow \mathcal{T}^{*}$ which is $(-1)^{w}$-Hermitian. In the graded category, we ask Q to be graded. We say Q is non-degenerate if it is an isomorphism.

Let us make this definition more explicit. We write $\mathrm{Q}=\left((-1)^{w} \mathcal{Q}, Q\right)$, where $\mathbb{Q}$ is a morphism $\mathcal{M}^{\prime} \rightarrow \mathcal{M}^{\prime \prime}$ which satisfies

$$
\left.\mathfrak{c}\left(m_{1}^{\prime}, \overline{\mathcal{Q} m_{2}^{\prime}}\right)=(-1)^{w} \mathfrak{c}^{*}\left(\underline{Q} m_{1}^{\prime}, \overline{m_{2}^{\prime}}\right)\right):=(-1)^{w} \mathfrak{c}\left(m_{2}^{\prime}, \overline{\mathcal{Q} m_{1}^{\prime}}\right),
$$

and the sesquilinear pairing on $\mathcal{M}^{\prime}$

$$
\left(m_{1}^{\prime}, m_{2}^{\prime}\right) \longmapsto \mathfrak{c}\left(m_{1}^{\prime}, \overline{2 m_{2}^{\prime}}\right)
$$

is $(-1)^{w}$-Hermitian in the usual sense. In the graded case, $\mathrm{Q}_{k}=\left(\mathcal{Q}_{k},(-1)^{w} \mathcal{Q}_{-k}\right)$, where each $Q_{k}$ is an isomorphism $\mathcal{M}_{k}^{\prime} \rightarrow \mathcal{I}_{-k}^{\prime \prime}$ which satisfies

$$
\left.\mathfrak{c}_{k}\left(m_{k}^{\prime}, \overline{\mathcal{Q}_{-k} m_{-k}^{\prime}}\right)=(-1)^{w} \overline{\mathfrak{c}_{-k}\left(m_{-k}^{\prime}, \overline{Q_{k} m_{k}^{\prime}}\right.}\right),
$$

so the graded sesquilinear pairing $\mathcal{M}_{k}^{\prime} \otimes \overline{\mathcal{M}_{-k}^{\prime \prime}} \rightarrow \mathfrak{C}_{X}\left(\right.$ or $\left.\mathfrak{D b}{ }_{X}\right)$

$$
\left(m_{k}^{\prime}, m_{-k}^{\prime}\right) \longmapsto \mathfrak{c}_{k}\left(m_{k}^{\prime}, \overline{Q_{-k} m_{-k}^{\prime}}\right)
$$

is $(-1)^{w}$-Hermitian in the graded sense.
The following is shown as in Remark 3.2.26.
Lemma 10.2.13. Any non-degenerate pre-polarized triple ( $\mathcal{T}, \mathrm{Q})$ of weight $w$ is isomorphic to a triple $\mathcal{T}_{1}=\left(\mathcal{M}_{1}, \mathcal{M}_{1}, \mathfrak{c}_{1}\right)$ such that $\mathfrak{c}_{1}^{*}=(-1)^{w} \mathfrak{c}_{1}$ (i.e., with polarization $\left.\left((-1)^{w} \mathrm{Id}, \mathrm{Id}\right)\right)$.

Similarly, any non-degenerate graded pre-polarized triple ( $\mathcal{T}_{.}$, Q. $^{\circ}$ ) of weight $w$ is isomorphic to a graded triple $\mathcal{T}_{1 \bullet}=\left(\mathcal{M}_{1 \bullet}, \mathcal{M}_{1 \bullet}, \mathfrak{c}_{1 \bullet}\right)$ such that $\mathfrak{c}_{1, k}^{*}=(-1)^{w} \mathfrak{c}_{1,-k}$ : $\mathcal{M}_{1,-k} \otimes \overline{\mathcal{N}_{1, k}} \rightarrow \mathfrak{C}_{X} \quad\left(\right.$ or $\left.\mathfrak{D b}_{X}\right)$.

In other words, working with non-degenerate pre-polarized triples of weight $w$ amounts to working with pairs $(\mathcal{M}, \mathfrak{c})$, where $\mathfrak{c}$ is a $(-1)^{w}$-Hermitian pairing $\mathcal{M} \otimes$ $\overline{\mathcal{M}} \rightarrow \mathfrak{C}_{X}\left(\right.$ or $\left.\mathfrak{D b}_{X}\right)$. Given such a pair, we associate to it the object $(\mathcal{M}, \mathcal{M}, \mathfrak{c})$ of $\mathscr{D}$-Triples $(X)$ with pre-polarization $\left((-1)^{w} \mathrm{Id}, \mathrm{Id}\right)$.

## Example 10.2.14 (Two basic examples).

(1) (Left case) The triple ${ }_{\mathrm{T}} \mathscr{O}_{X}=\left(\mathscr{O}_{X}, \mathscr{O}_{X}, \mathfrak{c}_{n}\right)$ is the smooth left triple with $\mathfrak{c}_{n}(1,1)=1$. It satisfies $\left({ }_{\mathrm{T}} \mathscr{O}_{X}\right)^{*}={ }_{\mathrm{T}} \mathscr{O}_{X}$. The identity morphism $\mathrm{Q}=(\mathrm{Id}, \mathrm{Id})$ : ${ }_{\mathrm{T}} \mathscr{O}_{X} \rightarrow\left({ }_{\mathrm{T}} \mathscr{O}_{X}\right)^{*}$ is an Hermitian non-degenerate pre-polarization.
(2) (Right case) The triple ${ }_{\mathrm{T}} \omega_{X}=\left(\omega_{X}, \omega_{X}, \mathfrak{c}_{n}\right)$ is the smooth right triple with $\mathfrak{c}_{n}\left(\omega^{\prime}, \overline{\omega^{\prime \prime}}\right)=\omega^{\prime} \wedge \overline{\omega^{\prime \prime}}$. We have $\mathfrak{c}_{n}^{*}=(-1)^{n} \mathfrak{c}_{n}$, and the morphism $\mathrm{Q}=\left((-1)^{n} \mathrm{Id}, \mathrm{Id}\right)$ : ${ }_{\mathrm{T}} \omega_{X} \rightarrow\left({ }_{\mathrm{T}} \omega_{X}\right)^{*}$ is a $(-1)^{n}$-Hermitian non-degenerate pre-polarization.

Definition 10.2.15 (Side-changing for a pre-polarization). Let $\mathcal{T}$ be a left $\mathscr{D}_{X}$-triple and let $\mathrm{Q}=\left(\mathbb{Q}^{\prime}, \mathbb{Q}^{\prime \prime}\right)$ be a pre-polarization of weight $w$ of it. We then set $\mathrm{Q}^{\text {right }}:=$ $\left((-1)^{n} \operatorname{Id} \otimes \mathbb{Q}^{\prime}, \operatorname{Id} \otimes \mathbb{Q}^{\prime \prime}\right)$, which is a pre-polarization of $\mathscr{T}^{\text {right }}$, with the natural morphisms

$$
\operatorname{Id} \otimes \mathfrak{Q}^{\prime}, \operatorname{Id} \otimes \mathfrak{Q}^{\prime \prime}: \omega_{X} \otimes \mathcal{M}^{\prime} \longrightarrow \omega_{X} \otimes \mathcal{M}^{\prime \prime}
$$

This defines a pre-polarization of weight $w+n$ of $\mathcal{T}^{\text {right }}$. Moreover, Q is non-degenerate if and only if $\mathrm{Q}^{\text {right }}$ is so.

Note that, for a $(-1)^{w}$-Hermitian left pair ( $\mathcal{M}, \mathfrak{c}$ ), the associated right pair is $\left(\omega_{X} \otimes \mathcal{M}, \mathfrak{c}^{\text {right }}\right)$.

Remark 10.2.16 (Lefschetz triples). The notion of (graded) Lefschetz structure ( $\mathcal{T}, \mathrm{N}$ ) in the abelian category $\mathscr{D}$-Triples $(X)$ is obtained from Definition 3.1.2. Using adjunction in $\mathscr{D}$-Triples $(X)$ (Definition 10.2.9), we obtain as in Definition 3.1.10 the notion of adjunction and pre-polarization of weight $w$ of a (graded) Lefschetz $\mathscr{D}$-triple ( $\mathcal{T}, \mathrm{N})$.

Any Lefschetz $\mathscr{D}$-triple $(\mathcal{T}, \mathrm{N})$ with a $(-1)^{w}$-Hermitian non-degenerate prepolarization Q is isomorphic to one of the form $((\mathcal{M}, \mathcal{M}, \mathfrak{c}),(\mathrm{N},-\mathrm{N}))$, with $\mathfrak{c}^{*}=(-1)^{w} \mathfrak{c}$, and $\mathrm{Q}=\left((-1)^{w} \mathrm{Id}, \mathrm{Id}\right)$. It is thus determined by the data $(\mathcal{M}, \mathfrak{c}, \mathrm{N})$, such that $\mathfrak{c}$ is $(-1)^{w}$-Hermitian and N is skew-adjoint with respect to $\mathfrak{c}$.

### 10.3. Pushforward in the category $\mathscr{D}$-Triples $(X)$

10.3.a. Pushforward of currents. Let $\eta$ be a $C^{\infty}$ form of maximal degree on $X$. If $f: X \rightarrow Y$ is a proper holomorphic map which is smooth, then the integral of $\eta$ in the fibres of $f$ is a $C^{\infty}$ form of maximal degree on $Y$, that one denotes by $\int_{f} \eta$.

If $f$ is not smooth, then $\int_{f} \eta$ is only defined as a current of degree 0 on $Y$, and the definition extends to the case where $\eta$ is itself a current of degree 0 on $X$ (see Appendix A.4.d for the notion of current).

## Exercise 10.3.1 (Pushforward of the sheaf of currents as a right $\mathscr{D}_{X, \bar{X}}$-module)

Extend the notion and properties of direct image of a right (resp. left) $\mathscr{D}_{X, \bar{X}^{-}}$ module, by introducing the transfer module $\mathscr{D}_{X \rightarrow Y, \bar{X} \rightarrow \bar{Y}}=\mathscr{D}_{X \rightarrow Y} \otimes_{\mathbb{C}} \mathscr{D}_{\bar{X} \rightarrow \bar{Y}}$. One denotes these direct images by ${ }_{\mathrm{D}, \overline{\mathrm{D}}} f_{*}$ or $\mathrm{D}, \overline{\mathrm{D}}^{f_{!}}$. In particular,

$$
{ }_{\mathrm{D}, \overline{\mathrm{D}}} f_{!} \mathfrak{C}_{X}:=\boldsymbol{R} f_{!}\left(\mathfrak{C}_{X} \otimes_{\mathscr{D}_{X, \bar{X}}} \mathrm{Sp}_{X \rightarrow Y, \bar{X} \rightarrow \bar{Y}}\left(\mathscr{D}_{X, \bar{X}}\right)\right)
$$

Definition 10.3.2 (Integration of currents of degree $(p, q)$ ). Let $f: X \rightarrow Y$ be a proper holomorphic map and let $u$ be a current of degree $(p, q)$ on $X$. The current $\int_{f} u$ of degree $(p, q)$ on $Y$ is defined by

$$
\left\langle\int_{f} u, \eta\right\rangle=\langle u, \eta \circ f\rangle, \quad \forall \eta \in \mathscr{E}_{\mathrm{c}}^{p, q}(Y) .
$$

This definition extends in a straightforward way if $f$ is only assumed to be proper on the support of $u$.

We continue to assume that $f$ is proper. We will now show how the integration of currents is used to defined a natural $\mathscr{D}_{Y, \bar{Y}}$ morphism $\mathscr{H}^{0}{ }_{\mathrm{D}, \overline{\mathrm{D}}} f_{*} \mathfrak{C}_{X} \rightarrow \mathfrak{C}_{Y}$. Let us first treat as an exercise the case of a closed embedding.

Exercise 10.3.3. Assume that $X$ is a closed submanifold of $Y$ and denote by $\iota: X \hookrightarrow Y$ the embedding (which is a proper map). Denote by 1 the canonical section of $\mathscr{D}_{X \rightarrow Y, \bar{X} \rightarrow \bar{Y}}$. Show that the natural map

$$
\mathscr{H}_{\mathrm{D}, \overline{\mathrm{D}} \iota_{*}}^{0} \mathfrak{C}_{X}=\iota_{*}\left(\mathfrak{C}_{X} \otimes_{\mathscr{D}_{X, \bar{X}}} \mathscr{D}_{X \rightarrow Y, \bar{X} \rightarrow \bar{Y}}\right) \longrightarrow \mathfrak{C}_{Y}, \quad u \otimes \mathbf{1} \longmapsto \int_{\iota} u
$$

induces an isomorphism of the right $\mathscr{D}_{Y, \bar{Y}}$-module $\mathscr{H}^{0}{ }_{\mathrm{D}, \overline{\mathrm{D}}} \iota_{*} \mathfrak{C}_{X}$ with the submodule of $\mathfrak{C}_{Y}$ consisting of currents supported on $X$. [Hint: use a local computation.]

For example, consider the case $\iota: X=X \times\{0\} \hookrightarrow X \times \mathbb{C}$, with coordinate $t$ on $\mathbb{C}$ and identify $\mathscr{H}^{0}{ }_{\mathrm{D}, \overline{\mathrm{D}}} \iota_{*} \mathfrak{C}_{X}$ with $\iota_{*} \mathfrak{C}_{X}\left[\partial_{t}, \partial_{\bar{t}}\right]$.

The integration of currents is a morphism

$$
\int_{f}: f_{*} \mathfrak{D b}_{X, p, q} \longrightarrow \mathfrak{D b}_{Y, p, q}
$$

which is compatible with the $\mathrm{d}^{\prime}$ and $\mathrm{d}^{\prime \prime}$ differentials of currents on $X$ and $Y$. In other words, taking the associated simple complex, it is a morphism of complexes

$$
\int_{f}: f_{*} \mathfrak{D b}_{X}^{\bullet}[2 n] \longrightarrow \mathfrak{D b}_{Y}^{\bullet}[2 m] .
$$

Let us notice that the integration of currents is compatible with conjugation. Namely, given a current $u_{p, q} \in \Gamma\left(X, \mathfrak{D} \mathfrak{b}_{X}^{n-p, n-q}\right)$, its conjugate $\overline{u_{p, q}} \in \Gamma\left(X, \mathfrak{D} \mathfrak{b}_{X}^{n-q, n-p}\right)$ is defined by the relation

$$
\left\langle\overline{u_{p, q}}, \eta^{q, p}\right\rangle:=\overline{\left\langle u_{p, q}, \overline{\eta^{q, p}}\right\rangle}
$$

for any test form $\eta^{q, p}$. Then we clearly have

$$
\begin{equation*}
\int_{f} \overline{u_{p, q}}=\overline{\int_{f} u_{p, q}} \tag{10.3.4}
\end{equation*}
$$

Since $\mathfrak{C}_{X}=\left(\mathfrak{D b}_{X}\right)^{\text {right,right }}$ as a right $\mathscr{D}_{X, \bar{X}}$-module, we can apply Exercise A.8.24(4) to get, since $f$ is proper,

$$
\begin{equation*}
\mathrm{D}, \overline{\mathrm{D}} f_{*} \mathfrak{C}_{X} \simeq f_{*}\left(\mathfrak{D b}_{X}^{\bullet}[2 n] \otimes_{f^{-1}} \mathscr{O}_{Y, \bar{Y}} f^{-1} \mathscr{D}_{Y, \bar{Y}}\right)=f_{*} \mathfrak{D b}_{X}^{\bullet}[2 n] \otimes_{\mathscr{O}_{Y, \bar{Y}}} \mathscr{D}_{Y, \bar{Y}} \tag{10.3.5}
\end{equation*}
$$

The integration of currents $\int_{f}$ induces then a $\mathscr{D}_{Y, \bar{Y}}$-linear morphism of complexes

$$
\begin{equation*}
\int_{f}::_{\mathrm{D}, \overline{\mathrm{D}}} f_{*} \mathfrak{C}_{X} \longrightarrow{\mathfrak{D} \mathfrak{b}_{Y}^{\bullet}[2 m] \otimes_{\mathscr{O}_{Y, \bar{Y}}} \mathscr{D}_{Y, \bar{Y}} \simeq \mathfrak{C}_{Y}, ~}_{\text {. }} \tag{10.3.6}
\end{equation*}
$$

where we recall that the differential on the complex $\mathfrak{D b}_{Y}^{\bullet}[2 m] \otimes_{\mathscr{O}_{Y, \bar{Y}}} \mathscr{D}_{Y, \bar{Y}}$ uses the universal connection $\nabla^{Y}+\overline{\nabla^{Y}}$ on $\mathscr{D}_{Y, \bar{Y}}$, and the isomorphism with $\mathfrak{C}_{Y}$ is given by (10.2.2). If we star from distributions, we have a morphism

$$
\begin{equation*}
\int_{f}:_{\mathrm{D}, \overline{\mathrm{D}}} f_{*} \mathfrak{D b}_{X}={ }_{\mathrm{D}, \overline{\mathrm{D}}} f_{*} \mathfrak{C}_{X}[2(m-n)] \longrightarrow \mathfrak{C}_{Y}[2(m-n)] \tag{10.3.7}
\end{equation*}
$$

Exercise 10.3.8. Extend the result of Exercise A.8.20 to the case of right $\mathscr{D}_{X, \bar{X}}$-modules and show that the composed map

$$
f_{*} \mathfrak{C}_{X} \longrightarrow \mathscr{H}_{\mathrm{D}, \overline{\mathrm{D}}}^{0} f_{*} \mathfrak{C}_{X} \longrightarrow \mathfrak{C}_{Y}
$$

is the integration of currents of Definition 10.3.2.
Exercise 10.3.9. Let $f: X \rightarrow Y$ be a holomorphic map and let $Z \subset X$ be a closed subset on which $f$ is proper.
(1) Define the sub- $\mathscr{D}_{X, \bar{X}}$-module $\mathfrak{C}_{X, Z}$ of $\mathfrak{C}_{X}$ consisting of currents supported on $Z$.
(2) Show that the integration of currents $\int_{f}$ induces a $\mathscr{D}_{Y, \bar{Y}^{-}}$-linear morphism of complexes

$$
\int_{f}:_{\mathrm{D}, \overline{\mathrm{D}}} f_{!} \mathfrak{C}_{X, Z} \longrightarrow \mathfrak{D b}_{Y}^{\bullet}[2 m] \otimes_{\mathscr{O}_{Y, \bar{Y}}} \mathscr{D}_{Y, \bar{Y}} \simeq \mathfrak{C}_{Y}
$$

10.3.b. Pushforward of a sesquilinear pairing. In order to define the pushforward of a sesquilinear pairing in the case of right $\mathscr{D}$-modules, it will be convenient to start from left $\mathscr{D}_{X}$-modules as in Exercise A.8.24, and use side changing at the source to get the right definition.

Let $\mathcal{M}^{\prime \text { left }}, \mathcal{M}^{\prime \prime \text { left }}$ be left $\mathscr{D}_{X}$-modules, let $\mathcal{M}^{\prime \text { right }}, \mathcal{N}^{\prime \prime \text { right }}$ be the associated $\mathscr{D}_{X}$-modules and let $f: X \rightarrow Y$ be a holomorphic map which is proper when restricted to $Z:=\operatorname{Supp} \mathcal{N}^{\prime} \cup \operatorname{Supp} \mathcal{M}^{\prime \prime}$. Let $\mathfrak{c}^{\text {left }}: \mathcal{M}^{\prime \text { left }} \otimes_{\mathbb{C}} \overline{\mathcal{N}^{\prime \prime l} \text { left }} \rightarrow \mathfrak{D b}_{X, Z}$ be a sesquilinear pairing and let $\mathfrak{c}^{\text {right }}$ the corresponding right sesquilinear pairing.

Our aim is to define, for every $k \in \mathbb{Z}$, a sesquilinear pairing:

$$
\begin{equation*}
{ }_{\mathrm{D}, \overline{\mathrm{D}}} f_{!}^{k} \mathfrak{c}^{\text {right }}: \mathscr{H}^{k}\left({ }_{\mathrm{D}} f_{!} \mathcal{M}^{\prime \text { right }}\right) \otimes_{\mathbb{C}} \overline{\mathscr{H}^{-k}\left({ }_{\mathrm{D}} f_{!} \mathcal{N}^{1 / \text { right }}\right)} \longrightarrow \mathfrak{C}_{Y} \tag{10.3.10}
\end{equation*}
$$

In order to integrate differential forms (and not poly-vector fields) we will use the formula of Exercise A.8.24(1) for computing the direct image as a complex of right $\mathscr{D}_{Y}$-modules, namely,

$$
{ }_{\mathrm{D}} f_{!} \mathcal{M}^{\text {right }} \xrightarrow{\sim} \boldsymbol{R} f_{!} \Omega_{X}^{\bullet}\left(\mathcal{M}^{\text {left }} \otimes_{f^{-1}} \mathscr{O}_{Y} f^{-1} \mathscr{D}_{Y}\right)[n],
$$

where the isomorphism is induced termwise by the morphism in Lemma A.5.8. Moreover, it will be convenient to compute the direct image $\boldsymbol{R} f$ ! by using flabby sheaves more adapted to the computation than the Godement sheaves. According to Remark 10.2.4, it is enough to define the $C^{\infty}$ extension of $_{\mathrm{D}, \overline{\mathrm{D}}} f_{!}^{k} \mathfrak{c}$, so we will use the formula

$$
\boldsymbol{R} f_{!} \Omega_{X}^{\bullet}\left(\mathcal{M}^{\text {left }} \otimes_{f-1} \mathscr{O}_{Y} f^{-1} \mathscr{D}_{Y}\right) \otimes_{\mathscr{O}_{Y}} \mathscr{C}_{Y}^{\infty} \xrightarrow{\sim} f_{!} \mathscr{E}_{X}^{\bullet}\left(\mathcal{N}^{\text {left }} \otimes_{f-1} \mathscr{O}_{Y} f^{-1} \mathscr{D}_{Y}\right)
$$

obtained from the Dolbeault resolution $\Omega_{X}^{k} \xrightarrow{\sim}\left(\mathscr{E}^{(k, \bullet)}, \mathrm{d}^{\prime \prime}\right)$ and by taking the associated simple complex. Lastly, we identify each term of this complex with

$$
\begin{equation*}
f_{!}\left(\mathscr{E}_{X}^{\bullet} \otimes_{\mathscr{O}_{X}} \mathcal{M}^{\mathrm{left}}\right) \otimes_{\mathscr{O}_{Y}} \mathscr{D}_{Y} \tag{10.3.11}
\end{equation*}
$$

and, with this identification, the differential is given by the formula

$$
\left[\left(\mathrm{d} \otimes \mathrm{Id}_{\mathcal{M}^{\text {left }}}\right) \otimes \mathrm{Id}\right]+[(\mathrm{Id} \otimes \nabla) \otimes \mathrm{Id}]+\left[(\mathrm{Id} \otimes \mathrm{Id}) \otimes f_{!} f^{*} \nabla^{Y}\right]
$$

where $\nabla^{Y}$ is the universal connection on $\mathscr{D}_{Y}$.
The $C^{\infty}$ extension of $\mathfrak{c}^{\text {left }}$ is denoted by $\mathfrak{c}_{\infty}^{\text {left }}$, and it induces a morphism

$$
\begin{align*}
& \mathfrak{c}_{\infty}^{\text {left }}:\left(\mathscr{E}_{X}^{k} \otimes_{\mathscr{O}_{X}} \mathcal{M}^{\prime \text { left }}\right) \otimes_{\mathscr{C}_{X}^{\infty}} \overline{\left(\mathscr{E}_{X}^{\ell} \otimes_{\mathscr{O}_{X}} \mathcal{M}^{\prime \prime \text { left }}\right)}  \tag{10.3.12}\\
& \longrightarrow \mathfrak{D}_{X, Z}^{k+\ell} \\
&\left(\eta^{\prime k} \otimes m^{\prime}\right) \otimes \overline{\eta^{\prime \prime \ell} \otimes m^{\prime \prime}} \longmapsto \eta^{\prime k} \wedge \overline{\eta^{\prime \prime \ell}} \mathfrak{c}^{\text {left }}\left(m^{\prime}, \overline{m^{\prime \prime}}\right)
\end{align*}
$$

and by applying $f_{!}$,

$$
f_{!} \mathrm{c}_{\infty}^{\text {left }}: f_{!}\left(\mathscr{E}_{X}^{k} \otimes_{\mathscr{O}_{X}} \mathcal{M}^{\prime \text { left }}\right) \otimes_{f_{!} \mathscr{C}_{X}^{\infty}} f_{!}\left(\overline{\mathscr{E}_{X}^{\ell} \otimes_{\mathscr{O}_{X}} \mathcal{M}^{\prime \prime \text { left }}}\right) \longrightarrow f_{!} \mathfrak{D b}_{X, Z}^{k+\ell}
$$

so, by right $\mathscr{D}_{Y, \bar{Y}}$-linearity, a morphism

$$
\begin{aligned}
& \left({ }_{\mathrm{D}, \overline{\mathrm{D}}} f_{!} \mathrm{c}_{\infty}^{\text {left }}\right)^{\text {right }}:\left[f_{!}\left(\mathscr{E}_{X}^{k} \otimes_{\mathscr{O}_{X}} \mathcal{M}^{\text {fleft }}\right) \otimes_{\mathscr{O}_{Y}} \mathscr{D}_{Y}\right] \otimes_{\mathscr{C}_{Y}^{\infty}} \overline{f_{!}\left(\mathscr{E}_{X}^{\ell} \otimes_{\mathscr{O}_{X}} \mathcal{M}^{\prime \prime \text { left }}\right) \otimes_{\mathscr{O}_{Y}} \mathscr{D}_{Y}} \\
& \longrightarrow f_{!}\left(\mathfrak{D b}_{X, Z}^{k+\ell}\right) \otimes_{\mathscr{O}_{Y, \bar{Y}}} \mathscr{D}_{Y, \bar{Y}} .
\end{aligned}
$$

The compatibility of $\mathfrak{c}^{\text {left }}$ with the connections on $\mathcal{M}^{\prime \text { left }}, \mathcal{M}^{\prime \prime l}$ left implies that this morphism is compatible with the differentials, so that, with respect to the identifications
above and according to (10.3.5), we get a morphism of complexes of right $\mathscr{D}_{Y, \bar{Y}^{-}}$ modules

$$
\mathrm{D}, \overline{\mathrm{D}} f_{!} \mathrm{r}_{\infty}^{\text {right }}:\left({ }_{\mathrm{D}} f_{!} \mathcal{M}^{\prime \text { right }} \otimes_{\mathscr{O}_{Y}} \mathscr{C}_{Y}^{\infty}\right) \otimes_{\mathscr{C}_{Y}^{\infty}} \overline{\left(\mathrm{D} f_{!} \mathcal{M}^{1 / \mathrm{right}} \otimes_{\mathscr{O}_{Y}} \mathscr{C}_{Y}^{\infty}\right)} \longrightarrow_{\mathrm{D}, \overline{\mathrm{D}}} f_{!} \mathfrak{C}_{X, Z}
$$

Composing with the integration of currents (see Exercise 10.3.9)

$$
\int_{f}:_{\mathrm{D}, \overline{\mathrm{D}}} f_{!} \mathfrak{C}_{X, Z} \longrightarrow \mathfrak{C}_{Y}
$$

we finally get a morphism of complexes of right $\mathscr{D}_{Y, \bar{Y}}$-modules (where $\mathfrak{C}_{Y}$ is regarded as a complex having a single term in degree zero) that we denote by the same symbol:

$$
{ }_{\mathrm{D}, \overline{\mathrm{D}}} f_{1} \mathrm{c}_{\infty}^{\text {right }}:\left({ }_{\mathrm{D}} f_{!} \mathcal{M}^{\prime \mathrm{right}} \otimes_{\mathscr{O}_{Y}} \mathscr{C}_{Y}^{\infty}\right) \otimes_{\mathscr{C}_{Y}^{\infty}} \overline{\left({ }_{\mathrm{D}} f_{!} \mathcal{M}^{\prime / \text { right }} \otimes_{\mathscr{O}_{Y}} \mathscr{C}_{Y}^{\infty}\right)} \longrightarrow \mathfrak{C}_{Y}
$$

By restricting to the holomorphic/anti-holomorphic part (and side changing from right to left in the left case), we obtain the pushforward morphism

$$
{ }_{\mathrm{D}, \overline{\mathrm{D}}} f_{!} \mathrm{c}^{\text {right }}:_{\mathrm{D}} f_{!} \mathcal{M}^{\prime \text { right }} \otimes_{\mathbb{C}} \overline{{ }_{\mathrm{D}} f_{!} \mathcal{M}^{\prime / \text { right }}} \longrightarrow \mathfrak{C}_{Y}
$$

Forgetting now the "right" exponent on $\mathfrak{c}$, we denote by ${ }_{\mathrm{D}, \overline{\mathrm{D}}} f_{!}^{\bullet} \mathfrak{c}$ the induced morphism

$$
\begin{equation*}
{ }_{\mathrm{D}, \overline{\mathrm{D}}} f_{!}^{k} \mathfrak{c}: \mathscr{H}_{\mathrm{D}}^{k} f_{!} \mathcal{M}^{\prime \text { right }} \otimes_{\mathbb{C}} \overline{\mathscr{H}^{-k}{ }_{\mathrm{D}} f_{!} \mathcal{M}^{\prime / \mathrm{right}}} \longrightarrow \mathfrak{C}_{Y} \tag{10.3.13}
\end{equation*}
$$

Remark 10.3.14 (Making explicit the pairing ${ }_{\mathrm{D}, \overline{\mathrm{D}}} f_{!}^{k} \mathfrak{c}$ ). We assume that $f$ is proper on $\operatorname{Supp} \mathcal{M}^{\prime}$ and $\operatorname{Supp} \mathcal{M}^{\prime \prime}$, so that we will consider the ordinary pushforward $f_{*}$. Let $U$ be an open set in $Y$, and let

$$
m_{\infty}^{\prime n+k} \in \Gamma\left(U, f_{*}\left(\mathscr{E}_{X}^{n+k} \otimes_{\mathscr{O}_{X}} \mathcal{N}^{\prime \mathrm{left}}\right)\right), \quad m_{\infty}^{\prime \prime n-k} \in \Gamma\left(U, f_{*}\left(\mathscr{E}_{X}^{n-k} \otimes_{\mathscr{O}_{X}} \mathcal{N}^{\prime \prime \mathrm{left}}\right)\right)
$$

Then $\int_{f} f_{*} \mathfrak{c}_{\infty}^{\text {left }}\left(m_{\infty}^{\prime n+k}, \overline{m_{\infty}^{\prime \prime n-k}}\right)$ belongs to $\Gamma\left(U, \mathfrak{C}_{Y}\right)$. If the section $m_{\infty}^{\prime n+k} \otimes 1$ of $f_{*}\left(\mathscr{E}_{X}^{n+k} \otimes_{\mathscr{O}_{X}} \mathcal{M}^{\prime \text { left }}\right) \otimes \mathscr{D}_{Y}\left(\right.$ resp. $\left.m_{\infty}^{\prime \prime n-k} \otimes 1\right)$ is closed with respect to the differential of the complex (10.3.11), then, denoting by [•] the cohomology class, we get

$$
\mathrm{D}, \overline{\mathrm{D}}^{f_{*}^{k}} \mathfrak{c}\left(\left[m_{\infty}^{\prime n+k} \otimes 1\right], \overline{\left[m_{\infty}^{\prime \prime n-k} \otimes 1\right]}\right)=\int_{f} f_{*} \mathrm{c}_{\infty}^{\text {left }}\left(m_{\infty}^{\prime n+k}, \overline{m_{\infty}^{\prime \prime n-k}}\right) \in \Gamma\left(U, \mathfrak{C}_{Y}\right)
$$

Remark 10.3.15 (Pushforward and adjunction). Let us denote by $\mathfrak{c}_{\infty}^{k, \ell}$ the sesquilinear form (10.3.12). Due to the relation $\overline{\eta^{\prime k}} \wedge \eta^{\prime \prime \ell}=(-1)^{k \ell} \eta^{\prime \prime \ell} \wedge \overline{\eta^{\prime k}}$, we have

$$
\begin{aligned}
\left(\mathfrak{c}_{\infty}^{*}\right)^{\ell, k}\left(\eta^{\prime \prime \ell} \otimes m^{\prime \prime}, \overline{\eta^{\prime k} \otimes m^{\prime}}\right) & =\eta^{\prime \prime \ell} \wedge \overline{\eta^{\prime k}} \mathfrak{c}^{*}\left(m^{\prime \prime}, \overline{m^{\prime}}\right) \\
& \left.=(-1)^{k, \ell} \overline{\mathfrak{c}_{\infty}^{k, \ell}\left(\eta^{\prime k} \otimes m^{\prime}, \overline{\eta^{\prime \prime \ell} \otimes m^{\prime \prime}}\right.}\right) \\
& =(-1)^{k, \ell}\left(\mathfrak{c}_{\infty}^{k, \ell}\right)^{*}\left(\eta^{\prime \prime \ell} \otimes m^{\prime \prime}, \overline{\eta^{\prime k} \otimes m^{\prime}}\right)
\end{aligned}
$$

It follows that the behaviour with respect to adjunction of $\left({ }_{\mathrm{D}, \mathrm{D}} f_{!}^{n-k} \boldsymbol{c}^{\text {left }}\right)^{\text {right }}$ (defined in a way similar to the previous formulas) is given by the relation

$$
\left(\mathrm{D}, \overline{\mathrm{D}} f_{!}^{n+k} \mathfrak{c}^{\text {left }}\right)^{\text {right } *}=(-1)^{n+k}\left({ }_{\mathrm{D}, \overline{\mathrm{D}}} f_{!}^{n-k} \mathfrak{c}^{\text {left } *}\right)^{\text {right }}
$$

Since we have

$$
{ }_{\mathrm{D}, \overline{\mathrm{D}}} f_{!}^{k} \mathfrak{c}^{\text {right }}=\left(\mathrm{D}, \overline{\mathrm{D}} f_{!}^{n+k} \mathfrak{c}^{\text {left }}\right)^{\text {right }}
$$

the behaviour with respect to adjunction is given by the formula

$$
\begin{equation*}
{ }_{\mathrm{D}, \overline{\mathrm{D}}} f_{!}^{k}\left(\mathfrak{c}^{*}\right)=(-1)^{k}\left(\mathrm{D}, \overline{\mathrm{D}} f_{!}^{-k} \mathfrak{c}\right)^{*} . \tag{10.3.15*}
\end{equation*}
$$

## 10.3.c. Pushforward of $\mathscr{D}$-triples

Remark 10.3.16 (Rule of signs for the pushforward). Before defining the proper pushforward of an object of $\mathscr{D}$-Triples $(X)_{\text {coh }}$, we will modify the definition of the pushforward ${ }_{\mathrm{D}, \overline{\mathrm{D}}} f_{*}^{k} \mathfrak{c}$ of the sesquilinear pairing in order to eliminate various signs. For every $k$, we set (recall that $\varepsilon(k)=(-1)^{k(k-1) / 2}$ ):

$$
\begin{equation*}
{ }_{\mathrm{\tau}} f_{*}^{k} \mathfrak{c}:=\varepsilon(n-m+k) \cdot{ }_{\mathrm{D}, \overline{\mathrm{D}}} f_{*}^{k}, \tag{10.3.16*}
\end{equation*}
$$

so that, since $\varepsilon(n-m-k)=(-1)^{k} \varepsilon(n-m+k),(10.3 .15 *)$ becomes

$$
\begin{equation*}
{ }_{\mathrm{r}} f_{!}^{k}\left(\mathfrak{c}^{*}\right)=\left({ }_{\mathrm{r}} f_{!}^{-k} \mathfrak{c}\right)^{*} \tag{10.3.16**}
\end{equation*}
$$

(The choice of $\varepsilon(n-m+k)$ instead of $\varepsilon(k)$ will be justified by the side-changing formula.)

Definition 10.3.17 (Proper pushforward). Let $\mathcal{T}$ be an object of $\mathscr{D}$-Triples $(X)_{\text {coh }}$ supported on $Z$ and let $f: X \rightarrow X^{\prime}$ be a holomorphic map which is proper on $Z$. Then ${ }_{\text {т }} f_{*}^{k} \mathcal{T}$ is the object

$$
\left(\mathscr{H}^{k}{ }_{\mathrm{D}} f_{*} \mathcal{M}^{\prime}, \mathscr{H}^{-k}{ }_{\mathrm{D}} f_{*} \mathcal{M}^{\prime \prime},{ }_{\mathrm{T}} f_{*}^{k} \mathfrak{c}\right)
$$

of $\mathscr{D}$-Triples $\left(X^{\prime}\right)_{\text {coh }}$. The total pushforward is the graded object $\bigoplus_{k \mathrm{~T}} f_{*}^{k} \mathcal{T}$, with ${ }_{\mathrm{T}} f_{*}^{k} \mathcal{T}$ in degree $k$.

Example 10.3.18 (Kashiwara's equivalence in $\mathscr{D}$-Triples). Assume that $\iota: X \hookrightarrow Y$ is a closed immersion and let $\mathcal{M}=\mathcal{M}^{\prime}, \mathcal{N}^{\prime \prime}$ be right $\mathscr{D}_{X}$-modules. We then have ${ }_{\mathrm{D}} \iota_{*} \mathcal{M}=\mathscr{H}^{0}{ }_{\mathrm{D}} \iota_{*} \mathcal{M}=\iota_{*}\left(\mathcal{M} \otimes_{\mathscr{D}_{X}} \mathscr{D}_{X \rightarrow Y}\right)$. Let 1 denote the canonical section of $\mathscr{D}_{X \rightarrow Y}=\mathscr{O}_{X} \otimes_{\iota^{-1}} \mathscr{O}_{Y} \iota^{-1} \mathscr{D}_{Y}$. It is a generator of $\mathscr{D}_{X \rightarrow Y}$ as a right $\iota^{-1} \mathscr{D}_{Y}$-module. Any sesquilinear pairing $\mathfrak{c}_{Y}: \mathscr{H}^{0}{ }_{\mathrm{D}} \iota_{*} \mathcal{M}^{\prime} \otimes \mathscr{H}^{0}{ }_{\mathrm{D}} \iota_{*} \overline{\mathcal{N}^{\prime \prime}} \rightarrow \mathfrak{C}_{Y}$ takes values in $\mathfrak{C}_{Y, X}$ and is determined by its restriction $\mathfrak{c}_{Y \mid X}$ to $\iota_{*}\left(\mathcal{M}^{\prime} \otimes \overline{\mathbf{1}}\right) \otimes \iota_{*}\left(\overline{\mathcal{N}^{\prime \prime}} \otimes \mathbf{1}\right)$. Hence it takes the form ${ }_{\mathrm{D}, \overline{\mathrm{D}} \iota_{*}^{0} \mathfrak{c}_{X} \text {. For local sections } m^{\prime}, m^{\prime \prime} \text { of } \mathcal{M}^{\prime}, \mathcal{M}^{\prime \prime} \text {, the current of degree } 0}$ ${ }_{\mathrm{D}, \overline{\mathrm{D}}}{ }^{0} \mathfrak{c}_{X}\left(m^{\prime} \otimes \mathbf{1}, \overline{m^{\prime \prime} \otimes \mathbf{1}}\right)$ is the pushforward by $\iota$ of the current $\mathfrak{c}\left(m^{\prime}, \overline{m^{\prime \prime}}\right)$. Together with the rule of signs (10.3.16*), we conclude that

$$
{ }_{\mathrm{T}} \iota_{*}^{0}: \mathscr{D} \text {-Triples }(X)_{\mathrm{coh}} \longmapsto \mathscr{D} \text {-Triples }_{X}(Y)_{\mathrm{coh}}
$$

is an equivalence of categories, compatible with adjunction.
Remark 10.3.19 (Pushforward of a $(-1)^{w}$-Hermitian pair). The behaviour of a prepolarized triple of weight $w$ by pushforward is determined by the behaviour of the associated $(-1)^{w}$-Hermitian pair (see Lemma 10.2.13).

If $(\mathcal{M}, \mathfrak{c})$ is a $(-1)^{w}$-Hermitian pair, then the pushforward $\left({ }_{\mathrm{D}} f_{*}^{\bullet} \mathcal{M},{ }_{\mathrm{T}} f_{*}^{\bullet} \mathfrak{c}\right)$ graded $(-1)^{w}$-Hermitian pair.

Remark 10.3.20 (The Lefschetz morphism). In the previous setting, let $\eta$ be a closed $(1,1)$-form on $X$ which is real, i.e., such that $\bar{\eta}=\eta$. This condition is satisfied if the cohomology class of $\eta$ is equal to $c_{1}(\mathscr{L})$ for some line bundle $\mathscr{L}$ on $X$. The corresponding Lefschetz morphism $\mathcal{L}_{\eta}:{ }_{\mathrm{D}} f_{*} \mathcal{M} \rightarrow{ }_{\mathrm{D}} f_{*} \mathcal{M}[2]$ with $\mathcal{M}=\mathcal{N}^{\prime}, \mathcal{N}^{\prime \prime}$ (see Definition A.8.16 and Remark A.8.18) satisfies then

$$
\mathrm{D}, \overline{\mathrm{D}} f_{*}^{k+2} \mathfrak{c}\left(\mathcal{L}_{\eta} m^{\prime}, \overline{m^{\prime \prime}}\right)={ }_{\mathrm{D}, \overline{\mathrm{D}}} f_{*}^{k} \mathfrak{c}\left(m^{\prime}, \overline{\mathcal{L}_{\eta} m^{\prime \prime}}\right)
$$

if $m^{\prime}$ (resp. $m^{\prime \prime}$ ) is a local section of ${ }_{\mathrm{D}} f_{*}^{k} \mathcal{M}^{\prime}$ (resp. of ${ }_{\mathrm{D}} f_{*}^{-k-2} \mathcal{M}^{\prime \prime}$ ), that is, because of $\varepsilon(n-m+k+2)=-\varepsilon(n-m+k)$,

$$
\begin{equation*}
{ }_{\mathrm{T}} f_{*}^{k+2} \mathfrak{c}\left(\mathcal{L}_{\eta} m^{\prime}, \overline{m^{\prime \prime}}\right)=-_{\mathrm{T}} f_{*}^{k} \mathfrak{c}\left(m^{\prime}, \overline{\mathcal{L}_{\eta} m^{\prime \prime}}\right) . \tag{10.3.20*}
\end{equation*}
$$

We can thus define a Lefschetz morphism by anti-symmetrization

$$
\begin{align*}
& \mathrm{L}_{\eta}=\left(\mathcal{L}_{\eta}^{\prime}, \mathcal{L}_{\eta}^{\prime \prime}\right):{ }_{\mathrm{T}} f_{*}^{k} \mathcal{T} \longrightarrow{ }_{\mathrm{T}} f_{*}^{k+2} \mathcal{T},  \tag{10.3.20**}\\
& \begin{cases}\mathcal{L}_{\eta}^{\prime}:=\mathcal{L}_{\eta} & \text { on } \mathscr{H}^{k}{ }_{\mathrm{D}} f_{*} \mathcal{N}^{\prime}, \\
\mathcal{L}_{\eta}^{\prime \prime}:=-\mathcal{L}_{\eta} & \text { on } \mathscr{H}^{-k-2}{ }_{\mathrm{D}} f_{*} \mathcal{N}^{\prime \prime}\end{cases}
\end{align*}
$$

It is functorial with respect to $\mathcal{T}$ and satisfies $\mathrm{L}_{\eta}^{*}=-\mathrm{L}_{\eta}$.
Let $(\mathcal{M}, \mathfrak{c})$ be a $(-1)^{w}$-Hermitian pair (corresponding to a pre-polarized triple of weight $w$ ). Then $(10.3 .20 *)$ implies that, for $k \geqslant 0$, the sesquilinear pairing

$$
\left({ }_{\mathrm{r}} f_{*} \mathfrak{c}\right)^{(-k)}={ }_{\mathrm{T}} f_{*}^{-k} \mathfrak{c}\left(\cdot, \overline{\mathcal{L}_{\eta}^{k}}\right):{ }_{\mathrm{r}} f_{*}^{-k} \mathcal{M} \otimes_{\mathrm{T}} f_{*}^{-k} \mathcal{M} \longrightarrow \mathfrak{C}_{Y}
$$

is $(-1)^{w+k}$-Hermitian.
10.3.d. Pushforward of left $\mathscr{D}$-triples. In a way analogous to (10.3.10), we first define a sesquilinear pairing

$$
\begin{equation*}
{ }_{\mathrm{D}, \overline{\mathrm{D}}} f_{!}^{n-m+k} \mathfrak{c}^{\text {left }}: \mathscr{H}^{n-m+k}\left({ }_{\mathrm{D}} f_{!} \mathcal{M}^{\prime \mathrm{left}}\right) \otimes_{\mathbb{C}} \overline{\mathscr{H}^{n-m-k}\left({ }_{\mathrm{D}} f_{!} \mathcal{M}^{\prime / \mathrm{left}}\right)} \longrightarrow \mathfrak{D}_{Y} \tag{10.3.21}
\end{equation*}
$$

According to Exercise A.8.24(1), we have

$$
\left({ }_{\mathrm{D}} f_{!} \mathcal{M}^{\text {left }}\right)^{\text {right }} \xrightarrow{\sim} \boldsymbol{R} f_{!} \Omega_{X}^{\bullet}\left(\mathcal{N}^{\text {left }} \otimes_{f^{-1} \mathscr{O}_{Y}} f^{-1} \mathscr{D}_{Y}\right)[m]
$$

By using (10.3.12) we obtain a morphism of complexes of right $\mathscr{D}_{Y, \bar{Y}}$-modules

$$
\begin{aligned}
&\left.\left({ }_{\mathrm{D}, \overline{\mathrm{D}}} f_{!} \mathfrak{c}_{\infty}^{\text {left }}\right)^{\text {right }}:\left(\left({ }_{\mathrm{D}} f_{!} \mathcal{M}^{\prime \text { left }}\right)^{\text {right }} \otimes_{\mathscr{O}_{Y}} \mathscr{C}_{Y}^{\infty}\right) \otimes_{\mathscr{C}_{Y}^{\infty}} \overline{\left(\left(_{\mathrm{D}} f_{!} \mathcal{M}^{\prime \prime \text { left }}\right)^{\text {right }}\right.} \otimes_{\mathscr{O}_{Y}} \mathscr{C}_{Y}^{\infty}\right) \\
& \longrightarrow \mathrm{D}, \overline{\mathrm{D}} f_{!} \mathfrak{C}_{X, Z}[2(m-n)]
\end{aligned}
$$

and by composing with the integration of currents, we finally get a morphism of complexes of right $\mathscr{D}_{Y, \bar{Y}}$-modules

$$
\begin{aligned}
& \left({ }_{\mathrm{D}, \overline{\mathrm{D}}} f_{!} \mathfrak{c}_{\infty}^{\text {left }}\right)^{\text {right }}:\left(\left(\left(_{\mathrm{D}} f_{!} \mathcal{M}^{\prime \text { left }}\right)^{\text {right }} \otimes_{\mathscr{O}_{Y}} \mathscr{C}_{Y}^{\infty}\right) \otimes_{\mathscr{C}_{Y}^{\infty}} \overline{\left(\left({ }_{\mathrm{D}} f_{!} \mathcal{M}^{\prime \prime \text { left }}\right)^{\text {right }} \otimes_{\mathscr{O}_{Y}} \mathscr{C}_{Y}^{\infty}\right)}\right. \\
& \longrightarrow \mathfrak{C}_{Y}[2(m-n)],
\end{aligned}
$$

which restrict to the pushforward morphism, after going from right to left in the target $Y$ :

$$
{ }_{\mathrm{D}, \overline{\mathrm{D}}} f_{!} \mathrm{c}^{\text {left }}:_{\mathrm{D}} f_{!} \mathcal{M}^{\prime \text { left }} \otimes_{\mathbb{C}} \overline{\mathrm{D} f_{!} \mathcal{N}^{\prime \prime \mathrm{left}}} \longrightarrow \mathfrak{D}_{Y}[2(m)] .
$$

At the cohomology level, and omitting the "left" exponent on $\mathfrak{c}$, we denote by ${ }_{\mathrm{D}, \overline{\mathrm{D}}} f_{!}^{\bullet} \mathfrak{c}$ the induced morphism

$$
\begin{equation*}
{ }_{\mathrm{D}, \overline{\mathrm{D}}} f_{!}^{n-m+k_{\mathrm{c}}}: \mathscr{H}^{n-m+k_{\mathrm{D}}} f_{!} \mathcal{M}^{\prime \mathrm{left}} \otimes_{\mathbb{C}} \overline{\mathscr{H}}^{n-m-k_{\mathrm{D}} f_{!} \mathcal{M}^{\prime \prime \mathrm{left}}} \longrightarrow \mathfrak{D b}_{Y} \tag{10.3.22}
\end{equation*}
$$

Remark 10.3.23 (Pushforward and side-changing). With respect to the side-changing functor of Remark 10.2.5, the right pairing (10.3.13) is obtained by side changing from (10.3.22), if $\mathfrak{c}^{\text {right }}$ is obtained by side-changing from $\mathfrak{c}^{\text {left }}$. This follows from the definition of both sesquilinear pairings, since both are defined from ( $\left.{ }_{\mathrm{D}, \overline{\mathrm{D}}} f_{!} \mathrm{c}_{\infty}^{\text {left }}\right)^{\text {right }}$, namely, (10.3.13) by applying the side-changing at the source, and (10.3.22) at the target. In other words, for $\mathfrak{c}=\mathfrak{c}^{\text {left }}$,

$$
\left({ }_{\mathrm{D}, \overline{\mathrm{D}}} f_{!}^{n-m+k} \mathfrak{c}\right)^{\text {right }}={ }_{\mathrm{D}, \overline{\mathrm{D}}} f_{!}^{k}\left(\mathfrak{c}^{\text {right }}\right) .
$$

If we define ${ }_{\mathrm{T}} f_{*}^{k} \mathfrak{c}$ in the left case by

$$
\begin{equation*}
{ }_{\mathrm{\tau}} f_{*}^{k} \mathfrak{c}:=\varepsilon(k) \cdot{ }_{\mathrm{D}, \overline{\mathrm{D}}} f_{*}^{k}, \tag{10.3.23*}
\end{equation*}
$$

the side-changing formula become
(10.3.23 **)

$$
\left({ }_{\mathrm{T}} f_{!}^{n-m+k} \mathfrak{c}\right)^{\mathrm{right}}={ }_{\mathrm{T}} f_{!}^{k}\left(\mathfrak{c}^{\text {right }}\right) .
$$

Remark 10.3.24 (Pushforward and adjunction). As in Remark 10.3.15, we use that

$$
{ }_{\mathrm{D}, \overline{\mathrm{D}}} f_{!}^{n-m+k} \mathfrak{c}^{\text {left }}=\left[\left(\left(\mathrm{D}, \overline{\mathrm{D}} f_{!}^{n+k} \mathfrak{c}^{\text {left }}\right)^{\text {right }}\right]^{\text {left }}\right.
$$

to obtain

$$
\begin{equation*}
{ }_{\mathrm{D}, \overline{\mathrm{D}},} f_{!}^{n-m+k}\left(\mathfrak{c}^{*}\right)=(-1)^{n-m+k}\left({ }_{\mathrm{D}, \overline{\mathrm{D}}} f_{!}^{n-m-k} \mathfrak{c}\right)^{*} . \tag{10.3.24*}
\end{equation*}
$$

Now (10.3.24*) becomes

$$
\begin{equation*}
{ }_{\mathrm{T}} f_{!}^{n-m+k}\left(\mathfrak{c}^{*}\right)=(-1)^{n-m}\left({ }_{\mathrm{T}} f_{!}^{n-m-k} \mathfrak{c}\right)^{*} . \tag{10.3.25}
\end{equation*}
$$

Then, if $\mathfrak{c}$ is $(-1)^{w}$-Hermitian, then ${ }_{\mathrm{T}} f_{*}^{\bullet} \mathfrak{c}$ is graded $(-1)^{w+n-m}$-Hermitian. As a consequence, the pushforward of a $(-1)^{w}$-Hermitian pair (left case) is a graded $(-1)^{w+n-m_{-}}$ Hermitian pair.

The definition of the pushforward of a $\mathscr{D}$-triple in the left case is similar to Definition 10.3.17. The only difference is the grading, which is shifted by $n-m$. We thus set

$$
{ }_{\mathrm{T}} f_{*}^{n-m+k} \mathcal{T}^{\text {left }}=\left(\mathscr{H}^{n-m+k}{ }_{\mathrm{D}} f_{*} \mathcal{M}^{\prime}, \mathscr{H}^{n-m-k}{ }_{\mathrm{D}} f_{*} \mathcal{M}^{\prime \prime},{ }_{\mathrm{T}} f_{*}^{n-m+k} \mathfrak{c}\right) .
$$

The total pushforward is the graded object ${ }_{\mathrm{T}} f_{*}^{n-m+\bullet} \mathfrak{T}=\bigoplus_{k}{ }_{\mathrm{T}} f_{*}^{n-m+k} \mathcal{T}$, with ${ }_{\mathrm{\tau}} f_{*}^{n-m+k \mathcal{T}}$ in degree $k$.

Similarly, the pushforward of a left $(-1)^{w}$-Hermitian pair $(\mathcal{M}, \mathfrak{c})$ is the graded pair $\left({ }_{\mathrm{T}} f_{*}^{n-m+} \cdot \mathcal{M},{ }_{\mathrm{T}} f_{*}^{n-m+} \cdot \mathfrak{c}\right)$, which is a graded $(-1)^{w+n-m}$-Hermitian pair.

Lastly, for $k \geqslant 0$, the Lefschetz morphism $\mathcal{L}_{\eta}$ induces a sesquilinear pairing

$$
\left({ }_{\mathrm{T}} f_{*} \mathfrak{c}\right)^{(n-m-k)}={ }_{\mathrm{T}} f_{*}^{n-m-k} \mathfrak{c}\left(\cdot, \overline{\mathcal{L}_{\eta}^{k}}\right):{ }_{\mathrm{T}} f_{*}^{n-m-k} \mathcal{M} \otimes_{\mathrm{T}} \overline{f_{*}^{n-m-k} \mathcal{M}} \longrightarrow \mathfrak{D b}_{Y}
$$

which is $(-1)^{w+n-m+k}$-Hermitian.

Example 10.3.26 (Kashiwara's equivalence in $\mathscr{D}$-Triples ${ }^{\text {left }}$ ). In the setting of Example 10.3.18, we obtain that

$$
{ }_{\mathrm{r}}^{\iota_{*}^{n-m}}: \mathscr{D} \text {-Triples }(X)_{\mathrm{coh}}^{\mathrm{left}} \longmapsto \mathscr{D} \text {-Triples }{ }_{X}(Y)_{\mathrm{coh}}^{\mathrm{left}}
$$

is an equivalence of categories.

### 10.4. Specialization in $\mathscr{D}$-Triples

10.4.a. Specialization of a sesquilinear pairing. Let $g: X \rightarrow \mathbb{C}$ be a holomorphic function on $X$ and let $\mathcal{N}^{\prime}, \mathcal{M}^{\prime \prime}$ be $\mathscr{D}_{X}$-modules which are specializable along $g=0$. Assume that $\mathfrak{c}$ is a sesquilinear pairing between $\mathcal{M}^{\prime}$ and $\mathcal{M}^{\prime \prime}$ with values in $\mathfrak{C}_{X}$ (right case) or $\mathfrak{D} \mathfrak{b}_{X}$ (left case). We wish to define sesquilinear pairings between the $\mathscr{D}_{X}$-modules $\psi_{g, \lambda} \mathcal{M}^{\prime}$ and $\psi_{g, \lambda} \mathcal{M}^{\prime}$ with values in $\mathfrak{C}_{X}$ or $\mathfrak{D b}_{X}$. We start with the case where $g$ is the projection $X=X_{0} \times \mathbb{C} \rightarrow \mathbb{C}$ and $\mathcal{M}^{\prime}, \mathcal{M}^{\prime \prime}$ are specializable $\mathscr{D}_{X}$-modules along $X_{0}$ equipped with a sesquilinear pairing. We will denote by $t$ the coordinate on $\mathbb{C}$. In order to define a sesquilinear pairing on nearby cycles, we will use a Mellin transform device by considering the residue of $\mathfrak{c}\left(m^{\prime}, \overline{m^{\prime \prime}}\right)|t|^{2 s}$ at various values of $s$. It is important to notice that, while we need to restrict the category of coherent $\mathscr{D}_{X^{-}}$ modules in order to define nearby and vanishing cycles (i.e., to consider $\mathbb{R}$-specializable coherent $\mathscr{D}_{X}$-modules only), the specialization of a sesquilinear pairing between them does not need any new restriction: any sesquilinear pairing between such $\mathscr{D}_{X}$-modules can be specialized.

We assume that $\mathcal{M}^{\prime}, \mathcal{M}^{\prime \prime}$ are right $\mathscr{D}_{X}$-modules which are $\mathbb{R}$-specializable along $X_{0}$. Let $\mathfrak{c}: \mathcal{N}^{\prime} \otimes_{\mathbb{C}} \overline{\mathcal{N}^{\prime \prime}} \rightarrow \mathfrak{C}_{X}$ be a sesquilinear pairing. Fix $x_{o} \in X_{0}$. For local sections $m^{\prime}, m^{\prime \prime}$ of $\mathcal{M}^{\prime}, \mathcal{M}^{\prime \prime}$ defined in some neighbourhood of $x_{o}$ in $X$, the current of degree 0 $\mathfrak{c}\left(m^{\prime}, \overline{m^{\prime \prime}}\right)$ has some finite order $p$ on some neighbourhood $\operatorname{nb}_{X}\left(x_{o}\right)$. For $2 \operatorname{Re} s>p$, the function $t \mapsto|t|^{2 s}$ is $C^{p}$, so for every such $s, \mathfrak{c}\left(m^{\prime}, \overline{m^{\prime \prime}}\right)|t|^{2 s}$ is a section of $\mathfrak{C}_{X}$ on $\operatorname{nb}_{X}\left(x_{o}\right)$. Moreover, for any test function $\eta$ with compact support in $\mathrm{nb}_{X}\left(x_{o}\right)$, the function $\left.\left.\left.s \mapsto\left\langle\mathfrak{c}\left(m^{\prime}, \overline{m^{\prime \prime}}\right)\right| t\right|^{2 s}, \eta\right\rangle:=\left.\left\langle\mathfrak{c}\left(m^{\prime}, \overline{m^{\prime \prime}}\right),\right| t\right|^{2 s} \eta\right\rangle$ is holomorphic on the halfplane $\{2 \operatorname{Re} s>p\}$. We say that $\mathfrak{c}\left(m^{\prime}, \overline{m^{\prime \prime}}\right)|t|^{2 s}$ depends holomorphically on $s$ on $\operatorname{nb}_{X}\left(x_{o}\right) \times\{2 \operatorname{Re} s>p\}$.

Let $\chi(t)$ be a real $C^{\infty}$ function with compact support, which is $\equiv 1$ near $t=0$. In the following, we will consider test functions $\eta_{o} \cdot \chi(t)$, where $\eta_{o}$ is a test function on a neighbourhood $\mathrm{nb}_{X_{0}}\left(x_{o}\right)$ of $x_{o}$ in $X_{0}$.

Proposition 10.4.1. Let $\mathcal{N}^{\prime}, \mathcal{M}^{\prime \prime}, \mathfrak{c}$ be as above. Then, for every $x_{o} \in X_{0}$, there exists an integer $L \geqslant 0$ and a finite set of real numbers $\gamma$ satisfying $\psi_{t, \exp (2 \pi \mathrm{i} \gamma)} \mathcal{M}_{x_{o}}^{\prime} \neq 0$ and $\psi_{t, \exp (2 \pi \mathrm{i} \gamma)} \mathcal{M}_{x_{o}}^{\prime \prime} \neq 0$, such that, for every element $m^{\prime}$ of $\mathcal{N}_{x_{o}}^{\prime}$ and $m^{\prime \prime}$ of $\mathcal{N}_{x_{o}}^{\prime \prime}$, the correspondence

$$
\begin{equation*}
\left.\left.\eta_{o} \longmapsto \prod_{\gamma} \Gamma(s-\gamma)^{-L} \cdot\left\langle\mathfrak{c}\left(m^{\prime}, \overline{m^{\prime \prime}}\right)\right| t\right|^{2 s}, \eta_{o} \cdot \chi(t)\right\rangle \tag{10.4.1*}
\end{equation*}
$$

defines, for every $s \in \mathbb{C}$, a section of $\mathfrak{C}_{X_{0}}$ on $\mathrm{nb}_{X_{0}}\left(x_{o}\right)$ which is holomorphic with respect to $s \in \mathbb{C}$.

The proposition asserts that the current of degree 0

$$
\left.\left.\eta_{o} \longmapsto\left\langle\mathfrak{c}\left(m^{\prime}, \overline{m^{\prime \prime}}\right)\right| t\right|^{2 s}, \eta_{o} \cdot \chi(t)\right\rangle
$$

extends as a current of degree 0 on $\mathrm{nb}_{X_{0}}\left(x_{o}\right)$ depending meromorphically on $s$, with poles at $s=-k+\gamma(k \in \mathbb{N})$ at most, and with a bounded order. We note that changing the function $\chi$ will modify the previous meromorphic current of degree 0 by a holomorphic one, as $|t|^{2 s}$ is $C^{\infty}$ for every $s$ away from $t=0$. The proposition is a consequence of the following more precise lemma.

Lemma 10.4.2. Let $x_{o} \in X_{0}$ and let $\alpha^{\prime}, \alpha^{\prime \prime} \in \mathbb{R}$. There exist $L \geqslant 0$ and a finite set of real numbers $\gamma$ satisfying

$$
\begin{equation*}
\psi_{t, \exp (2 \pi \mathrm{i} \gamma)} \mathcal{M}_{x_{o}}^{\prime}, \psi_{t, \exp (2 \pi \mathrm{i} \gamma)} \mathcal{M}_{x_{o}}^{\prime \prime} \neq 0, \quad \text { and } \quad \gamma \leqslant \min \left(\alpha^{\prime}, \alpha^{\prime \prime}\right) \tag{10.4.2*}
\end{equation*}
$$

such that, for any sections $m^{\prime} \in V_{\alpha^{\prime}} \mathcal{M}_{x_{o}}^{\prime}$ and $m^{\prime \prime} \in V_{\alpha^{\prime \prime}} \mathcal{M}_{x_{o}}^{\prime \prime}$, the correspondence

$$
\begin{equation*}
\left.\left.\eta_{o} \longmapsto \prod_{\gamma} \Gamma(s-\gamma)^{-L} \cdot\left\langle\mathfrak{c}\left(m^{\prime}, \overline{m^{\prime \prime}}\right)\right| t\right|^{2 s}, \eta_{o} \cdot \chi(t)\right\rangle \tag{10.4.2**}
\end{equation*}
$$

defines, for every $s \in \mathbb{C}$, a section of $\mathfrak{C}_{X_{0}}$ on $\operatorname{nb}_{X_{0}}\left(x_{o}\right)$ which is holomorphic with respect to $s \in \mathbb{C}$.

Proof. Let $b_{m^{\prime}}(S)=\prod_{\gamma \in R\left(m^{\prime}\right)}(S-\gamma)^{\nu(\gamma)}$ be the Bernstein polynomial of $m^{\prime}$ (see Definition 7.3.10), with $\nu(\gamma)$ bounded by the nilpotency index $L$ of $\mathrm{E}-\gamma z$. It is enough to prove that the product $\prod_{\gamma \in R\left(m^{\prime}\right)} \Gamma(s-\gamma)^{-\nu(\gamma)}$ of $\Gamma$ factors can be used in (10.4.2**) (recall that the $\Gamma$ function has no zeros and has simple poles at the non-positive integers, and no other poles). Indeed, arguing similarly for $m^{\prime \prime}$ and using that the set of roots $R\left(m^{\prime \prime}\right)$ of $b_{m^{\prime \prime}}(S)$ is real, one obtains that the product of $\Gamma$ factors indexed by $R\left(m^{\prime}\right) \cap R\left(m^{\prime \prime}\right)$ can also be used in (10.4.2**). It is then easy to check that Conditions $(10.4 .2 *)$ on $\gamma$ are satisfied by any $\gamma \in R\left(m^{\prime}\right) \cap R\left(m^{\prime \prime}\right)$.

We note first that, for every germ $Q \in V_{0} \mathscr{D}_{X, x_{o}}$ and any test function $\eta$ on $\operatorname{nb}_{X}\left(x_{o}\right)$, the function $Q \cdot\left(|t|^{2 s} \eta\right)$ is $C^{p}$ with compact support if $2 \operatorname{Re} s>p$. Applying this to the Bernstein operator $Q=b_{m^{\prime}}(\mathrm{E})-P$ for $m^{\prime}$ (see Definition 7.3.10), one gets

$$
\begin{align*}
0 & \left.=\left.\left\langle\mathfrak{c}\left(m^{\prime}, \overline{m^{\prime \prime}}\right) \cdot\left[b_{m^{\prime}}(\mathrm{E})-P\right],\right| t\right|^{2 s} \eta\right\rangle \\
& =\left\langle\mathfrak{c}\left(m^{\prime}, \overline{m^{\prime \prime}}\right),\left[b_{m^{\prime}}(\mathrm{E})-P\right] \cdot\left(|t|^{2 s} \eta\right)\right\rangle  \tag{10.4.3}\\
& \left.\left.=\left.b_{m^{\prime}}(s)\left\langle\mathfrak{c}\left(m^{\prime}, \overline{m^{\prime \prime}}\right),\right| t\right|^{2 s} \eta\right\rangle+\left.\left\langle\mathfrak{c}\left(m^{\prime}, \overline{m^{\prime \prime}}\right),\right| t\right|^{2 s} t \eta_{1}\right\rangle
\end{align*}
$$

for some $\eta_{1}$, which is a polynomial in $s$ with coefficients being $C^{\infty}$ with compact support contained in that of $\eta$. As $|t|^{2 s} t$ is $C^{p}$ for $2 \operatorname{Re} s+1>p$, we can argue by induction to show that, for every $\eta$ and $k \in \mathbb{N}$,

$$
\begin{equation*}
\left.\left.s \longmapsto b_{m^{\prime}}(s+k-1) \cdots b_{m^{\prime}}(s)\left\langle\mathfrak{c}\left(m^{\prime}, \overline{m^{\prime \prime}}\right)\right| t\right|^{2 s}, \eta\right\rangle \tag{10.4.4}
\end{equation*}
$$

extends as a holomorphic function on $\{s \mid 2 \operatorname{Re} s>p-k\}$, and thus, letting $k \rightarrow \infty$,

$$
\left.\left.s \longmapsto \prod_{\gamma} \Gamma(s-\gamma)^{-\nu(\gamma)} \cdot\left\langle\mathfrak{c}\left(m^{\prime}, \overline{m^{\prime \prime}}\right)\right| t\right|^{2 s}, \eta\right\rangle
$$

extends as an entire function. We apply this result to $\eta=\eta_{o} \cdot \chi(t)$ to get the lemma.

Remark 10.4.5. The previous proof also applies if we only assume that $\mathfrak{c}$ is $\mathscr{D}_{X, \bar{X}}$-linear away from $\{t=0\}$. Indeed, this implies that $\mathfrak{c}\left(m^{\prime}, \overline{m^{\prime \prime}}\right) \cdot\left[b_{m^{\prime}}(\mathrm{E})-P\right]$ is supported on $\{t=0\}$, and (10.4.3) only holds for $\operatorname{Re} s$ big enough, maybe $\gg p$. Then, (10.4.4) coincides with a holomorphic current of degree 0 defined on $\{s \mid 2 \operatorname{Re} s>p-k\}$ only for $\operatorname{Re} s \gg 0$. But, by uniqueness of analytic extension, it coincides with it on $\operatorname{Re} s>p$.

A current of degree 0 on $X_{0}$ which is holomorphic with respect to $s$ can be restricted as a current of degree 0 by setting $s=\alpha$. By a similar argument, the polar coefficients at $s=\alpha$ of the meromorphic current of degree $\left.\left.0\left\langle\mathfrak{c}\left(m^{\prime}, \overline{m^{\prime \prime}}\right)\right| t\right|^{2 s}, \cdots \chi(t)\right\rangle$ exist as currents of degree 0 on $\mathrm{nb}_{X_{0}}\left(x_{o}\right)$.

Lemma 10.4.6. Let $\left[m^{\prime}\right]$ be a germ of section of $\psi_{t, \exp (2 \pi \mathrm{i} \alpha)} \mathcal{M}^{\prime}$ near $x_{o}$ and $\left[m^{\prime \prime}\right] a$ germ of section of $\psi_{t, \exp (2 \pi \mathrm{i} \alpha)} \mathcal{M}^{\prime \prime}$ near $x_{o}$. Then the polar coefficients of the current of degree $\left.\left.0\left\langle\mathfrak{c}\left(m^{\prime}, \overline{m^{\prime \prime}}\right)\right| t\right|^{2 s}, \cdots \chi(t)\right\rangle$ at $s=\alpha$ do neither depend on the choice of the local liftings $m^{\prime}, m^{\prime \prime}$ of $\left[m^{\prime}\right],\left[m^{\prime \prime}\right]$ nor on the choice of $\chi$, and take value in $\mathfrak{C}_{X_{0}}$.

Proof. Indeed, any other local lifting of $m^{\prime}$ can be written as $m^{\prime}+\mu^{\prime}$, where $\mu^{\prime}$ is a germ of section of $V_{<\alpha} \mathcal{M}^{\prime}$. By the previous lemma, $\left.\left.\left\langle\mathfrak{c}\left(\mu^{\prime}, \overline{m^{\prime \prime}}\right)\right| t\right|^{2 s}, \cdots \chi(t)\right\rangle$ is holomorphic at $s=\alpha$. We note also that a different choice of the function $\chi$ does not modify the polar coefficients.

According to the lemma, for $\alpha \in[-1,0)$, we get a well-defined sesquilinear pairing

$$
\begin{align*}
\operatorname{gr}_{\alpha}^{V} \mathcal{M}^{\prime} & \otimes \mathbb{C} \overline{\operatorname{gr}_{\alpha}^{V} \mathcal{M}^{\prime \prime}} \\
\quad\left(\left[m^{\prime}\right], \overline{\operatorname{gr}_{\alpha}^{V}(\mathfrak{c})} \mathfrak{C}_{X_{0}}\right] & \left.\left.\longmapsto \frac{\mathrm{i}}{2 \pi} \operatorname{Res}_{s=\alpha}\left\langle\mathfrak{c}\left(m^{\prime}, \overline{m^{\prime \prime}}\right), \cdot\right| t\right|^{2 s} \chi(t)\right\rangle, \tag{10.4.7}
\end{align*}
$$

where $m^{\prime}, m^{\prime \prime}$ are local liftings of $\left[m^{\prime}\right],\left[m^{\prime \prime}\right]$. The nilpotent operator $\mathrm{N}:=2 \pi \mathrm{i}(\mathrm{E}-\alpha)$ is skew-adjoint with respect to this pairing, in the sense that

$$
\begin{equation*}
\operatorname{gr}_{\alpha}^{V}(\mathfrak{c})\left(\mathrm{N}\left[m^{\prime}\right], \overline{\left[m^{\prime \prime}\right]}\right)=-\operatorname{gr}_{\alpha}^{V}(\mathfrak{c})\left(\left[m^{\prime}\right], \overline{\mathrm{N}\left[m^{\prime \prime}\right]}\right) \tag{10.4.8}
\end{equation*}
$$

This is a consequence of the following properties:

- $\alpha$ is real,
- $\bar{t} \partial_{\bar{t}}|t|^{2 s}=t \partial_{t}|t|^{2 s}$,
- $\bar{t} \partial_{\bar{t}} \chi(t)$ and $t \partial_{t} \chi(t)$ are zero in a neighbourhood of $t=0$.

Exercise 10.4.9 (see Remark 7.4.12). Show that $\operatorname{gr}_{\alpha}^{V}(\mathfrak{c})$ induces pairings $(\ell \in \mathbb{Z})$ :

$$
\operatorname{gr}_{\ell}^{\mathrm{M}} \operatorname{gr}_{\alpha}^{V}(\mathfrak{c}):=\operatorname{gr}_{\ell}^{\mathrm{M}} \operatorname{gr}_{\alpha}^{V} \mathcal{M}^{\prime} \otimes_{\mathbb{C}} \overline{\operatorname{gr}_{-\ell}^{\mathrm{M}} \operatorname{gr}_{\alpha}^{V} \mathcal{M}^{\prime \prime}} \longrightarrow \mathfrak{C}_{X_{0}}
$$

and, for $\ell \geqslant 0$,

$$
\mathrm{P}_{\ell} \operatorname{gr}_{\alpha}^{V}(\mathfrak{c}):=\mathrm{P}_{\ell} \operatorname{gr}_{\alpha}^{V} \mathcal{M}^{\prime} \otimes_{\mathbb{C}} \overline{\mathrm{P}_{\ell} \operatorname{gr}_{\alpha}^{V} \mathcal{M}^{\prime \prime}} \longrightarrow \mathfrak{C}_{X_{0}}
$$

by composing with $\mathrm{N}^{\ell}$ on the $\mathcal{M}^{\prime \prime}$ side.
Exercise 10.4.10 (Adjunction and nearby cycles). Show that

$$
\operatorname{gr}_{\alpha}^{V}\left(\mathfrak{c}^{*}\right)=-\left(\operatorname{gr}_{\alpha}^{V} \mathfrak{c}\right)^{*}
$$

[Hint: use that $\alpha$ and $\chi$ are real.]
Exercise 10.4.11. Show that, if $\mathcal{M}^{\prime}$ or $\mathcal{M}^{\prime \prime}$ is supported on $X_{0}$, the right-hand side of $(10.4 .2 * *)$ is always zero, and the residue formula (10.4.7) returns the value zero for every $\alpha \in \mathbb{R}$.

Example 10.4.12 (The smooth case). We set $\mathcal{M}=\mathcal{N}^{\prime}$ or $\mathcal{N}^{\prime \prime}$. Assume that $\mathcal{N}^{\prime}, \mathcal{N}^{\prime \prime}$ are $\mathscr{O}_{X}$-locally free of finite rank, and let $\mathfrak{c}: \mathcal{M}^{\prime} \otimes_{\mathbb{C}} \overline{\mathcal{M}^{\prime \prime}} \rightarrow \mathfrak{C}_{X}$ be a sequilinear pairing. Let $m^{\prime}, m^{\prime \prime}$ be horizontal local sections of $\mathcal{M}^{\prime}, \mathcal{N}^{\prime \prime}$ (i.e., local sections annihilated by $\partial_{x_{i}}$ ). Then the current $\mathfrak{c}\left(m^{\prime}, \overline{m^{\prime \prime}}\right)$ satisfies $\mathfrak{c}\left(m^{\prime}, \overline{m^{\prime \prime}}\right) \partial_{x_{i}}=\mathfrak{c}\left(m^{\prime}, \overline{m^{\prime \prime}}\right) \partial_{\overline{x_{i}}}=0$ for every $i$, hence is a locally constant function by the Poincaré lemma for distributions. Since $\mathcal{M}=\omega_{X} \otimes_{\mathbb{C}} \mathcal{M}^{\nabla}$, we conclude that $\mathfrak{c}$ takes values in the sheaf of $C^{\infty}$ forms of maximal degree. For local sections $\omega^{\prime} \otimes \mu^{\prime}$ and $\omega^{\prime \prime} \otimes \mu^{\prime \prime}$, we thus have $\mathfrak{c}\left(\omega^{\prime} \otimes \mu^{\prime}, \overline{\omega^{\prime \prime}} \otimes \mu^{\prime \prime}\right)=$ $\mathfrak{c}^{\nabla}\left(\mu^{\prime}, \overline{\mu^{\prime \prime}}\right) \cdot \omega^{\prime} \wedge \overline{\omega^{\prime \prime}}$, where $\mathfrak{c}^{\nabla}: \mathcal{N}^{\prime \nabla} \otimes \overline{\mathcal{N}^{\prime \prime} \nabla} \rightarrow \mathbb{C}$ is the induced sesquilinear pairing on the underlying local systems.

Assume that $X=H \times \Delta_{t}$. Then $\mathcal{M}=V_{-1} \mathcal{M}, \operatorname{gr}_{\alpha}^{V} \mathcal{M}=0$ for $\alpha \notin-\mathbb{N}^{*}$, and $\operatorname{gr}_{-1}^{V} \mathcal{M}=\mathcal{M} / t \mathcal{M}$. For local sections as above, set $\omega^{\prime}=\omega_{o}^{\prime} \wedge \mathrm{d} t$ and $\omega^{\prime \prime}=\omega_{o}^{\prime \prime} \wedge \mathrm{d} t$. Then from Exercise 5.4.7 we obtain, by choosing $\chi(t)=\mu\left(|t|^{2}\right)$,

$$
\operatorname{gr}_{-1}^{V} \mathfrak{c}\left(\omega_{o}^{\prime} \otimes \mu^{\prime}, \overline{\omega_{o}^{\prime \prime} \otimes \mu^{\prime \prime}}\right)=(-1)^{n-1} \mathfrak{c}^{\nabla}\left(\mu^{\prime}, \overline{\mu^{\prime \prime}}\right)_{\mid H} \cdot \omega_{o}^{\prime} \wedge \overline{\omega_{o}^{\prime \prime}}
$$

since

$$
\begin{aligned}
\frac{\mathrm{i}}{2 \pi} \operatorname{Res}_{s=-1}\left\langle\mathfrak { c } \left(\omega_{o}^{\prime} \wedge \mathrm{d} t\right.\right. & \left.\left.\otimes \mu^{\prime}, \overline{\omega_{o}^{\prime \prime} \wedge \mathrm{d} t \otimes \mu^{\prime \prime}}\right)|t|^{2 s}, \eta_{o} \cdot \chi(t)\right\rangle \\
& =\frac{\mathrm{i}}{2 \pi} \operatorname{Res}_{s=-1} \mathfrak{c}^{\nabla}\left(\mu^{\prime}, \overline{\mu^{\prime \prime}}\right)_{\mid H} \int|t|^{2 s} \eta_{o} \chi(t) \omega_{o}^{\prime} \wedge \mathrm{d} t \wedge \overline{\omega_{o}^{\prime \prime} \wedge \mathrm{d} t} \\
& =(-1)^{n-1} \mathfrak{c}^{\nabla}\left(\mu^{\prime}, \overline{\mu^{\prime \prime}}\right)_{\mid H} \int \eta_{o} \omega_{o}^{\prime} \wedge \overline{\omega_{o}^{\prime \prime}} .
\end{aligned}
$$

We now take up the setting of Exercise 7.3.31(4), namely we consider the local embedding $\iota: X=H \times \Delta_{t} \times\{0\} \hookrightarrow X_{1}=H \times \Delta_{t} \times \Delta_{x}$. Let $\mathcal{N}^{\prime}, \mathcal{M}^{\prime \prime}$ be coherent $\mathscr{D}_{X}$-modules which are $\mathbb{R}$-specializable along $H$ and let $\mathfrak{c}: \mathcal{N}^{\prime} \otimes \overline{\mathcal{N}^{\prime \prime}} \rightarrow \mathfrak{C}_{X}$ be a sesquilinear pairing between them. We deduce a sesquilinear pairing ${ }_{\mathrm{D}, \overline{\mathrm{D}}} \iota_{*}^{0} \mathfrak{c}$ between ${ }_{\mathrm{D}} \iota_{*}^{0} \mathcal{M}^{\prime}$ and ${ }_{\mathrm{D}} \iota_{*}^{0} \mathcal{N}^{\prime \prime}$. Let us denote by $\iota_{o}: H \times\{0\} \hookrightarrow X$ the inclusion. We consider the $V$-filtrations along $(t)$ in $X$ and $X_{1}$.

Lemma 10.4.13 (Independence of the embedding). With these assumptions, we have for all $\alpha \in[-1,0)$,

$$
\operatorname{gr}_{\alpha}^{V}\left(\left(_{\left.\mathrm{D}, \overline{\mathrm{D}} \iota_{*}^{0} \mathfrak{c}\right)={ }_{\mathrm{D}, \overline{\mathrm{D}}} \iota_{o *}^{0} \operatorname{gr}_{\alpha}^{V}(\mathfrak{c}) \quad \text { and } \quad \operatorname{gr}_{\alpha}^{V}\left({ }_{\mathrm{T}} \iota_{*}^{0} \mathfrak{c}\right)={ }_{\mathrm{T}} \iota_{o *}^{0} \operatorname{gr}_{\alpha}^{V}(\mathfrak{c}) . . . .}\right.\right.
$$

Proof. The second equality follows from the first one, since the codimension of $H$ in $X$ and $H \times \Delta_{x}$ in $X_{1}$ is the same. Recall that if $\eta$ is any test function on $X_{1}$ and $m^{\prime}, m^{\prime \prime}$ are local sections of $\mathcal{M}^{\prime}, \mathcal{M}^{\prime \prime}$, so that $m^{\prime} \otimes 1, m^{\prime \prime} \otimes 1$ are local sections of $\iota_{*} \mathcal{M}^{\prime}\left[\partial_{x}\right], \iota_{*} \mathcal{N}^{\prime \prime}\left[\partial_{x}\right]$, then

$$
\left\langle{ }_{\mathrm{D}, \overline{\mathrm{D}}} \iota_{*}^{0} \mathfrak{c}\left(m^{\prime} \otimes 1, \overline{m^{\prime \prime} \otimes 1}\right), \eta\right\rangle:=\left\langle\mathfrak{c}\left(m^{\prime}, \overline{m^{\prime \prime}}\right), \eta_{\mid X}\right\rangle
$$

For $m^{\prime}, m^{\prime \prime}$ in $V_{\alpha} \mathcal{M}^{\prime}, V_{\alpha} \mathcal{M}^{\prime \prime}, \eta$ a test function on $H \times \Delta_{x}$ and $\chi(t)$ as above, we thus have (arguing for Re $s \gg 0$ first and then using analytic continuation)

$$
\begin{aligned}
\left\langle\operatorname{gr}_{\alpha}^{V}\left(\mathrm{D}, \overline{\mathrm{D}} \iota_{*}^{0} \mathfrak{c}\right)\left(\left[m^{\prime} \otimes 1\right], \overline{\left[m^{\prime \prime} \otimes 1\right]}\right), \eta\right\rangle & \left.=\left.\frac{\mathrm{i}}{2 \pi} \operatorname{Res}_{s=\alpha}\left\langle_{\mathrm{D}, \overline{\mathrm{D}} \iota_{*}^{0}} \mathfrak{c}\left(m^{\prime} \otimes 1, \overline{m^{\prime \prime} \otimes 1}\right)\right| t\right|^{2 s}, \eta \chi(t)\right\rangle \\
& \left.=\left.\frac{\mathrm{i}}{2 \pi} \operatorname{Res}_{s=\alpha}\left\langle\mathfrak{c}\left(m^{\prime}, \overline{m^{\prime \prime}}\right)\right| t\right|^{2 s}, \eta_{\mid H} \chi(t)\right\rangle \\
& =\left\langle\operatorname{gr}_{\alpha}^{V} \mathfrak{c}\left(m^{\prime}, \overline{m^{\prime \prime}}\right), \eta_{\mid X}\right\rangle .
\end{aligned}
$$

## Exercise 10.4.14 (Non-characteristic restriction of a sesquilinear pairing)

Assume that $X=H \times \Delta_{t}$ and let $\mathcal{N}^{\prime}, \mathcal{N}^{\prime \prime}$ be coherent $\mathscr{D}_{X}$-modules such that the hypersurface $H=\{t=0\}$ is non-characteristic for them (see Section 7.5). Let $\mathfrak{c}: \mathcal{M}^{\prime} \otimes_{\mathbb{C}} \overline{\mathcal{M}^{\prime \prime}} \rightarrow \mathfrak{C}_{X}$ be a sesquilinear pairing. Then $\mathrm{gr}_{-1}^{V} \mathfrak{c}$, as defined by (10.4.7), is a sesquilinear pairing between $\psi_{t, 1} \mathcal{M}^{\prime}=\mathcal{M}^{\prime} / t \mathcal{M}^{\prime}$ and $\psi_{t, 1} \mathcal{M}^{\prime \prime}=\mathcal{M}^{\prime} / t \mathcal{M}^{\prime \prime}$. We denote it by $\iota_{H}^{*} \mathfrak{c}$.

Check that ${ }_{\mathrm{D}} \iota_{H}^{*} \mathfrak{c}$ only depends on the embedding $H \hookrightarrow X$ and not on the product decomposition $X \simeq H \times \Delta_{t}$. Conclude that it is a well-defined sesquilinear pairing ${ }_{\mathrm{D}} \iota_{H}^{*} \mathfrak{c}:{ }_{\mathrm{D}} \iota_{H}^{*} \mathcal{M}^{\prime} \otimes \mathbb{C} \overline{\mathrm{D}^{\iota_{H}^{*} \mathcal{N}^{\prime \prime}}} \rightarrow \mathfrak{C}_{H}$.

## Proposition 10.4.15 (Uniqueness along a non-characteristic divisor)

Let $\mathcal{M}^{\prime}, \mathcal{M}^{\prime \prime}$ be coherent $\mathscr{D}_{X}$-modules and let $H$ be an hypersurface which is noncharacteristic for them. If two sesquilinear pairings $\mathfrak{c}_{1}, \mathfrak{c}_{2}: \mathcal{N}^{\prime} \otimes_{\mathbb{C}} \overline{\mathcal{N}^{\prime \prime}} \rightarrow \mathfrak{C}_{X}$ coincide when restricted to the open set $X \backslash H$, then they coincide.

Proof. The question is local, so we can assume that $X=H \times \Delta_{t}$ and we can shrink $\Delta_{t}$ if needed. Set $\mathfrak{c}=\mathfrak{c}_{1}-\mathfrak{c}_{2}$ and let $m^{\prime}, m^{\prime \prime}$ be local sections of $\mathcal{N}^{\prime}, \mathcal{N}^{\prime \prime}$ defined on some neighbourhood $\operatorname{nb}\left(x_{o}\right)=\mathrm{nb}_{H} \times \Delta_{t}$ of $x_{o} \in H \times\{0\}$. Let $\eta \in C_{\mathrm{c}}^{\infty}\left(\mathrm{nb}\left(x_{o}\right)\right)$, and let $p$ be the order of $\mathfrak{c}\left(m^{\prime}, \overline{m^{\prime \prime}}\right)$ on the compact set Supp $\eta$. We aim at proving that $\left\langle\mathfrak{c}\left(m^{\prime}, \overline{m^{\prime \prime}}\right), \eta\right\rangle=0$.

We consider the current on $\Delta_{t}$ defined by

$$
\chi \longmapsto \mathfrak{c}\left(m^{\prime}, \overline{m^{\prime \prime}}\right)_{\eta}(\chi):=\left\langle\mathfrak{c}\left(m^{\prime}, \overline{m^{\prime \prime}}\right), \chi \cdot \eta\right\rangle \quad \text { for } \chi \in C_{c}^{\infty}\left(\Delta_{t}\right) .
$$

It is enough to prove that $\mathfrak{c}\left(m^{\prime}, \overline{m^{\prime \prime}}\right)_{\eta}=0$ (by choosing $\chi \equiv 1$ on the projection to $\Delta_{t}$ of Supp $\eta$ ). This current has order $\leqslant p$ and is supported at the origin, hence can be
written in a unique way, by using the Dirac current $\delta_{0}$ at the origin, as

$$
\mathfrak{c}\left(m^{\prime}, \overline{m^{\prime \prime}}\right)_{\eta}=\sum_{0 \leqslant a+b \leqslant p} c_{a, b}(\eta) \delta_{0} \partial_{t}^{a} \partial_{\bar{t}}^{b}, \quad c_{a, b}(\eta) \in \mathbb{C} .
$$

We will prove that all the coefficients $c_{a, b}(\eta)$ vanish. This is obvious if $\eta=t^{q} \bar{t}^{r} \eta_{q, r}$ with $q+r>p$ and if $\eta_{q, r}$ is $C^{\infty}$, so that we can reduce to the case where $\eta$ does not depend on $t, \bar{t}$.

We claim that there exists $N$ large enough such that $m^{\prime}$ satisfies an equation of the form

$$
m^{\prime} \cdot b\left(t \partial_{t}\right):=m^{\prime} \cdot \prod_{k=1}^{N}\left(t \partial_{t}+k\right)=m^{\prime} \cdot t^{p+1} \sum_{j} P_{j}\left(t, x, \partial_{x}\right)\left(t \partial_{t}\right)^{j}
$$

where $x$ are local coordinates on $H$. Indeed, $H$ is also non-characteristic for the coherent sub-module $m^{\prime} \cdot \mathscr{D}_{X}$, and the filtration $m^{\prime} \cdot V_{k} \mathscr{D}_{X}$ is comparable with the $V$-filtration $V_{\bullet}\left(m^{\prime} \cdot \mathscr{D}_{X}\right)$, so there exists $N$ such that $V_{-N-1}\left(m^{\prime} \cdot \mathscr{D}_{X}\right) \subset m^{\prime} \cdot V_{-(p+1)} \mathscr{D}_{X}$. Since $m^{\prime} \partial_{t}^{N} \in\left(m^{\prime} \cdot \mathscr{D}_{X}\right)=V_{-1}\left(m^{\prime} \cdot \mathscr{D}_{X}\right)$, we have $m^{\prime} \partial_{t}^{N} t^{N} \in V_{-N-1}\left(m^{\prime} \cdot \mathscr{D}_{X}\right)$, hence the assertion.

We thus have $\mathfrak{c}\left(m^{\prime}, \overline{m^{\prime \prime}}\right)_{\eta} \cdot b\left(t \partial_{t}\right)=0$. Since $\delta_{0} \partial_{t}^{a} \partial_{\bar{t}}^{b} \cdot\left(t \partial_{t}+k\right)=(a+k) \delta_{0} \partial_{t}^{a} \partial_{\bar{t}}^{b}$, we conclude that for every $a, b, c_{a, b}(\eta) \cdot \prod_{k=1}^{N}(a+k)=0$, so $c_{a, b}(\eta)=0$.

We also have an analogue of Corollary 7.7.13 for sesquilinear pairings.
Proposition 10.4.16. Let $\mathcal{M}^{\prime}, \mathcal{N}^{\prime \prime}$ be two holonomic $\mathscr{D}_{X}$-modules which are $S$-decomposable and let $\left(Z_{i}\right)_{i \in I}$ be the family of their strict components. Then any sesquilinear pairing $\mathfrak{c}: \mathcal{M}_{Z_{i}}^{\prime} \otimes_{\mathbb{C}} \overline{\mathcal{M}_{Z_{j}}^{\prime \prime}} \rightarrow \mathfrak{C}_{X}$ vanishes identically if $Z_{i} \neq Z_{j}$.

We will first prove a similar result related to the S-decomposition along a function.
Lemma 10.4.17. Let $g: X \rightarrow \mathbb{C}$ be a holomorphic function and let $\mathcal{M}^{\prime}, \mathcal{M}^{\prime \prime}$ be two coherent $\mathscr{D}_{X}$-modules which are $\mathbb{R}$-specializable along $(g)$. Assume that one of them, say $\mathcal{M}^{\prime}$, is a minimal extension along $(g)$, and the other one, say $\mathcal{M}^{\prime \prime}$, is supported on $g^{-1}(0)$. Then any sesquilinear pairing $\mathfrak{c}: \mathcal{M}^{\prime} \otimes \overline{\mathcal{N}^{\prime \prime}} \rightarrow \mathfrak{C}_{X}$ vanishes indentically.

Proof. By Kashiwara's equivalence 10.3.18, we can assume that $g$ is the projection $X_{0} \times \mathbb{C} \rightarrow \mathbb{C}$, and we choose a coordinate $t$ on $\mathbb{C}$. We work locally near $x_{o} \in X_{0}$. Consider $\mathfrak{c}$ as a morphism $\mathcal{M}^{\prime} \rightarrow \mathscr{H}^{\prime} m_{\mathscr{D}_{\bar{X}}}\left(\overline{\mathcal{N}^{\prime \prime}}, \mathfrak{C}_{X}\right)$. Fix local $\mathscr{D}_{X}$-generators $m_{1}^{\prime \prime}, \ldots, m_{\ell}^{\prime \prime}$ of $\mathcal{M}_{x_{o}}^{\prime \prime}$. By Kashiwara's equivalence 7.6.1, there exists $q \geqslant 0$ such that $m_{k}^{\prime \prime} t^{q}=0$ for all $k=1, \ldots, \ell$. Let $m^{\prime} \in \mathcal{M}_{x_{o}}^{\prime}$ and let $p$ be the maximum of the orders of $\mathfrak{c}\left(m^{\prime}\right)\left(\overline{m_{k}^{\prime \prime}}\right)$ on some neighbourhood of $x_{o}$. As $t^{p+1+q} / \bar{t}^{q}$ is $C^{p}$, we have, for every $k=1, \ldots, \ell$,

$$
\mathfrak{c}\left(m^{\prime}\right)\left(\overline{m_{k}^{\prime \prime}}\right) t^{p+1+q}=\mathfrak{c}\left(m^{\prime}\right)\left(\overline{m_{k}^{\prime \prime}}\right) \bar{t}^{q} \cdot \frac{t^{p+1+q}}{\bar{t}^{q}}=\mathfrak{c}\left(m^{\prime}\right)\left(\overline{m_{k}^{\prime \prime} t^{q}}\right) \cdot \frac{t^{p+1+q}}{\bar{t}^{q}}=0
$$

hence $\mathfrak{c}\left(m^{\prime}\right) t^{p+1+q} \equiv 0$. Applying this to generators of $\mathcal{M}_{x_{o}}^{\prime}$ shows that all local sections of $\mathfrak{c}\left(\mathcal{M}_{x_{o}}^{\prime}\right)$ are killed by some power of $t$.

As $\mathcal{M}^{\prime}$ is a minimal extension along $(t)$, we know from Proposition 7.7.2(2) that $V_{<0} \mathcal{M}_{x_{o}}^{\prime}$ generates $\mathcal{M}_{x_{o}}^{\prime}$ over $\mathscr{D}_{X}$. It is therefore enough to show that $\mathfrak{c}\left(V_{<0} \mathcal{M}_{x_{o}}^{\prime}\right)=0$.

On the one hand, we have $\mathfrak{c}\left(V_{\alpha} \mathcal{M}_{x_{o}}^{\prime}\right)=0$ for $\alpha \ll 0$ : indeed, $t: \mathfrak{c}\left(V_{\alpha} \mathcal{M}_{x_{o}}^{\prime}\right) \rightarrow$ $\mathfrak{c}\left(V_{\alpha-1} \mathcal{N}_{x_{o}}^{\prime}\right)$ is an isomorphism for $\alpha<0$, hence acts injectively on $\mathfrak{c}\left(V_{\alpha} \mathcal{N}_{x_{o}}^{\prime}\right)$, therefore the conclusion follows, as $t$ is also nilpotent by the argument above.

Let now $\alpha<0$ be such that $\mathfrak{c}\left(V_{<\alpha} \mathcal{N}_{x_{o}}^{\prime}\right)=0$, and let $m^{\prime}$ be a section of $V_{\alpha} \mathcal{N}_{x_{o}}^{\prime}$; there exists $\nu_{\alpha} \geqslant 0$ such that, setting $b(s)=\left(t \partial_{t}-\alpha\right)^{\nu_{\alpha}}$, we have $m^{\prime} b(s) \in V_{<\alpha} \mathcal{N}_{x_{o}}^{\prime}$, hence $\mathfrak{c}\left(m^{\prime}\right) b\left(t \partial_{t}\right)=0$; on the other hand, we have seen that there exists $N$ such that $\mathfrak{c}\left(m^{\prime}\right) t^{N}=0$, hence, putting $B(s)=\prod_{\ell=0}^{N-1}(s-\ell)$, it also satisfies $\mathfrak{c}\left(m^{\prime}\right) B\left(t \partial_{t}\right)=0$; notice now that $b(s)$ and $B(s)$ have no common root, so $\mathfrak{c}\left(m^{\prime}\right)=0$.

Proof of Proposition 10.4.16. The assertion is local on $X$, so we fix $x_{o} \in X$ and we work with germs at $x_{o}$. Assume for example that $Z_{i}$ is not contained in $Z_{j}$ and consider a germ $g$ of analytic function, such that $g \equiv 0$ on $Z_{j}$ and $g \not \equiv 0$ on $Z_{i}$. Then we can apply Lemma 10.4 .17 to $\mathcal{M}_{Z_{i}}^{\prime}$ and $\mathcal{M}_{Z_{j}}^{\prime \prime}$.
Definition 10.4.18 (Sesquilinear pairing on nearby cycles). Let $g: X \rightarrow \mathbb{C}$ be a holomorphic function. Assume that $\mathcal{N}^{\prime}, \mathcal{N}^{\prime \prime}$ are $\mathbb{R}$-specializable along $(g)$. For a sesquilinear pairing $\mathfrak{c}: \mathcal{M}^{\prime} \otimes \overline{\mathcal{M}^{\prime \prime}} \rightarrow \mathfrak{C}_{X}$ and for every $\lambda \in S^{1}$ and $\alpha \in[-1,0)$ such that $\lambda=\exp (2 \pi \mathrm{i} \alpha)$, we define (note that ${ }_{\mathrm{T}} \iota_{g *}^{0} \mathfrak{c}=\varepsilon(-1)_{\mathrm{D}, \overline{\mathrm{D}}} \iota^{0}{ }_{g *} \mathfrak{c}=-_{\mathrm{D}, \overline{\mathrm{D}}} \iota_{g *}^{0} \mathfrak{c}$ )

$$
\begin{equation*}
\psi_{g, \lambda} \mathfrak{c}:=(-1)^{n-1} \operatorname{gr}_{\alpha}^{V}\left({ }_{\mathrm{T}} \iota_{g *}^{0} \mathfrak{c}\right): \psi_{g, \lambda} \mathcal{M}^{\prime} \otimes \overline{\psi_{g, \lambda} \mathcal{M}^{\prime \prime}} \rightarrow \mathfrak{C}_{X} \tag{10.4.18*}
\end{equation*}
$$

Remark 10.4.19 (The basic example). The $\operatorname{sign}(-1)^{n-1}$ is justified by the calculation in Example 10.4.12. Namely, if $X=X_{0} \times \mathbb{C}$, and if $\mathcal{N}^{\prime}=\mathcal{N}^{\prime \prime}=\omega_{X}$, with $\mathfrak{c}_{n}$ being the natural pairing 10.2.14(2) in dimension $n$, then $\operatorname{gr}_{-1}^{V} \omega_{X} \simeq \omega_{X_{0}}$ and $(-1)^{n-1} \operatorname{gr}_{-1}^{V} \mathfrak{c}_{n}=\mathfrak{c}_{n-1}$. Setting now $g=t$ and denoting by $\iota: X_{0} \hookrightarrow X$ the inclusion, we have $\operatorname{gr}_{-1}^{V}\left({ }_{\mathrm{T}} \iota_{g *}^{0} \mathfrak{c}_{n}\right)={ }_{\mathrm{r}} \iota_{*}^{0} \mathrm{gr}_{-1}^{V} \mathfrak{c}_{n}$, according to Lemma 10.4.13, so with our definition, $\psi_{g, 1} \mathfrak{c}_{n}={ }_{\mathrm{T}} \iota_{*}^{0} \mathfrak{c}_{n-1}$.

Remark 10.4.20 (Properties of $\psi_{g, \lambda} \mathfrak{c}$ ). The following properties are obviously obtained from similar properties for $\operatorname{gr}_{\alpha \text { т }}^{V} \iota_{*}^{0} \mathbf{c}$.
(1) $\psi_{g, \lambda} \mathfrak{c}\left(\mathrm{~N} m^{\prime}, \overline{m^{\prime \prime}}\right)=-\psi_{g, \lambda} \mathfrak{c}\left(m^{\prime}, \overline{\mathrm{N} m^{\prime \prime}}\right) \quad\left(m^{\prime} \in \psi_{g, \lambda} \mathcal{M}_{x_{o}}^{\prime}, m^{\prime \prime} \in \psi_{g, \lambda} \mathcal{M}_{x_{o}}^{\prime \prime}\right)$.
(2) We have induced pairings $\operatorname{gr}_{\ell}^{\mathrm{M}} \psi_{g, \lambda} \mathfrak{c}: \operatorname{gr}_{\ell}^{\mathrm{M}} \psi_{g, \lambda} \mathcal{M}^{\prime} \otimes \operatorname{gr}_{-\ell}^{\mathrm{M}} \psi_{g, \lambda} \mathcal{M}^{\prime \prime} \rightarrow \mathfrak{C}_{X}$ and, for every $\ell \geqslant 0, \mathrm{P}_{\ell} \psi_{g, \lambda} \mathfrak{c}: \mathrm{P}_{\ell} \psi_{g, \lambda} \mathcal{M}^{\prime} \otimes \overline{\mathrm{P}_{\ell} \psi_{g, \lambda} \mathcal{M}^{\prime \prime}} \rightarrow \mathfrak{C}_{X}$ is induced by $\left.\operatorname{gr}_{\ell}^{\mathrm{M}} \psi_{g, \lambda} \mathfrak{c}\left(\bullet, \overline{\mathrm{~N}^{\ell}}\right)\right)$.
(3) $\psi_{g, \lambda}\left(\mathfrak{c}^{*}\right)=-\left(\psi_{g, \lambda} \mathfrak{c}\right)^{*}$.

Remark 10.4.21 (Sesquilinear pairing on vanishing cycles). It is possible to give the definition of a sesquilinear pairing $\phi_{t, 1} \mathfrak{c}: \phi_{t, 1} \mathcal{N}^{\prime} \otimes \overline{\phi_{t, 1} \mathcal{N}^{\prime \prime}} \rightarrow \mathfrak{C}_{X}$ for every $\mathcal{M}^{\prime}, \mathcal{M}^{\prime \prime}$ which are coherent and $\mathbb{R}$-specializable along $t=0$. However, such a general definition is not needed for our further purpose, since we will mainly have to work with S-decomposable $\mathscr{D}_{X}$-modules. We now focus on this case.

Assume first that $\mathcal{M}=\mathcal{N}^{\prime}, \mathcal{N}^{\prime \prime}$ is a minimal extension along $X_{0}$ (see 7.7.3). Then $\phi_{t, 1} \mathcal{M}=\operatorname{Im}\left[\mathrm{N}: \psi_{t, 1} \mathcal{M} \rightarrow \psi_{t, 1} \mathcal{M}\right]$. Then, for $\left[\mu^{\prime}\right] \in \phi_{t, 1} \mathcal{N}^{\prime}$ and $\left[\mu^{\prime \prime}\right] \in \phi_{t, 1} \mathcal{M}^{\prime \prime}$, and

$$
\begin{aligned}
& {\left[\mu^{\prime}\right]=\mathrm{N}\left[m^{\prime}\right],\left[\mu^{\prime \prime}\right]=\mathrm{N}\left[m^{\prime \prime}\right], \text { we set }} \\
& \quad \phi_{t, 1} \mathfrak{c}\left(\left[\mu^{\prime}\right], \overline{\left[\mu^{\prime \prime}\right]}\right):=\psi_{t, 1} \mathfrak{c}\left(\left[m^{\prime}\right], \overline{\mathrm{N}\left[m^{\prime \prime}\right]}\right)=\psi_{t, 1} \mathfrak{c}\left(\mathrm{~N}\left[m^{\prime}\right], \overline{\left[m^{\prime \prime}\right]}\right),
\end{aligned}
$$

where the latter equality is due to $10.4 .20(1)$, as well as the well-definedness, since $\psi_{t, 1} \mathfrak{c}\left(\left[m^{\prime}\right], \overline{\mathrm{N}\left[m^{\prime \prime}\right]}\right)=0$ for $\left[m^{\prime}\right] \in$ Ker N . Then all properties of Remark 10.4.20 also holds for $\phi_{t, 1} \mathfrak{c}$ instead of $\psi_{t, \lambda} \mathfrak{c}$.

Assume now that $\mathcal{M}^{\prime}, \mathcal{M}^{\prime \prime}$ are S-decomposable. Then $\mathfrak{c}$ only pairs components with the same pure support (see Proposition 10.4.16). If the pure support is not contained in $X_{0}$, we apply the previous construction. If the pure support is contained in $X_{0}$, then $\phi_{t, 1} \mathcal{M}=\mathcal{M}$ for $\mathcal{M}=\mathcal{M}^{\prime}, \mathcal{M}^{\prime \prime}$. Then we obviously set $\phi_{t, 1} \mathfrak{c}:=\mathfrak{c}$.
10.4.b. Specialization of objects of $\mathscr{D}$-Triples $(X)$. We say that an object $\mathcal{T}=$ $\left(\mathcal{M}^{\prime}, \mathcal{M}^{\prime \prime}, \mathfrak{c}\right)$ of $\mathscr{D}$-Triples $(X)$ is $\mathbb{R}$-specializable along $(g)$ if $\mathcal{M}^{\prime}, \mathcal{M}^{\prime \prime}$ are so. We then define, for $\lambda \in S^{1}$,

$$
\begin{equation*}
\psi_{g, \lambda} \mathcal{T}:=\left(\psi_{g, \lambda} \mathcal{M}^{\prime}, \psi_{g, \lambda} \mathcal{M}^{\prime \prime}, \psi_{g, \lambda} \mathfrak{c}\right) \tag{10.4.22}
\end{equation*}
$$

Then $\psi_{g, \lambda}$ is a functor from the full subcategory of $\mathbb{R}$-specializable objects of $\mathscr{D}$-Triples $(X)$ to the category of objects supported on $g^{-1}(0)$.

Example 10.4.23. In the setting of Remark 10.4.19, and using Definition 10.2.14(2) for ${ }_{\mathrm{T}} \omega_{X}$, we thus have $\psi_{t, \lambda}\left({ }_{\mathrm{T}} \omega_{X}\right)={ }_{\mathrm{T}} \iota_{*}^{0}\left({ }_{\mathrm{T}} \omega_{X_{0}}\right)$, if $\iota: X_{0} \times\{0\} \hookrightarrow X=X_{0} \times \mathbb{C}$ denotes the inclusion.

## Remark 10.4.24 (Rule of signs for the nilpotent endomorphism)

We did not define the nilpotent operator on $\psi_{g, \lambda} \mathcal{T}$ since we wish to emphasize the rule of signs in its definition. This rule is motivated by two properties we want to realize.

- As for the Lefschetz operator $\mathrm{L}_{\mathscr{L}}(2.3 .10)$, we wish that the nilpotent comes from geometry, i.e., is defined from the monodromy. Definition 7.4.11 is the "raison d'être" of the definition of N as $2 \pi \mathrm{i}(\mathrm{E}-\alpha)$.
- Let $\mathrm{N}=\left(\mathrm{N}^{\prime}, \mathrm{N}^{\prime \prime}\right)$ be the nilpotent operator to be defined on $\psi_{g, \lambda} \mathcal{T}$. Due to the skew-adjointness of N with respect to $\psi_{g, \lambda} \mathfrak{c}$, we need to anti-symmetrize the action of N on $\psi_{g, \lambda} \mathcal{M}^{\prime}$ and $\psi_{g, \lambda} \mathcal{M}^{\prime \prime}$. We are thus led to define the nilpotent operator N on $\psi_{g, \lambda} \mathcal{T}$ as
$(10.4 .24 *) \quad \mathrm{N}:=\left(\mathrm{N}^{\prime}, \mathrm{N}^{\prime \prime}\right), \quad \mathrm{N}^{\prime}=-\mathrm{N}$ on $\psi_{g, \lambda} \mathcal{M}^{\prime}, \quad \mathrm{N}^{\prime \prime}=\mathrm{N}$ on $\psi_{g, \lambda} \mathcal{M}^{\prime \prime}$.
With this definition, N is indeed a morphism $\psi_{g, \lambda} \mathcal{T} \rightarrow \psi_{g, \lambda} \mathcal{T}$.


## Remark 10.4.25 (The graded Lefschetz object attached to $\left(\psi_{g, \lambda} \mathcal{T}, \mathrm{~N}\right)$ )

The monodromy filtration of N exists in the abelian category $\mathscr{D}$ - $\operatorname{Triples}(X)$, and we have, according to Remark 10.4.20(2),

$$
\begin{aligned}
\operatorname{gr}_{\ell}^{\mathrm{M}} \psi_{g, \lambda} \mathcal{T} & =\left(\operatorname{gr}_{\ell}^{\mathrm{M}} \psi_{g, \lambda} \mathcal{M}^{\prime}, \mathrm{gr}_{-\ell}^{\mathrm{M}} \psi_{g, \lambda} \mathcal{M}^{\prime \prime}, \operatorname{gr}_{\ell}^{\mathrm{M}} \psi_{g, \lambda} \mathfrak{c}\right), \\
\mathrm{P}_{\ell} \psi_{g, \lambda} \mathcal{T} & =\left(\mathrm{P}_{\ell} \psi_{g, \lambda} \mathcal{M}^{\prime}, \mathrm{P}_{\ell} \psi_{g, \lambda} \mathcal{M}^{\prime \prime}, \mathrm{P}_{\ell} \psi_{g, \lambda} \mathfrak{c}\right) \quad(\ell \geqslant 0)
\end{aligned}
$$

## Remark 10.4.26 (Specialization of a pre-polarization of weight $w$ )

Let $\mathrm{Q}=\left((-1)^{w} \mathbb{Q}, \mathbb{Q}\right): \mathcal{T} \rightarrow \mathcal{T}^{*}$ be a pre-polarization of weight $w$ of $\mathcal{T}$. Then $\left((-1)^{w-1} \psi_{g, \lambda} \mathcal{Q}, \psi_{g, \lambda} \mathbb{Q}\right)$ is a pre-polarization of weight $w-1$ of $\psi_{g, \lambda} \mathcal{T}$.

Remark 10.4.27 (Specialization of a $(-1)^{w}$-Hermitian pair). Let $(\mathcal{M}, \mathfrak{c})$ be a $(-1)^{w_{-}}$ Hermitian pair. Assume that $\mathcal{M}$ is $\mathbb{R}$-specializable along $(g)$. Then $\left(\psi_{g, \lambda} \mathcal{M}, \psi_{g, \lambda} \mathfrak{c}\right)$ is a $(-1)^{w-1}$-Hermitian pair and N is skew-adjoint with respect to $\psi_{g, \lambda} \mathfrak{c}$ and, for $\ell \geqslant 0$, $\left(\mathrm{P}_{\ell} \psi_{g, \lambda} \mathcal{M}, \mathrm{P}_{\ell} \psi_{g, \lambda} \mathfrak{c}\right)$ is a $(-1)^{w-1+\ell}$-Hermitian pair.

Example 10.4.28. Let us take up Example 10.2.14(2) with $X=\Delta_{t}$, the ( -1 )-Hermitian pair $\left(\omega_{X}, \mathfrak{c}\right)$. With respect to the coordinate $t$, we have $\operatorname{gr}_{-1}^{V} \omega_{X}=\mathbb{C}$ and, for $m^{\prime}=$ $a^{\prime} \mathrm{d} t, m^{\prime \prime}=a^{\prime \prime} \mathrm{d} t$ with $a^{\prime}, a^{\prime \prime} \in \mathbb{C}$ (so that $\left[m^{\prime}\right]=a^{\prime},\left[m^{\prime \prime}\right]=a^{\prime \prime}$ in $\operatorname{gr}{ }_{-1}^{V} \omega_{X}$ ),

$$
\operatorname{gr}_{-1}^{V} \mathfrak{c}\left(\left[m^{\prime}\right], \overline{\left[m^{\prime \prime}\right]}\right):=a^{\prime} \overline{a^{\prime \prime}} \frac{\mathrm{i}}{2 \pi} \operatorname{Res}_{s=-1} \int|t|^{2 s} \chi(t) \mathrm{d} t \wedge \mathrm{~d} \bar{t}=a^{\prime} \overline{a^{\prime \prime}}
$$

## Definition 10.4.29 (Middle extension quiver of a $\mathbb{R}$-specializable $\mathscr{D}$-triple)

Assume that $\mathcal{T}$ is $\mathbb{R}$-specializable along $(g)$.
(1) We say that it is a minimal extension along $(g)$ if $\mathcal{M}^{\prime}, \mathcal{M}^{\prime \prime}$ are so (see 7.7.3).
(2) If $\mathcal{T}$ is a minimal extension along $(g)$, we define
with

$$
\begin{aligned}
\phi_{g, 1} \mathcal{T} & :=\left(\phi_{g, 1} \mathcal{N}^{\prime}, \phi_{g, 1} \mathcal{M}^{\prime \prime}, \phi_{g, 1} \mathfrak{c}\right) \\
\phi_{g, 1} \mathcal{M}^{\prime} & :=\operatorname{Im} \mathrm{N}^{\prime}, \\
\phi_{g, 1} \mathcal{M}^{\prime \prime} & :=\operatorname{Coim} \mathrm{N}^{\prime \prime}:=\mathcal{M}^{\prime \prime} / \operatorname{Ker} \mathrm{N}^{\prime \prime}, \\
\phi_{g, 1} \mathfrak{c} & :=\psi_{g, 1} \mathfrak{c}_{\mid \operatorname{Im} \mathrm{N}^{\prime} \otimes \overline{\operatorname{Coim} \mathrm{N}^{\prime \prime}}} .
\end{aligned}
$$

In other words, $\phi_{g, 1} \mathcal{T}=\operatorname{Im} \mathrm{N}$ in the category $\mathscr{D}$-Triples $(X)$.
Let us make this definition more explicit. Since $\psi_{g, 1} \mathfrak{c}\left(\mathrm{~N}^{\prime} m^{\prime}, \overline{m^{\prime \prime}}\right)=\psi_{g, 1} \mathfrak{c}\left(m^{\prime}, \overline{\mathrm{N}^{\prime \prime} m^{\prime \prime}}\right)$, we have $\psi_{g, 1} \mathfrak{c}\left(\mathrm{~N}^{\prime} m^{\prime}, \overline{m^{\prime \prime}}\right)=0$ for $m^{\prime \prime} \in \operatorname{Ker} \mathrm{N}^{\prime \prime}$, and $\phi_{g, 1} \mathfrak{c}$ is well-defined.

We define the middle extension diagram when $\mathcal{T}$ is a minimal extension along $(g)$ :

where the epimorphism can : $\psi_{g, 1} \mathcal{T} \rightarrow \phi_{g, 1} \mathcal{T}$ is given by

$$
\operatorname{can}^{\prime}=\mathrm{N}^{\prime}: \mathcal{M}^{\prime} \longrightarrow \operatorname{Im} \mathrm{N}^{\prime}, \quad \operatorname{can}^{\prime \prime}=\mathrm{N}^{\prime \prime}: \mathcal{N}^{\prime \prime} /{\text { Ker } \mathrm{N}^{\prime \prime}} \longrightarrow \mathcal{M}^{\prime \prime}
$$

and the monomorphism var : $\phi_{g, 1} \mathcal{T} \rightarrow \psi_{g, 1} \mathcal{T}$ is given by

$$
\operatorname{var}^{\prime}=\text { incl. }: \operatorname{Im} \mathrm{N}^{\prime} \longleftrightarrow \mathcal{M}^{\prime}, \quad \operatorname{var}^{\prime \prime}=\text { proj. }: \mathcal{N}^{\prime \prime} \longrightarrow \mathcal{M}^{\prime \prime} / \text { Ker } \mathrm{N}^{\prime \prime}
$$

## Remark 10.4.30 (Middle extension quiver of a $\mathbb{R}$-specializable $(-1)^{w}$-Hermitian pair)

Let $(\mathcal{M}, \mathfrak{c})$ be a $(-1)^{w}$-Hermitian pair which is $\mathbb{R}$-specializable along $(g)$ and such that $\mathcal{M}$ is a minimal extension along $(g)$. We thus have

$$
\phi_{g, 1} \mathcal{M}=\operatorname{Im}\left[\mathrm{N}: \psi_{g, 1} \mathcal{M} \longrightarrow \psi_{g, 1} \mathcal{M}\right] .
$$

For local sections $m^{\prime}, m^{\prime \prime}$ of $\mathcal{M}$, we set

$$
\phi_{g, 1} \mathfrak{c}\left(\mathrm{~N} m^{\prime}, \overline{\mathrm{N} m^{\prime \prime}}\right):=\psi_{g, 1} \mathfrak{c}\left(m^{\prime}, \overline{\mathrm{N} m^{\prime \prime}}\right)
$$

Then $\phi_{g, 1} \mathfrak{c}$ is $(-1)^{w-1}$-Hermitian.
Definition 10.4.31 (S-decomposable $\mathscr{D}$-triples). We say that a coherent $\mathscr{D}$-triple $\mathcal{T}$ is $S$-decomposable if its components $\mathcal{N}^{\prime}, \mathcal{M}^{\prime \prime}$ are so. It then has a decomposition $\mathcal{T}=\bigoplus_{i} \mathcal{T}_{Z_{i}}$ with $\mathcal{T}_{Z_{i}}$ having pure support the irreducible closed analytic subset $Z_{i} \subset X$ (see Proposition 10.4.16).
10.4.c. Specialization of left $\mathscr{D}$-triples. While the question of signs is better behaved for the pushforward functor in the case of right $\mathscr{D}_{X}$-modules, the situation is reversed for the specialization functors.

Let us start with the case of a projection $X=X_{0} \times \mathbb{C} \rightarrow \mathbb{C}$ and $\mathbb{R}$-specializable left $\mathscr{D}_{X}$-modules $\mathcal{M}^{\prime}, \mathcal{M}^{\prime \prime}$ equipped with a sesquilinear pairing $\mathfrak{c}: \mathcal{M}^{\prime} \otimes \overline{\mathcal{M}^{\prime \prime}} \rightarrow \mathfrak{D b}_{X}$ between them. For every $\beta \in(-1,0]$ and each test form $\eta_{o}$ of maximal degree on $X_{0}$, the formula

$$
\begin{equation*}
\left.\left\langle\operatorname{gr}_{V}^{\beta} \mathfrak{c}\left(\left[m^{\prime}\right], \overline{\left[m^{\prime \prime}\right]}\right), \eta_{o}\right\rangle:=\left.\operatorname{Res}_{s=-\beta-1}\langle | t\right|^{2 s} \mathfrak{c}^{\text {left }}\left(m^{\prime}, \overline{m^{\prime \prime}}\right), \eta_{o} \wedge \chi(t) \frac{\mathrm{i}}{2 \pi} \mathrm{~d} t \wedge \mathrm{~d} \bar{t}\right\rangle \tag{10.4.32}
\end{equation*}
$$

defines sesquilinear pairing $\operatorname{gr}_{V}^{\beta} \mathfrak{c}: \operatorname{gr}_{V}^{\beta} \mathcal{N}^{\prime} \otimes \overline{\operatorname{gr}_{V}^{\beta} \mathcal{N}^{\prime \prime}} \rightarrow \mathfrak{D b}_{X_{0}}$. This is proved as for (10.4.7). Since $\beta$ and $\chi$ are real, and since the form $\frac{i}{2 \pi} \mathrm{~d} t \wedge \mathrm{~d} \bar{t}$ is real, we have

$$
\begin{equation*}
\operatorname{gr}_{V}^{\beta}\left(\mathfrak{c}^{*}\right)=\left(\operatorname{gr}_{V}^{\beta} \mathfrak{c}\right)^{*} \tag{10.4.33}
\end{equation*}
$$

On the other hand, N is skew-adjoint with respect to $\mathfrak{c}$ (same proof as for (10.4.8) in the right case).

## Definition 10.4.34 (Sesquilinear pairing on nearby cycles, left case)

Let $g: X \rightarrow \mathbb{C}$ be a holomorphic function. Assume that $\mathcal{M}^{\prime}, \mathcal{M}^{\prime \prime}$ are $\mathbb{R}$-specializable along $(g)$. For a sesquilinear pairing $\mathfrak{c}: \mathcal{M}^{\prime} \otimes \overline{\mathcal{M}^{\prime \prime}} \rightarrow \mathfrak{D b}_{X}$ and for every $\lambda \in S^{1}$ and $\beta \in(-1,0]$ such that $\lambda=\exp (-2 \pi \mathrm{i} \beta)$, we define
$(10.4 .34 *) \quad \psi_{g, \lambda} \mathfrak{c}:=\operatorname{gr}_{V}^{\beta}\left({ }_{\mathrm{D}, \overline{\mathrm{D}}} \iota_{g *}^{-1} \mathfrak{c}\right)=-\operatorname{gr}_{V}^{\beta}{\left({ }_{\mathrm{T}} \iota_{g *}^{-1} \mathfrak{c}\right): \psi_{g, \lambda} \mathcal{M}^{\prime} \otimes \overline{\psi_{g, \lambda} \mathcal{M}^{\prime \prime}} \longrightarrow \mathfrak{D b}_{X} .}$
Given a left $\mathscr{D}$-triple $\mathcal{T}=\left(\mathcal{M}^{\prime}, \mathcal{M}^{\prime \prime}, \mathfrak{c}\right)$ which is $\mathbb{R}$-specializable along $(g)$, we set
(10.4.34**)

$$
\psi_{g, \lambda} \mathcal{T}:=\left(\psi_{g, \lambda} \mathcal{M}^{\prime}, \psi_{g, \lambda} \mathcal{M}^{\prime \prime}, \psi_{g, \lambda} \mathfrak{c}\right)
$$

Remark 10.4.35 (Side-changing for $\left.\psi_{g, \lambda} \mathfrak{c}\right)$. If $X=H \times \Delta_{t}$ and $\mathcal{T}=\left(\mathcal{M}^{\prime}, \mathcal{M}^{\prime \prime}, \mathfrak{c}\right)$ is a left $\mathscr{D}$-triple which is $\mathbb{R}$-specializable along $H$, then, for holomorphic forms $\omega_{o}^{\prime}, \omega_{o}^{\prime \prime}$ of maximal degree on $X_{0}$, for a $C^{\infty}$ function $\eta_{o}$ on $X_{0}$ and for $\operatorname{Re} s \gg 0$, we have

$$
\begin{aligned}
&\left.\left.\left\langle\mathfrak{c}^{\mathrm{right}}\left(\left(\omega_{o}^{\prime} \wedge \mathrm{d} t\right) \otimes m^{\prime}, \overline{\left(\omega_{o}^{\prime \prime} \wedge \mathrm{d} t\right) \otimes m^{\prime \prime}}\right), \eta_{o} \chi(t)\right| t\right|^{2 s}\right\rangle \\
&\left.=\left.(-1)^{n-1}\left\langle\mathfrak{c}\left(m^{\prime}, \overline{m^{\prime \prime}}\right), \eta_{o} \omega_{o}^{\prime} \wedge \overline{\omega_{o}^{\prime \prime}} \cdot \chi(t)\right| t\right|^{2 s} \mathrm{~d} t \wedge \mathrm{~d} \bar{t}\right\rangle
\end{aligned}
$$

We deduce that, for $\beta=-\alpha-1$,

$$
\left(\operatorname{gr}_{V}^{\beta} \mathfrak{c}\right)^{\text {right }}=(-1)^{n-1} \operatorname{gr}_{\alpha}^{V}\left(\mathfrak{c}^{\text {right }}\right)
$$

As a consequence, for every $\lambda \in S^{1}$, we have

$$
\begin{equation*}
\psi_{g, \lambda}\left(\mathfrak{c}^{\text {right }}\right)=\left(\psi_{g, \lambda} \mathfrak{c}\right)^{\text {right }} \tag{10.4.35*}
\end{equation*}
$$

Indeed, we have

$$
\begin{aligned}
\psi_{g, \lambda}\left(\mathfrak{c}^{\text {right }}\right) & =(-1)^{n-1} \operatorname{gr}_{\alpha}^{V}\left({ }_{\mathrm{T}} \iota_{*}^{0} \mathrm{c}^{\text {right }}\right) \quad \text { (Definition 10.4.18) } \\
& =(-1)^{n-1} \operatorname{gr}_{\alpha}^{V}\left(\left(_{\mathrm{T}} \iota_{*}^{-1} \mathfrak{c}\right)^{\text {right }}\right) \quad(\text { see }(10.3 .23 * *)) \\
& =-\operatorname{gr}_{V}^{\beta}{ }_{V}\left({ }_{\mathrm{T}} \iota_{*}^{-1} \mathfrak{c}\right)^{\text {right }} \quad(\text { see above for } X \longrightarrow X \times \mathbb{C}) \\
& =\left(\psi_{g, \lambda} \mathfrak{c}\right)^{\text {right }} .
\end{aligned}
$$

Example 10.4.36. Let $\iota: X_{0} \times\{0\} \hookrightarrow X_{0} \times \mathbb{C}$ denote the inclusion. Using Definition 10.2.14(1) for ${ }_{\mathrm{T}} \mathscr{O}_{X \times \mathbb{C}}$, we thus have $\psi_{t, \lambda}\left({ }_{\mathrm{T}} \mathscr{O}_{X \times \mathbb{C}}\right)={ }_{\mathrm{T}} \iota_{*}^{-1}\left({ }_{\mathrm{T}} \mathscr{O}_{X}\right)$. Indeed, this follows from Example 10.4.23, and from the side changing formulas (10.3.23**) and $(10.4 .35 *)$, since ${ }_{\mathrm{T}} \omega_{X}=\left({ }_{\mathrm{T}} \mathscr{O}_{X}\right)^{\text {right }}$.

Example 10.4.37 (The smooth case). Let us take up the notation of Example 10.4.12 in the left case. We have $\mathcal{M}=\mathscr{O}_{X} \otimes_{\mathbb{C}} \mathcal{N}^{\nabla}$ and $\mathfrak{c}$ takes values in $C^{\infty}$ functions. For local horizontal sections $\mu^{\prime}, \mu^{\prime \prime}$, we have $\mathfrak{c}\left(1 \otimes \mu^{\prime}, \overline{1 \otimes \mu^{\prime \prime}}\right)=\mathfrak{c}^{\nabla}\left(\mu^{\prime}, \overline{\mu^{\prime \prime}}\right)$. If $X=H \times \Delta_{t}$, then $\operatorname{gr}_{V}^{\beta} \mathcal{M}=0$ for $\alpha \notin \mathbb{N}$ and $\operatorname{gr}_{V}^{0} \mathcal{M}=\mathcal{M} / t \mathcal{M}$. For local holomorphic functions $f^{\prime}, f^{\prime \prime}$ and local horizontal sections $\mu^{\prime}, \mu^{\prime \prime}$, indicating by an index $o$ the restriction to $t=0$, and for a $C^{\infty}$ test form $\eta_{o}$ on $H$ of maximal degree, we obtain

$$
\left\langle\operatorname{gr}_{V}^{0} \mathfrak{c}\left(f_{o}^{\prime} \otimes \mu_{o}^{\prime}, \overline{f_{o}^{\prime \prime} \otimes \mu_{o}^{\prime \prime}}\right), \eta_{o}\right\rangle=\mathfrak{c}^{\nabla}\left(\mu^{\prime}, \overline{\mu^{\prime \prime}}\right)_{o} \int f_{o}^{\prime} \overline{f_{o}^{\prime \prime}} \eta_{o}
$$

according to Exercise 5.4.7.

### 10.5. Localization and dual localization of a sesquilinear pairing

10.5.a. Moderate distributions. We refer to [Mal66, Chap. VII] for the results in this subsection.

Let $D$ be a reduced divisor in $X$ and let $\mathscr{O}_{X}(* D)$ be the sheaf of meromorphic functions on $X$ with poles along $D$. The subsheaf $\mathfrak{D b}_{X, D}$ of $\mathfrak{D b}_{X}$ consists of distributions supported on $D$ (i.e., vanishing when applied to any test form with compact support in $X \backslash D$ ).

On the other hand, let $j: X \backslash D \hookrightarrow X$ denote the open inclusion. By definition, there is an exact sequence of left $\mathscr{D}_{X, \bar{X}}$-modules

$$
0 \longrightarrow \mathfrak{D b}_{X, D} \longrightarrow \mathfrak{D b}_{X} \longrightarrow j_{*} \mathfrak{D b}_{X \backslash D}
$$

The image of the latter morphism is the sheaf on $X$ of distributions on $X \backslash D$ which are extendable as distributions on $X$. It can be characterized as the subsheaf of $j_{*} \mathfrak{D b}_{X \backslash D}$ consisting of distributions which can be tested along $C^{\infty}$ forms of maximal degree on $X \backslash D$ having rapid decay along $D$. It is denoted by $\mathfrak{D b}_{X}^{\bmod D}$ (sheaf on $X$
of distributions having moderate growth along $D$ ). It can be characterized more algebraically. Indeed, we have

$$
\mathfrak{D b}_{X}^{\bmod D}=\mathscr{O}_{X}(* D) \otimes_{\mathscr{O}_{X}} \mathfrak{D} \mathfrak{b}_{X}=\mathscr{O}_{\bar{X}}(* \bar{D}) \otimes_{\mathscr{O}_{X}} \mathfrak{D} \mathfrak{b}_{X} .
$$

In other words, $\mathfrak{D b}_{X, D}$ is equal to the subsheaf of $\mathfrak{D b}_{X}$ consisting of local sections annihilated some power of $g$ ( or $\bar{f}$ ), and we have a short exact sequence

$$
0 \longrightarrow \mathfrak{D b}_{X, D} \longrightarrow \mathfrak{D b}_{X} \longrightarrow \mathfrak{D b}_{X}^{\bmod D} \longrightarrow 0
$$

The previous results apply to currents of degree 0 as well, and we keep similar notation.

Example $\mathbf{1 0 . 5 . 1}$ (The case where $D$ is smooth). If $D$ is smooth, the sheaf $\mathfrak{D b}_{X, D}$ is identified with the push-forward, in the sense of $\mathscr{D}_{X, \bar{X}}$-modules, of $\mathfrak{D b}_{D}$. If for example $X=D \times \mathbb{C}$, then, according to Exercise 10.3.3, we find exact sequences

$$
\begin{gathered}
0 \longrightarrow \iota_{*} \mathfrak{D b}_{D}\left[\partial_{t}, \partial_{\bar{t}}\right] \longrightarrow \mathfrak{D b}_{X} \longrightarrow \mathfrak{D b}_{X}[1 / t] \longrightarrow 0 \\
0 \longrightarrow \iota_{*} \mathfrak{C}_{D}\left[\partial_{t}, \partial_{\bar{t}}\right] \longrightarrow \mathfrak{C}_{X} \longrightarrow \mathfrak{C}_{X}[1 / t] \longrightarrow 0
\end{gathered}
$$

10.5.b. Localization of a sesquilinear pairing. Let $\mathfrak{c}: \mathcal{N}^{\prime} \otimes_{\mathbb{C}} \overline{\mathcal{M}^{\prime \prime}} \rightarrow \mathfrak{C}_{X}$ be a sesquilinear pairing between right $\mathscr{D}_{X}$-modules. Recall that localization and dual localization are defined for $\mathscr{D}_{X}$-modules which are $\mathbb{R}$-specializable along $D$ and that we have natural morphisms (see Corollaries 9.3.6(3) and 9.4.9(3))

$$
\mathcal{M}(!D) \xrightarrow{\iota} \mathcal{M} \xrightarrow{\iota^{\vee}} \mathcal{M}(* D)
$$

According to the results recalled above, it defines a moderate sesquilinear pairing by localization:

$$
\mathfrak{c}^{\bmod D}: \mathcal{M}^{\prime}(* D) \otimes_{\mathbb{C}} \overline{\mathcal{M}^{\prime \prime}(* D)} \longrightarrow \mathfrak{C}_{X}^{\bmod D}
$$

Our aim is to refine it as a pairing taking values in $\mathfrak{C}_{X}$.

Proposition 10.5.2. Assume that $\mathcal{M}^{\prime}, \mathcal{M}^{\prime \prime}$ are $\mathbb{R}$-specializable along $D$. Then $\mathfrak{c}^{\bmod D}$ naturally induces sesquilinear pairings

$$
\begin{aligned}
& \mathfrak{c}^{(* D)}: \mathcal{M}^{\prime}(* D) \otimes_{\mathbb{C}} \overline{\mathcal{M}^{\prime \prime}(!D)} \longrightarrow \mathfrak{C}_{X} \\
& \mathfrak{c}^{(!D)}: \mathcal{M}^{\prime}(!D) \otimes_{\mathbb{C}} \overline{\mathcal{N}^{\prime \prime}(* D)} \longrightarrow \mathfrak{C}_{X}
\end{aligned}
$$

Moreover, the second one is obtained by adjunction of the first one, that is,

$$
\mathfrak{c}^{(* D)}=\left[\mathfrak{c}^{*(!D)}\right]^{*} .
$$

Lastly, $\mathfrak{c}^{(* D)}$ and $\mathfrak{c}^{(!D)}$ are compatible with $\mathfrak{c}$, in the sense that the following diagram commutes:


Proof. The question is local, and we can reduce to the case where $X=D \times \mathbb{C}$, with $D$ smooth. The $V$-filtration is then well-defined for an $\mathbb{R}$-specializable $\mathscr{D}_{X}$-module. Since the morphisms $\mathcal{N}(!D) \rightarrow \mathcal{M}$ and $\mathcal{M} \rightarrow \mathcal{N}(* D)$ have kernels and cokernels supported in $D$, they induce isomorphisms between the $V_{<0}$ of these modules. In particular, the restriction of $\mathfrak{c}\left(\right.$ hence of $\left.\mathfrak{c}^{\bmod D}\right)$ to $V_{<0} \mathcal{M}^{\prime} \otimes_{\mathbb{C}} \overline{V_{<0} \mathcal{M}^{\prime \prime}}$ takes values in $\mathfrak{C}_{X}$. We will construct $\mathfrak{c}^{(* D)}$, the case of $\mathfrak{c}^{(!D)}$ being similar.

For every $\ell \geqslant 1$, we first extend $\mathfrak{c}$ as a sesquilinear pairing

$$
\mathfrak{c}_{\ell}: V_{<0} \mathcal{M}^{\prime} \otimes_{\mathbb{C}} \overline{V_{<0} \mathcal{N}^{\prime \prime}} \cdot t^{-\ell} \longrightarrow \mathfrak{C}_{X}
$$

We argue exactly as in the proof of Lemma 10.4 .2 by extending, for every test function $\eta$ on $\mathrm{nb}\left(x_{o}\right)$ and each local section $m^{\prime}$ of $V_{<0} \mathcal{M}^{\prime}$ and $m^{\prime \prime}$ of $V_{<0} \mathcal{M}^{\prime \prime}$, the holomorphic function (for $\operatorname{Re} s \gg 0$ )

$$
\left.\left.s \longmapsto\left\langle\mathfrak{c}\left(m^{\prime}, \overline{m^{\prime \prime}}\right)\right| t\right|^{2(s-\ell)} \bar{t}^{\ell}, \eta\right\rangle
$$

as a meromorphic function on $\mathbb{C}$, and by checking that it has no pole at $s=0$ since $m^{\prime \prime} \in V_{<0} \mathcal{M}_{x_{o}}^{\prime \prime}$. Taking the value of this function at $s=0$ gives the desired extension of $\mathfrak{c}$, since $|t|^{2(s-\ell)} \bar{t}^{\ell}=|t|^{2 s} t^{-\ell}$. Moreover, one checks that $\mathfrak{c}_{\ell}$ restricts to $\mathfrak{c}_{\ell-1}$ on $\left(V_{<0} \mathcal{M}^{\prime} \cdot t^{-\ell+1}\right) \otimes_{\mathbb{C}} \overline{V_{<0} \mathcal{M}^{\prime \prime}}$, and thus defines a sesquilinear pairing

$$
\mathfrak{c}^{(* D)}: \mathcal{M}^{\prime}(* D) \otimes_{\mathbb{C}} \overline{V_{<0} \mathcal{M}^{\prime \prime}} \longrightarrow \mathfrak{C}_{X}
$$

This pairing can be extended in at most one way as a pairing

$$
\mathfrak{c}^{(* D)}: \mathcal{M}^{\prime}(* D) \otimes_{\mathbb{C}} \overline{\mathcal{N}^{\prime \prime}(!D)} \longrightarrow \mathfrak{C}_{X}
$$

due to the $\mathscr{D}_{\bar{X}}$-linearity and the equality $\mathcal{M}^{\prime \prime}(!D)=V_{<0} \mathcal{M}^{\prime \prime} \otimes_{V_{0} \mathscr{D}_{X}} \mathscr{D}_{X}$. However, since $\mathscr{D}_{X}$ is not locally free as a $V_{0} \mathscr{D}_{X}$-module, the existence of such an extension is not a priori obvious. Such an extension will exist near $x_{o}$ if, for any finite family $\left(m_{j}^{\prime \prime}\right)$ of elements of $V_{<0} \mathcal{M}_{x_{o}}^{\prime \prime}$, any finite family $\left(P_{j}\right)_{j}$ of germs of differential operators at $x_{o}$, and any $m^{\prime} \in \mathcal{M}(* D)_{x_{o}}$, the condition $\sum_{j} m_{j}^{\prime \prime} \otimes P_{j}=0$ implies $\sum_{j} \mathfrak{c}^{* D)}\left(m^{\prime}, \overline{m_{j}^{\prime \prime}}\right) \cdot \bar{P}_{j}=0$. This holds by definition if all $P_{j}$ belong to $V_{0} \mathscr{D}_{X, x_{o}}$. Therefore, one can reduce to the case where $j=0, \ldots, N$ and $P_{j}=\partial_{t}^{j}$.

We argue by induction on $N$, the case where $N=0$ being clear. We first claim that $m_{N}^{\prime \prime} \otimes \partial_{t} \in V_{<0} \mathcal{M}^{\prime \prime}(!D)$. Indeed, $\mathcal{M}^{\prime \prime}(!D)$ has the property that $\partial_{t}: \operatorname{gr}_{\alpha}^{V} \mathcal{M}^{\prime \prime}(!D) \rightarrow$ $\operatorname{gr}_{\alpha+1}^{V} \mathcal{M}^{\prime \prime}(!D)$ is an isomorphism if $\alpha=-1$, and on the other hand it is an isomorphism
for any other $\alpha$ (this holds for any $\mathbb{R}$-specializable coherent $\mathscr{D}_{X}$-module). This implies that

$$
\partial_{t}^{N}: V_{<0} \mathcal{M}^{\prime \prime}(!D) / V_{<-1} \mathcal{N}^{\prime \prime}(!D) \longrightarrow V_{<N} \mathcal{M}^{\prime \prime}(!D) / V_{<N-1} \mathcal{M}^{\prime \prime}(!D)
$$

is an isomorphism. Since

$$
m_{N}^{\prime \prime} \otimes \partial_{t}^{N}=-\sum_{j=0}^{N-1} m_{j}^{\prime \prime} \otimes \partial_{t}^{j} \in V_{<N-1} \mathcal{N}^{\prime \prime}(!D)_{x_{o}}
$$

we conclude that $m_{N}^{\prime \prime} \otimes 1 \in V_{<-1} \mathcal{N}^{\prime \prime}(!D)_{x_{o}}$, hence the assertion.
By induction, we thus have

$$
\sum_{j=0}^{N-1} \mathfrak{c}^{(* D)}\left(m^{\prime}, \overline{m_{j}^{\prime \prime}}\right) \cdot \bar{\partial}_{t}^{j}+\mathfrak{c}^{(* D)}\left(m^{\prime}, \overline{m_{N}^{\prime \prime} \otimes \partial_{t}}\right) \cdot \bar{\partial}_{t}^{N-1}=0 \in \mathfrak{C}_{X}
$$

It is therefore enough to check that, for $m^{\prime} \in \mathcal{M}^{\prime}(* D)_{x_{o}}$ and $m^{\prime \prime} \in V_{<-1} \mathcal{N}_{x_{o}}^{\prime \prime}$, we have

$$
\mathfrak{c}^{(* D)}\left(m^{\prime}, \overline{m^{\prime \prime} \otimes \partial_{t}}\right)=\mathfrak{c}^{(* D)}\left(m^{\prime}, \overline{m^{\prime \prime}}\right) \cdot \bar{\partial}_{t}
$$

Notice now that $t: V_{<0} \mathcal{M}_{x_{o}}^{\prime \prime} \rightarrow V_{<-1} \mathcal{M}_{x_{o}}^{\prime \prime}$ is an isomorphism, hence $m^{\prime \prime}=n^{\prime \prime} t$ for some $n^{\prime \prime} \in V_{<0} \mathcal{M}_{x_{o}}^{\prime \prime}$. We thus have

$$
\begin{aligned}
\mathfrak{c}^{(* D)}\left(m^{\prime}, \overline{m^{\prime \prime} \otimes \partial_{t}}\right) & =\mathfrak{c}^{(* D)}\left(m^{\prime}, \overline{n^{\prime \prime} t \otimes \partial_{t}}\right)=\mathfrak{c}^{(* D)}\left(m^{\prime}, \overline{n^{\prime \prime} \otimes t \partial_{t}}\right) \\
& =\mathfrak{c}^{(* D)}\left(m^{\prime}, \overline{n^{\prime \prime} t \partial_{t} \otimes 1}\right)=\mathfrak{c}^{(* D)}\left(m^{\prime}, \overline{n^{\prime \prime} \otimes 1}\right) \cdot \overline{t \partial_{t}} \\
& =\mathfrak{c}^{(* D)}\left(m^{\prime}, \overline{n^{\prime \prime} t \otimes 1}\right) \cdot \overline{\partial_{t}}=\mathfrak{c}^{* D)}\left(m^{\prime}, \overline{m^{\prime \prime} \otimes 1}\right) \cdot \overline{\partial_{t}}
\end{aligned}
$$

The remaining assertions are straightforward, since $\left(\mathfrak{c}^{*}\right)^{\bmod D}=\left(\mathfrak{c}^{\bmod D}\right)^{*}$.

## Definition 10.5.3 (Localization and dual localization of $\mathscr{D}$-triples)

Let $D$ be an effective divisor in $X$ and let $\mathcal{T}=\left(\mathcal{M}^{\prime}, \mathcal{M}^{\prime \prime}, \mathfrak{c}\right)$ be an object of $\mathscr{D}$-Triples $(X)$ which is $\mathbb{R}$-specializable along $D$ (i.e., its components $\mathcal{M}^{\prime}, \mathcal{M}^{\prime \prime}$ are so). If $D=(g)$, we then set

$$
\begin{aligned}
\mathcal{T}(* D) & :=\left(\mathcal{M}^{\prime}(* D), \mathcal{M}^{\prime \prime}(!D), \mathfrak{c}^{(* D)}\right), \\
\mathcal{T}(!D) & :=\left(\mathcal{M}^{\prime}(!D), \mathcal{M}^{\prime \prime}(* D), \mathfrak{c}^{(!D)}\right)
\end{aligned}
$$

These functors satisfy obvious identities with respect to Adjunction 10.2.9.

### 10.6. Pushforward, specialization and localization of sesquilinear pairings

Let $f: X \rightarrow Y$ be an holomorphic map between complex manifolds and let $g^{\prime}$ : $Y \rightarrow \mathbb{C}$ be an holomorphic function. Set $g=g^{\prime} \circ f$. Let $\mathcal{M}^{\prime}, \mathcal{M}^{\prime \prime}$ be right $\mathscr{D}_{X}$-modules which are $\mathbb{R}$-specializable along $(g)$. Let $\mathfrak{c}: \mathcal{M}^{\prime} \otimes \overline{\mathcal{M}^{\prime \prime}} \rightarrow \mathfrak{C}_{X}$ be a sesquilinear pairing. Assume that $f$ is proper on the support of $\mathcal{M}^{\prime}, \mathcal{M}^{\prime \prime}$.
10.6.a. Pushforward and specialization of sesquilinear pairings. Recall that Theorem 7.8.5 implies:

- for every $k \in \mathbb{Z}, \mathscr{H}_{\mathrm{D}}^{k} f_{*} \mathcal{M}$ is $\mathbb{R}$-specializable along $\left(g^{\prime}\right)$,
- for every $\alpha \in \mathbb{R}$, the natural morphism $\mathscr{H}_{\mathrm{D}}^{k} f_{*} V_{\alpha} \mathcal{M} \rightarrow \mathscr{H}_{\mathrm{D}}^{k} f_{*} \mathcal{M}$ is injective and its image is equal to $V_{\alpha} \mathscr{H}_{\mathrm{D}}^{k} f_{*} \mathcal{M}$.

Theorem 10.6.1. With respect to the previous natural morphism, we have

$$
{ }_{\mathrm{r}} f_{*}^{k} \psi_{g, \lambda} \mathfrak{c}=(-1)^{n-m} \psi_{g^{\prime}, \lambda}\left({ }_{\mathrm{T}} f_{*}^{k} \mathfrak{c}\right) .
$$

Proof. It is enough to argue with ${ }_{\mathrm{D}, \overline{\mathrm{D}}} f_{*}^{k}$. We start with the case of a map $f \times \mathrm{Id}$ : $X \times \mathbb{C} \rightarrow Y \times \mathbb{C}$ and we take for the function $g^{\prime}: Y \times \mathbb{C} \rightarrow \mathbb{C}$ the second projection. We assume that $\mathcal{M}^{\prime}, \mathcal{N}^{\prime \prime}$ are right $\mathscr{D}_{X \times \mathbb{C}}$-modules.

Lemma 10.6.2. With these assumptions, for every $\alpha \in \mathbb{R}$ and $k \in \mathbb{Z}$,

$$
\mathrm{D}, \overline{\mathrm{D}} f_{*}^{k}\left(\operatorname{gr}_{\alpha}^{V} \mathfrak{c}\right)=\operatorname{gr}_{\alpha}^{V}\left(\mathrm{D}, \overline{\mathrm{D}}(f \times \operatorname{Id})_{*}^{k} \mathfrak{c}\right)
$$

Proof. Set $\beta=-\alpha-1$ and let

$$
\begin{aligned}
m_{\infty}^{\prime n+1+k} & \in \Gamma\left(U, f_{*}\left(\mathscr{E}_{X \times C}^{n+1+k} \otimes_{\mathscr{O}_{X \times \mathbb{C}}} V^{\beta} \mathcal{N}^{\prime \mathrm{left}}\right)\right), \\
m_{\infty}^{\prime \prime n+1-k} & \in \Gamma\left(U, f_{*}\left(\mathscr{E}_{X \times \mathbb{C}}^{n+1-k} \otimes_{\mathscr{O}_{X \times C}} V^{\beta} \mathcal{M}^{\prime \prime \mathrm{left}}\right)\right) .
\end{aligned}
$$

The cohomology classes $\left[m_{\infty}^{\prime n+k}\right]$ and $\left[m_{\infty}^{\prime \prime n+1-k}\right]$ can be regarded as sections of $V_{\alpha}\left({ }_{\mathrm{D}} f_{*}^{k} \mathcal{M}^{\prime}\right) \otimes_{\mathscr{O}_{Y}} \mathscr{C}_{Y}^{\infty}$ and $V_{\alpha}\left({ }_{\mathrm{D}} f_{*}^{-k} \mathcal{N}^{\prime \prime}\right) \otimes_{\mathscr{O}_{Y}} \mathscr{C}_{Y}^{\infty}$ respectively, according to the result recalled above. We can then compute with these classes. Let us also denote by $\bullet_{\alpha}$ the class of $\bullet \in V_{\alpha}$ modulo $V_{<\alpha}$. We have, for $\eta \in \mathscr{C}_{Y}^{\infty}(U)$,

$$
\begin{aligned}
\left\langle\operatorname { g r } _ { \alpha } ^ { V } \left({ }_{\mathrm{D}, \overline{\mathrm{D}}}(f \times\right.\right. & \left.\left.\mathrm{Id})_{*}^{k} \mathfrak{c}\right)\left(\left[m_{\infty}^{\prime n+1+k}\right]_{\alpha}, \overline{\left[m_{\infty}^{\prime \prime n+1-k}\right]_{\alpha}}\right), \eta(y)\right\rangle \\
& =\frac{\mathrm{i}}{2 \pi} \operatorname{Res}_{s=\alpha}\left\langle\left(\left(\mathrm{D}, \overline{\mathrm{D}}(f \times \mathrm{Id})_{*}^{k} \mathfrak{c}\right)\left(\left[m_{\infty}^{\prime n+1+k}\right], \overline{\left[m_{\infty}^{\prime \prime n+1-k}\right]}\right), \eta(y)|t|^{2 s} \chi(t)\right\rangle\right. \\
& \left.=\left.\frac{\mathrm{i}}{2 \pi} \operatorname{Res}_{s=\alpha}\left\langle\mathfrak{c}\left(m_{\infty}^{\prime n+1+k}, \overline{m_{\infty}^{\prime \prime n+1-k}}\right), \eta \circ f(x)\right| t\right|^{2 s} \chi(t)\right\rangle \\
& =\left\langle\operatorname{gr}_{\alpha}^{V} \mathfrak{c}\left(\left(m_{\infty}^{\prime n+1+k}\right)_{\alpha}, \overline{\left(m_{\infty}^{\prime \prime n+1-k}\right)_{\alpha}}\right), \eta \circ f(x)\right\rangle \\
& =\left\langle{ }_{\mathrm{D}, \overline{\mathrm{D}}} f_{*}^{k} \operatorname{gr}_{\alpha}^{V} \mathfrak{c}\left(\left[\left(m_{\infty}^{\prime n+1+k}\right)_{\alpha}\right], \overline{\left[\left(m_{\infty}^{\prime \prime n+1-k}\right)_{\alpha}\right]}\right), \eta(y)\right\rangle,
\end{aligned}
$$

and we obtain the desired equality since, as recalled, $\left[m_{\infty}^{n+1+k}\right]_{\alpha}=\left[\left(m_{\infty}^{n+1+k}\right)_{\alpha}\right]$ in $\Gamma\left(U, \operatorname{gr}_{\alpha \mathrm{D}, \overline{\mathrm{D}}}^{V}(f \times \mathrm{Id})_{*}^{k} \mathcal{M}\right)=\Gamma\left(U,{ }_{\mathrm{D}, \overline{\mathrm{D}}} f_{*}^{k} \operatorname{gr}_{\alpha}^{V} \mathcal{M}\right)$.

We can now end the proof of Theorem 10.6.1. We have

$$
\begin{aligned}
\mathrm{T} f_{*}^{k} \psi_{g, \lambda} \mathfrak{c} & =\varepsilon(n-m-k)(-1)^{n}{ }_{\mathrm{D}, \overline{\mathrm{D}}} f_{*}^{k} \operatorname{gr}_{\alpha}^{V}\left(\mathrm{D}, \overline{\mathrm{D}} \iota_{g *}^{0} \mathfrak{c}\right) \\
& =\varepsilon(n-m-k)(-1)^{n} \operatorname{gr}_{\alpha}^{V}\left({ }_{\mathrm{D}, \overline{\mathrm{D}}}(f \times \mathrm{Id})_{* \mathrm{D}, \overline{\mathrm{D}}}^{k} \iota_{g *}^{0} \mathfrak{c}\right) \\
& =\varepsilon(n-m-k)(-1)^{n} \operatorname{gr}_{\alpha}^{V}\left({ }_{\left.\mathrm{D}, \overline{\mathrm{D}} \iota_{g^{\prime} * \mathrm{D}, \overline{\mathrm{D}}}^{0} f_{*}^{k} \mathfrak{c}\right)}\right. \\
& =(-1)^{n-1} \operatorname{gr}_{\alpha}^{V}\left({ }_{\mathrm{T}} \iota_{g^{\prime} * \mathrm{~T}}^{0} f_{*}^{k} \mathfrak{c}\right) \\
& =(-1)^{n-m} \psi_{g^{\prime}, \lambda}\left({ }_{\mathrm{T}} f_{*}^{k} \mathfrak{c}\right) .
\end{aligned}
$$

## Corollary 10.6.3 (Pushforward and specialization of $\mathscr{D}$-triples)

Let $\mathcal{T}$ be an object of $\mathscr{D}$-Triples $(X)$ which is $\mathbb{R}$-specializable along $(g)=\left(g^{\prime} \circ f\right)$, where $f: X \rightarrow Y$ is proper. Then we have an isomorphism

$$
\left((-1)^{n-m} \mathrm{Id}, \mathrm{Id}\right):_{\mathrm{T}} f_{*}^{k} \psi_{g, \lambda} \mathcal{T} \xrightarrow{\sim} \psi_{g^{\prime}, \lambda}\left({ }_{\mathrm{T}} f_{*}^{k} \mathcal{T}\right) .
$$

Remark 10.6.4 (The case of left $\mathscr{D}$-triples). Due to ( $10.3 .23 * *$ ) and (10.4.35*), the same isomorphism holds in the case of left $\mathscr{D}$-triples.
10.6.b. Pushforward and localization of sesquilinear pairings. Similarly, let $D^{\prime}$ be an effective divisor in $X^{\prime}$ and set $D=f^{*} D^{\prime}$. Assume that $\mathcal{M}$ is $\mathbb{R}$-specializable along $D$. Then we have natural morphisms $\mathscr{H}_{\mathrm{D}}^{k} f_{*} \mathcal{M}\left[!D^{\prime}\right] \rightarrow \mathscr{H}^{k}\left({ }_{\mathrm{D}} f_{*} \mathcal{M}[!D]\right)$ and $\mathscr{H}_{\mathrm{D}}^{k} f_{*}(\mathcal{M}[* D]) \rightarrow \mathscr{H}_{\mathrm{D}}^{k} f_{*} \mathcal{M}\left[* D^{\prime}\right]$.

Theorem 10.6.5. With respect to the previous natural morphism, the sesquilinear pairings ${ }_{\mathrm{D}, \overline{\mathrm{D}}} f_{*}^{k}\left(\mathfrak{c}^{(* D)}\right)$ and $\left({ }_{\mathrm{D}, \overline{\mathrm{D}}} f_{*}^{k} \mathfrak{c}\right)^{\left(* D^{\prime}\right)}$ coincide ( $\left.\star=!, *\right)$.

### 10.7. Beilinson's construction for sesquilinear pairings

Let $D=(g)$ be a principal divisor on $X$ and let $\mathcal{M}$ be a coherent $\mathscr{D}_{X}$-module which is $\mathbb{R}$-specializable along $D$. Beilinson's construction (see Section 9.6) produces two exact sequences (9.6.2!) and (9.6.2*) (recall that, for $\mathscr{D}_{X}$-modules, $\mathbb{R}$-specializability ensures maximalizability, due to the validity of Kashiwara's equivalence). We can also start from a $\mathscr{D}_{X}(* D)$-coherent module $\mathcal{N}_{*}$ which is $\mathbb{R}$-specializable along $D$. Given a sesquilinear pairing $\mathfrak{c}^{\bmod D}$ between $\mathcal{M}_{*}^{\prime}$ and $\mathcal{M}_{*}^{\prime \prime}$ with values in $\mathfrak{D} \mathfrak{b}_{X}^{\bmod D}$, our aim is to extend it as a sesquilinear pairing $\Xi_{g} \mathfrak{c}: \Xi_{g} \mathcal{M}^{\prime} \otimes_{\mathbb{C}} \overline{\Xi_{g} \mathcal{M}^{\prime \prime}} \rightarrow \mathfrak{D b}_{X}$ in such a way that
(1) $\Xi_{g} \mathfrak{c}\left(a^{\prime} \bullet, \overline{b^{\prime \prime v}}\right)=0$ and the pairing induced by means of $a^{\prime \prime \vee}$ on $\mathcal{M}^{\prime}(!D) \otimes_{\mathbb{C}}$ $\overline{\mathcal{N}^{\prime \prime}(* D)}$ is equal to $\mathfrak{c}^{(!D)}$,
(2) a similar property to recover the pairing $\mathfrak{c}^{(* D)}$,
(3) a similar property to recover $\psi_{t, 1} \mathbf{c}$.

### 10.8. Comments

Here come the references to the existing work which has been the source of inspiration for this chapter.

