APPENDIX A

TRAINING ON *D*-MODULES

Summary. In this chapter, we introduce the fundamental functors on \mathscr{D} -modules that we will use in order to define supplementary structures, and we also introduce various operations: pushforward and pullback by a holomorphic map between complex manifolds or a morphism between smooth algebraic varieties, and specialization along a divisor. Most results are presented as exercises. They only rely on Leibniz rule.

Although it would be natural to develop the theory of coherent \mathscr{D}_X -modules in a way similar to that of \mathscr{O}_X -modules, some points of the theory are not known to extend to \mathscr{D}_X -modules (the lemma on holomorphic matrices). The approach which is therefore classically used consists in using the \mathscr{O}_X -theory, and the main tools for that purpose are the coherent filtrations. The main references for this chapter are **[Bjö93]**, **[Kas03]** and **[GM93]**.

A.1. The sheaf of holomorphic differential operators

Let (X, \mathscr{O}_X) be a complex manifold endowed with its sheaf of holomorphic functions. We also denote by \mathscr{C}_X^{∞} the sheaf of complex-valued C^{∞} functions on the underlying C^{∞} manifold $X_{\mathbb{R}}$. This sheaf is a fine sheaf, hence is soft.

We will denote by Θ_X the sheaf of holomorphic vector fields on X. This is the \mathscr{O}_X -locally free sheaf generated in local coordinates by $\partial_{x_1}, \ldots, \partial_{x_n}$. It is a sheaf of \mathscr{O}_X -Lie algebras which is locally free as an \mathscr{O}_X -module, and vector fields act (on the left) on functions by derivation, in a way compatible with the Lie algebra structure: given a local vector field ξ acting on functions as a derivation $g \mapsto \xi(g)$, and a local holomorphic function $f, f\xi$ is the vector field acting as $f \cdot \xi(g)$, and given two vector fields ξ, η , their bracket as derivations $[\xi, \eta](g) := \xi(\eta(g)) - \eta(\xi(g))$ is still a derivation, hence defines a vector field.

Dually, we denote by Ω_X^1 the sheaf of holomorphic 1-forms on X. We will set $\Omega_X^k = \wedge^k \Omega_X^1$. We denote by $d: \Omega_X^k \to \Omega_X^{k+1}$ the differential.

Exercise A.1. Let \mathcal{E} be a locally free \mathscr{O}_X -module of rank d and let \mathcal{E}^{\vee} be its dual. Show that, given any local basis $e = (e_1, \ldots, e_d)$ of \mathcal{E} with dual basis e^{\vee} , the section $\sum_{i=1}^{d} e_i \otimes e_i^{\vee} \text{ of } \mathcal{E} \otimes_{\mathscr{O}_X} \mathcal{E}^{\vee} \text{ does not depend on the choice of the local basis } \boldsymbol{e} \text{ and extends} as a global section of } \mathcal{E} \otimes_{\mathscr{O}_X} \mathcal{E}^{\vee}.$ Show that it defines, up to a constant, an \mathscr{O}_X -linear section $\mathscr{O}_X \to \mathcal{E} \otimes_{\mathscr{O}_X} \mathcal{E}^{\vee}$ of the natural duality pairing $\mathcal{E} \otimes_{\mathscr{O}_X} \mathcal{E}^{\vee} \to \mathscr{O}_X.$ Conclude that we have a natural global section of $\Omega^1_X \otimes_{\mathscr{O}_X} \Theta_X$ given, in local coordinates, by $\sum_i \mathrm{d} x_i \otimes \partial_{x_i}.$

Recall that the *contraction* by a vector field ξ is the \mathscr{O}_X -linear morphism $\xi \sqcup: \Omega_X^k \to \Omega_X^{k-1}$ defined by $\eta \mapsto \eta(\xi, \bullet)$, where \bullet is an ordered (k-1)-tuple of vector fields. Set $n = \dim X$ and $\omega_X = \Omega_X^n$: this is the sheaf of forms of maximal degree. In local coordinates (x_1, \ldots, x_n) , set $d\boldsymbol{x} := dx_1 \wedge \cdots dx_n$ and $d\boldsymbol{x}_i := dx_1 \wedge \cdots \wedge dx_n \wedge dx_n$. Then we have $\partial_{x_i} \sqcup d\boldsymbol{x} = (-1)^{i-1} d\boldsymbol{x}_i$, and for $f \in \mathscr{O}_X$, we have $d(f\partial_{x_i} \sqcup d\boldsymbol{x}) = \partial f/\partial_{x_i} \cdot d\boldsymbol{x}$. Setting $\mathscr{L}_{\xi}(d\boldsymbol{x}) := \xi \sqcup d\boldsymbol{x}$, and using that $\mathscr{L}_{\partial_{x_i}}(d\boldsymbol{x}) = 0$, this relation can be written as

$$[\partial_{x_i}, f] \mathrm{d}\boldsymbol{x} = - \Big[f \mathscr{L}_{\partial_{x_i}}(\mathrm{d}\boldsymbol{x}) - \mathscr{L}_{f \partial_{x_i}}(\mathrm{d}\boldsymbol{x}) \Big].$$

We conclude that there is a natural *right* action (in a compatible way with the Lie algebra structure) of Θ_X on ω_X : the action is given by $\omega \cdot \xi = -\mathcal{L}_{\xi}\omega := -d(\xi \sqcup \omega)$ (\mathcal{L}_{ξ} is called the Lie derivative of ξ). The action is on the right due to the sign above which makes this definition compatible with bracket.

Definition A.1.1 (The sheaf of holomorphic differential operators)

For any open set U of X, the ring $\mathscr{D}_X(U)$ of holomorphic differential operators on U is the subring of $\operatorname{Hom}_{\mathbb{C}}(\mathscr{O}_U, \mathscr{O}_U)$ generated by

- multiplication by holomorphic functions on U,
- derivation by holomorphic vector fields on U.

The sheaf \mathscr{D}_X is defined by $\Gamma(U, \mathscr{D}_X) = \mathscr{D}_X(U)$ for every open set U of X.

By construction, the sheaf \mathscr{D}_X acts on the left on \mathscr{O}_X , i.e., \mathscr{O}_X is a left \mathscr{D}_X -module.

Definition A.1.2 (The filtration of \mathscr{D}_X by the order). The increasing family of subsheaves $F_k \mathscr{D}_X \subset \mathscr{D}_X$ is defined inductively:

- $F_k \mathscr{D}_X = 0$ if $k \leq -1$,
- $F_0 \mathscr{D}_X = \mathscr{O}_X$ (via the canonical injection $\mathscr{O}_X \hookrightarrow \mathscr{H}om_{\mathbb{C}}(\mathscr{O}_X, \mathscr{O}_X)$),

• the local sections P of $F_{k+1}\mathcal{D}_X$ are characterized by the fact that [P, f] is a local section of $F_k\mathcal{D}_X$ for any holomorphic function g.

Exercise A.2. Show that a differential operator P of order ≤ 1 satisfying P(1) = 0 is a derivation of \mathcal{O}_X , i.e., a section of Θ_X .

Exercise A.3 (Local computations). Let U be an open set of \mathbb{C}^n with coordinates x_1, \ldots, x_n . Denote by $\partial_{x_1}, \ldots, \partial_{x_n}$ the corresponding vector fields.

(1) Show that the following relations are satisfied in $\mathscr{D}(U)$:

$$[\partial_{x_i}, f] = \frac{\partial f}{\partial x_i}, \quad \forall f \in \mathscr{O}(U), \ \forall i \in \{1, \dots, n\}, \\ [\partial_{x_i}, \partial_{x_j}] = 0 \quad \forall i, j \in \{1, \dots, n\}.$$

with standard notation concerning multi-indices α, β .

(2) Show that any element $P \in \mathscr{D}(U)$ can be written in a unique way as $\sum_{\alpha} a_{\alpha} \partial_x^{\alpha}$ or $\sum_{\alpha} \partial_x^{\alpha} b_{\alpha}$ with $a_{\alpha}, b_{\alpha} \in \mathscr{O}(U)$. Conclude that \mathscr{D}_X is a locally free module over \mathscr{O}_X with respect to the action on the left and that on the right.

(3) Show that $\max\{|\alpha| : a_{\alpha} \neq 0\} = \max\{|\alpha| : b_{\alpha} \neq 0\}$. It is denoted by $\operatorname{ord}_{x} P$.

(4) Show that $\operatorname{ord}_x P$ does not depend on the coordinate system chosen on U.

(5) Show that PQ = 0 in $\mathscr{D}(U) \Rightarrow P = 0$ or Q = 0.

(6) Identify $F_k \mathscr{D}_X$ with the subsheaf of local sections of \mathscr{D}_X having order $\leq k$ (in some or any local coordinate system). Show that it is a locally free \mathscr{O}_X -module of finite rank.

(7) Show that the filtration $F_{\bullet}\mathscr{D}_X$ is exhaustive (i.e., $\mathscr{D}_X = \bigcup_k F_k \mathscr{D}_X$) and that it satisfies

$$F_k \mathscr{D}_X \cdot F_\ell \mathscr{D}_X = F_{k+\ell} \mathscr{D}_X$$

(The left-hand term consists by definition of all sums of products of a section of $F_k \mathscr{D}_X$ and a section of $F_\ell \mathscr{D}_X$.)

(8) Show that the bracket [P,Q] := PQ - QP induces for every k, ℓ a \mathbb{C} -bilinear morphism $F_k \mathscr{D}_X \otimes_{\mathbb{C}} F_\ell \mathscr{D}_X \to F_{k+\ell-1} \mathscr{D}_X$.

(9) Conclude that the graded ring $\operatorname{gr}^F \mathscr{D}_X$ is commutative.

Exercise A.4 (The graded sheaf $\operatorname{gr}^F \mathscr{D}_X$). The goal of this exercise is to show that the sheaf of graded rings $\operatorname{gr}^F \mathscr{D}_X$ may be canonically identified with the sheaf of graded rings $\operatorname{Sym} \Theta_X$. If one identifies Θ_X with the sheaf of functions on the cotangent space T^*X which are linear in the fibres, then $\operatorname{Sym} \Theta_X$ is the sheaf of functions on T^*X which are polynomial in the fibres. In particular, $\operatorname{gr}^F \mathscr{D}_X$ is a sheaf of commutative rings.

(1) Identify the \mathscr{O}_X -module $\operatorname{Sym}^k \Theta_X$ with the sheaf of symmetric \mathbb{C} -linear forms $\boldsymbol{\xi} : \mathscr{O}_X \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathscr{O}_X \to \mathscr{O}_X$ on the *k*-fold tensor product, which behave like a derivation with respect to each factor.

(2) Show that $\operatorname{Sym} \Theta_X := \bigoplus_k \operatorname{Sym}^k \Theta_X$ is a sheaf of graded \mathscr{O}_X -algebras on X and identify it with the sheaf of functions on T^*X which are polynomial in the fibres.

(3) Show that the map $F_k \mathscr{D}_X \to \mathscr{H}om_{\mathbb{C}}(\otimes^k_{\mathbb{C}} \mathscr{O}_X, \mathscr{O}_X)$ which sends any section P of $F_k \mathscr{D}_X$ to

 $f_1 \otimes \cdots \otimes f_k \longmapsto [\cdots [[P, f_1]f_2] \cdots f_k]$

induces an isomorphism of \mathscr{O}_X -modules $\operatorname{gr}_k^F \mathscr{D}_X \to \operatorname{Sym}^k \Theta_X$.

(4) Show that the induced morphism

$$\operatorname{gr}^F \mathscr{D}_X := \bigoplus_k \operatorname{gr}^F_k \mathscr{D}_X \longrightarrow \operatorname{Sym} \Theta_X$$

is an isomorphism of sheaves of graded \mathcal{O}_X -algebras.

Exercise A.5 (The universal connection).

(1) Show that the natural left multiplication of Θ_X on \mathscr{D}_X can be written as a *connection*

$$\nabla:\mathscr{D}_X\longrightarrow\Omega^1_X\otimes_{\mathscr{O}_X}\mathscr{D}_X,$$

i.e., as a \mathbb{C} -linear morphism satisfying the Leibniz rule $\nabla(fP) = df \otimes P + f\nabla P$, where g is any local section of \mathscr{O}_X and P any local section of \mathscr{D}_X . [Hint: $\nabla(1)$ is the global section of $\Omega^1_X \otimes_{\mathscr{O}_X} \Theta_X$ considered in Exercise A.1.]

(2) Extend this connection for every $k \ge 1$ as a \mathbb{C} -linear morphism

$${}^{(k)}\nabla:\Omega^k_X\otimes_{\mathscr{O}_X}\mathscr{D}_X\longrightarrow\Omega^{k+1}_X\otimes_{\mathscr{O}_X}\mathscr{D}_X$$

satisfying the Leibniz rule written as

$$^{(k)}\nabla(\omega\otimes P) = d\omega\otimes P + (-1)^k\omega\wedge\nabla P$$

- (3) Show that ${}^{(k+1)}\nabla \circ {}^{(k)}\nabla = 0$ for every $k \ge 0$ (i.e., ∇ is *integrable* or *flat*).
- (4) Show that the morphisms ${}^{(k)}\nabla$ are right \mathscr{D}_X -linear (but not left \mathscr{O}_X -linear).

Exercise A.6. More generally, show that a left \mathscr{D}_X -module \mathscr{M} is nothing but an \mathscr{O}_X -module with an *integrable* connection $\nabla : \mathscr{M} \to \Omega^1_X \otimes_{\mathscr{O}_X} \mathscr{M}$. [*Hint*: to get the connection, tensor the left \mathscr{D}_X -action $\mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{M} \to \mathscr{M}$ by Ω^1_X on the left and compose with the universal connection to get $\mathscr{D}_X \otimes \mathscr{M} \to \Omega^1_X \otimes \mathscr{M}$; compose it on the left with $\mathscr{M} \to \mathscr{D}_X \otimes \mathscr{M}$ given by $m \mapsto 1 \otimes m$.] Define similarly the iterated connections ${}^{(k)}\nabla : \Omega^k_X \otimes_{\mathscr{O}_X} \mathscr{M} \to \Omega^{k+1}_X \otimes_{\mathscr{O}_X} \mathscr{M}$. Show that ${}^{(k+1)}\nabla \circ {}^{(k)}\nabla = 0$.

In conclusion:

Proposition A.1.3. Giving a left \mathscr{D}_X -module \mathfrak{M} is equivalent to giving an \mathscr{O}_X -module \mathfrak{M} together with an integrable connection ∇ .

Proof. Exercises A.1, A.5 and A.6.

A.2. Filtered objects and the graded Rees ring $R_F \mathscr{D}_X$

A.2.a. Filtered rings and modules

Definition A.2.1. Let $(\mathscr{A}, F_{\bullet})$ be a filtered \mathbb{C} -algebra. A filtered \mathscr{A} -module $(\mathfrak{M}, F_{\bullet}\mathfrak{M})$ is an \mathscr{A} -module \mathfrak{M} together with an increasing filtration indexed by \mathbb{Z} satisfying (for left modules for instance)

$$F_k \mathscr{A} \cdot F_\ell \mathfrak{M} \subset F_{k+\ell} \mathfrak{M} \quad \forall k, \ell \in \mathbb{Z}.$$

We always assume that the filtration is *exhaustive*, i.e., $\bigcup_{\ell} F_{\ell} \mathcal{M} = \mathcal{M}$. We also say that $F_{\bullet} \mathcal{M}$ is an $F_{\bullet} \mathscr{A}$ -filtration, or simply an F-filtration.

A filtered morphism between filtered \mathscr{A} -modules is a morphism of \mathscr{A} -modules which is compatible with the filtrations.

It is possible to apply the techniques of the previous sections to filtered objects. A simple way to do that is to introduce the Rees object associated to any filtered object. Introduce a new variable z. We will replace the base field \mathbb{C} with the polynomial ring $\mathbb{C}[z]$.

Caveat A.2.2. Since it is standard, when considering Hodge filtrations, to work with decreasing filtrations, and since the variable z is adapted to increasing ones, we set the degree of z to -1.

Definition A.2.3 (Rees ring and Rees module). If $(\mathscr{A}, F_{\bullet})$ is a filtered \mathbb{C} -algebra, we denote by $\widetilde{\mathscr{A}}$ (or $R_F \mathscr{A}$ if we want to insist on the dependence with respect to the filtration) the graded subring $\bigoplus_p F_p \mathscr{A} \cdot z^p$ of $\mathscr{A} \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}]$ (the term $F_p \mathscr{A} \cdot z^p$ is in degree -p). For example, if $F_p \mathscr{A} = 0$ for $p \leq -1$ and $F_p \mathscr{A} = \mathscr{A}$ for $p \geq 0$, we have $\widetilde{\mathscr{A}} = \mathscr{A} \otimes_{\mathbb{C}} \mathbb{C}[z]$. Any filtered module $(\mathcal{M}, F_{\bullet})$ on the filtered ring $(\mathscr{A}, F_{\bullet})$ gives rise similarly to a graded $\widetilde{\mathscr{A}}$ -module $R_F \mathscr{M} = \bigoplus_p F_p \mathscr{M} \cdot z^p \subset \mathfrak{M} \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}]$, and a filtered morphism gives rise to a graded morphism (of degree zero) between the associated Rees modules.

The category $\mathsf{Modgr}(\widetilde{\mathscr{A}})$ is the category whose objects are graded $\widetilde{\mathscr{A}}$ -modules and whose morphisms are graded morphisms of degree zero. It is an abelian category. It comes equipped with an automorphism σ : given an object $\mathscr{M} = \bigoplus_p \mathscr{M}^p$ of $\mathsf{Modgr}(\widetilde{\mathscr{A}})$ (where \mathscr{M}^p is in degree p), we set

(A.2.3*)
$$\sigma(\mathscr{M}) = \mathscr{M}(1) \text{ with } \mathscr{M}(1)^p = \mathscr{M}^{p+1}.$$

Remark A.2.4 (Shift of the filtration and twist of the Rees module)

(1) The shift F[k] of an increasing filtration is defined by

(A.2.4*)
$$F[k]_{\bullet}\mathcal{M} = F_{\bullet-k}\mathcal{M}$$

(2) If $\mathscr{M} = R_F \mathfrak{M}$, with $F_p \mathfrak{M} z^p$ in degree -p, then $\mathscr{M}(-k)$ has $F_{p+k} \mathfrak{M} z^{p+k}$ in degree -p, so that, for $k \ge 0$, z^k is a graded morphism of degree zero $\mathscr{M} \to \mathscr{M}(-k)$. On the other hand, for every $k \in \mathbb{Z}$, the isomorphism $z^k : \mathscr{M}[z^{-1}] \to \mathscr{M}[z^{-1}](-k)$ induces an isomorphism

$$(A.2.4**) \qquad \qquad R_{F[k]}\mathcal{M} \xrightarrow{\sim} \mathscr{M}(-k).$$

Notice also that, as $R_F \mathcal{M}$ is contained in $\mathcal{M} \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}]$, the multiplication by z is injective on $R_F \mathcal{M}$.

Exercise A.7.

(1) If \mathscr{M} is a graded \mathscr{A} -module, show that its $\mathbb{C}[z]$ -torsion is also graded and each torsion element is annihilated by some power of z.

(2) Conclude that $(z-a) : \mathcal{M} \to \mathcal{M}$ is injective for every $a \in \mathbb{C} \setminus \{0\}$, equivalently that $\mathcal{M}[z^{-1}] := \mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}[z]} \mathcal{M}$ is $\mathbb{C}[z, z^{-1}]$ -flat, and that a graded $\widetilde{\mathscr{A}}$ -module is $\mathbb{C}[z]$ -flat if and only if it has no z-torsion.

(3) Let $\varphi : \mathscr{M} \to \mathscr{N}$ be a morphism in $\mathsf{Modgr}(\mathscr{A})$. Assume that φ is injective. Show that the induced morphism $\varphi_a : \mathscr{M}/(z-a)\mathscr{M} \to \mathscr{N}/(z-a)\mathscr{N}$ is injective if $a \neq 0$. [*Hint*: use (2) for Coker φ .]

(4) Let \mathscr{M}^{\bullet} be a complex in $\mathsf{Modgr}(\mathscr{A})$. Show that, for every *i* and each $a \neq 0$, we have

$$\mathscr{H}^{i}(\mathscr{M}^{\bullet}/(z-a)\mathscr{M}^{\bullet})\simeq \mathscr{H}^{i}\mathscr{M}^{\bullet}/(z-a)\mathscr{H}^{i}\mathscr{M}^{\bullet}.$$

[Hint: Consider the long exact sequence

 $\cdots \mathscr{H}^{i}\mathscr{M}^{\bullet} \xrightarrow{z-a} \mathscr{H}^{i}\mathscr{M}^{\bullet} \longrightarrow \mathscr{H}^{i}(\mathscr{M}^{\bullet}/(z-a)\mathscr{M}^{\bullet}) \longrightarrow \cdots$

attached to the exact sequence of complexes (according to (3))

$$0 \longrightarrow \mathscr{M}^{\bullet} \xrightarrow{z-a} \mathscr{M}^{\bullet} \longrightarrow \mathscr{M}^{\bullet}/(z-a)\mathscr{M}^{\bullet} \longrightarrow 0$$

and apply (3).

(5) Show that the Rees construction gives an equivalence between the category of filtered $(\mathscr{A}, F_{\bullet})$ -modules and the subcategory of the category of graded $\widetilde{\mathscr{A}}$ -modules (the morphisms are graded of degree zero) whose objects have no z-torsion (equivalently, are $\mathbb{C}[z]$ -flat). [*Hint*: If $\mathscr{M} = \bigoplus \mathscr{M}^p$ is a graded $\widetilde{\mathscr{A}}$ -module, the property that $z : \mathscr{M} \to \mathscr{M}(-1)$ is injective is equivalent to $\mathscr{M}^p \subset \mathscr{M}^{p-1}$ for all p; set then $\mathscr{M} = \varinjlim_k \mathscr{M}^{-k}$.]

(6) Recover \mathcal{M} as $R_F \mathcal{M}/(z-1)R_F \mathcal{M}$, and $\operatorname{gr}^F \mathcal{M}$ as $R_F \mathcal{M}/zR_F \mathcal{M}$ (as a graded $\operatorname{gr}^F \mathscr{A}$ -module).

Exercise A.8. If $(\mathcal{M}, F_{\bullet}\mathcal{M})$ is a filtered object of $\mathsf{Mod}(\mathscr{A})$, then a subobject \mathcal{M}' of \mathcal{M} carries the induced filtration $(F_p\mathcal{M}\cap\mathcal{M}')_{p\in\mathbb{Z}}$, while a quotient object $\mathcal{M}/\mathcal{M}''$ carries the induced filtration $((F_p\mathcal{M}+\mathcal{M}'')/\mathcal{M}'')_{p\in\mathbb{Z}}$. Show the following properties.

- (1) $R_F \mathcal{M}' = R_F \mathcal{M} \cap \mathcal{M}'[z, z^{-1}]$ and $R_F (\mathcal{M}/\mathcal{M}'') = R_F \mathcal{M} \cap \mathcal{M}''[z, z^{-1}]/\mathcal{M}''[z, z^{-1}].$
- (2) The two possible induced filtrations on a subquotient $\mathcal{M}' \cap \mathcal{M}'' / \mathcal{M}''$ of \mathcal{M} agree.

(3) For every filtered complex $(\mathcal{M}^{\bullet}, F)$, the *i*-th cohomology of the complex is a subquotient of \mathcal{M}^i , hence it carries an induced filtration. Then there is a canonical morphism $\mathscr{H}^i(F_p\mathcal{M}^{\bullet}) \to \mathscr{H}^i(\mathcal{M}^{\bullet})$, whose image is denoted by $F_p\mathscr{H}^i(\mathcal{M}^{\bullet})$.

A.2.b. Strictness. Strictness is a property which enables one to faithfully pass properties from a filtered object to the associated graded object.

Definition A.2.5 (Strictness in $Mod(\widetilde{\mathscr{A}})$ and $Modgr(\widetilde{\mathscr{A}})$).

(1) An object of $\mathsf{Mod}(\widetilde{\mathscr{A}})$ is said to be *strict* if it has no $\mathbb{C}[z]$ -torsion.

(2) A morphism in $Mod(\mathscr{A})$ is said to be *strict* if its kernel and cokernel are strict (note that the composition of two strict morphisms need not be strict).

(3) A complex \mathscr{M}^{\bullet} of $\mathsf{Mod}(\mathscr{A})$ is said to be *strict* if each of its cohomology modules is a strict object of $\mathsf{Modgr}(\widetilde{\mathscr{A}})$.

An object, resp. morphism, resp. complex in $\mathsf{Modgr}(\widetilde{\mathscr{A}})$ is strict if it is so when considered in $\mathsf{Mod}(\widetilde{\mathscr{A}})$.

Exercise A.7 shows that an object of $\mathsf{Modgr}(\mathscr{A})$ is strict if and only if it comes from a filtered \mathscr{A} -module by the Rees construction.

Exercise A.9. Show the following properties in $Mod(\mathscr{A})$ or in $Modgr(\mathscr{A})$.

(1) A subobject of a strict object is strict.

(2) An extension in of two strict objects is strict.

(3) A morphism between two strict objects is strict if and only if its cokernel is strict.

(4) A complex which consists of strict objects and which is bounded from above is a strict complex if and only if each differential is a strict morphism.

A.2.c. Some basic results on filtrations in abelian categories

Let A be an abelian category. The category WA consisting of objects of A equipped with a finite exhaustive⁽¹⁾ increasing filtration indexed by \mathbb{Z} , and morphisms compatible with filtrations, is an additive category which has kernels and cokernels, but which is not abelian in general. For a filtered object $(H, W_{\bullet}H)$ and for every $k \leq \ell$, the object $(W_{\ell}H, W_{\bullet}H)_{\bullet \leq \ell}$ is a subobject of $(H, W_{\bullet}H)$ (i.e., the kernel of $(W_{\ell}H, W_{\bullet}H)_{\bullet \leq \ell} \rightarrow (H, W_{\bullet}H)$ is zero) and the object $(W_{\ell}H/W_kH, W_{\bullet}H/W_kH)_{k \leq \bullet \leq \ell}$ is a quotient object of $(W_{\ell}H, W_{\bullet}H)_{\bullet \leq \ell}$ (i.e., the cokernel of $(W_{\ell}H, W_{\bullet}H)_{\bullet \leq \ell} \rightarrow$ $(W_{\ell}H/W_kH, W_{\bullet}H/W_kH)_{k \leq \bullet \leq \ell}$ is zero).

Lemma A.2.6. We set $A = Modgr(\widetilde{\mathscr{A}})$.

(1) Let \mathscr{M} be a an object of WA. If each $\operatorname{gr}_k^{W} \mathscr{M}$ is strict, then \mathscr{M} is strict.

(2) Let $\varphi : \mathscr{M}_1 \to \mathscr{M}_2$ be a morphism in WA. If $\operatorname{gr}_k^W \mathscr{M}_1, \operatorname{gr}_k^W \mathscr{M}_2$ are strict for all k, and if φ is strictly compatible with W, i.e., satisfies $\varphi(W_k \mathscr{M}) = W_k \mathscr{N} \cap \varphi(\mathscr{M})$ for all k, then φ is strict.

Proof. The first point is treated in Exercise A.9(2). Let us prove (2). Let $W_{\bullet} \operatorname{Ker} \varphi$ and $W_{\bullet} \operatorname{Coker} \varphi$ be the induced filtrations. By strict compatibility, the sequence

$$0 \longrightarrow \operatorname{gr}_k^W \operatorname{Ker} \varphi \longrightarrow \operatorname{gr}_k^W \mathscr{M} \xrightarrow{\operatorname{gr}_k^W \varphi} \operatorname{gr}_k^W \mathscr{N} \longrightarrow \operatorname{gr}_k^W \operatorname{Coker} \varphi \longrightarrow 0$$

is exact. By strictness of $\operatorname{gr}_k^W \varphi$, and applying (1) to $\operatorname{Ker} \varphi$ and $\operatorname{Coker} \varphi$, one gets that $\operatorname{Ker} \varphi$ and $\operatorname{Coker} \varphi$ are strict, i.e., φ is strict.

Let A_j $(j \in \mathbb{Z})$ be full abelian subcategories which are stable by Ker and Coker in A an such that, for every j > k, $\operatorname{Hom}_A(A_j, A_k) = 0$. We will denote by A_{\bullet} the data $(A, (A_j)_{j \in \mathbb{Z}})$. Let WA_• be the full subcategory of WA consisting of objects such that for every j, $\operatorname{gr}_j^W \in A_j$.

Proposition A.2.7. The category WA_• is abelian, and morphisms are strictly compatible with W_{\bullet} .

^{1.} Exhaustivity means that, for a given object H in A, we have $W_{\ell}H=0$ for $\ell\ll 0$ and $W_{\ell}H=H$ for $\ell\gg 0.$

Proof. It suffices to show the second assertion. Let $\varphi : (H, W_{\bullet}H) \to (H', W_{\bullet}H')$ be a morphism. It is proved by induction on the length of W_{\bullet} . Consider the diagram of exact sequences in A:

(A.2.8)
$$\begin{array}{c} 0 \longrightarrow W_{j-1}H \longrightarrow W_{j}H \longrightarrow \operatorname{gr}_{j}^{W}H \longrightarrow 0 \\ \varphi_{j-1} \downarrow \qquad \qquad \downarrow \varphi_{j} \qquad \qquad \qquad \downarrow \operatorname{gr}_{j}^{W}\varphi \\ 0 \longrightarrow W_{j-1}H' \longrightarrow W_{j}H' \longrightarrow \operatorname{gr}_{j}^{W}H' \longrightarrow 0 \end{array}$$

Due to the inductive assumption, the assertion reduces to proving in A:

$$\operatorname{Im} \varphi_{j-1} = \operatorname{Im} \varphi_j \cap W_{j-1} H',$$

equivalently, $\operatorname{Coker} \varphi_{j-1} \to \operatorname{Coker} \varphi_j$ is a monomorphism. This follows from the assumption on the categories A_j and the snake lemma, which imply that the short sequences of Ker's and that of Coker's are exact.

A.2.d. The filtered ring $(\mathscr{D}_X, F_{\bullet}\mathscr{D}_X)$. Applying these constructions to the filtered ring $(\mathscr{D}_X, F_{\bullet}\mathscr{D}_X)$ and its (left or right) modules, we obtain the following properties:

•
$$\mathscr{O}_X := R_F \mathscr{O}_X = \mathscr{O}_X[z].$$

• in local coordinates, we have

(A.2.9)
$$\widetilde{\mathscr{D}}_X := R_F \mathscr{D}_X = \mathscr{O}_X[z] \langle \eth_{x_1}, \dots, \eth_{x_n} \rangle,$$

i.e., any germ of section of $\widetilde{\mathscr{D}}_X$ may be written in a unique way as

$$\sum_{\alpha} a_{\alpha}(x,z) \eth_{x}^{\alpha} = \sum_{\alpha} \eth_{x}^{\beta} b_{\alpha}(x,z),$$

where $a_{\alpha}, b_{\alpha} \in \widetilde{\mathscr{O}}_X$, and where we set

(A.2.10)
$$\eth_{x_i} := z \eth_{x_i}.$$

• The sheaf $\widetilde{\Theta}_X$ is the locally free graded $\widetilde{\mathscr{O}}_X$ -module locally generated by $\eth_{x_1}, \ldots, \eth_{x_n}$ (having degree -1, due to our convention A.2.2) and we have $[\eth_{x_i}, f] = z\eth f/\eth x_i$ for any local section g of $\widetilde{\mathscr{O}}_X$;

• $\widetilde{\Omega}^1_X$ is the locally free graded $\widetilde{\mathscr{O}}_X$ -module $z^{-1}\mathbb{C}[z] \otimes_{\mathbb{C}} \Omega^1_X$, and $\widetilde{\Omega}^k_X = \wedge^k \widetilde{\Omega}^1_X$; the differential \widetilde{d} is induced by $1 \otimes \widetilde{d}$ on $z^{-k}\mathbb{C}[z] \otimes_{\mathbb{C}} \widetilde{\Omega}^k_X$; we regard the differential as a graded morphism of degree zero

$$\widetilde{\mathbf{d}}: \widetilde{\Omega}^k_X \longrightarrow \widetilde{\Omega}^{k+1}_X;$$

the local basis $(\widetilde{d}x_i = z^{-1}\widetilde{d}x_i)_i$ (having degree 1) is dual to the basis $(\eth_{x_i})_i$ of $\widetilde{\Theta}_X$. • We also set $\widetilde{\mathscr{C}_X^{\infty}} := \mathscr{C}_X^{\infty}[z]$. This is a fine sheaf on the underlying C^{∞} manifold $X_{\mathbb{R}}$,

• We also set $\mathscr{C}_X^{\infty} := \mathscr{C}_X^{\infty}[z]$. This is a fine sheaf on the underlying C^{∞} manifold $X_{\mathbb{R}}$ hence a soft sheaf.

Example A.2.11 (Filtered flat local systems). Let (\mathcal{L}, ∇) be a flat bundle on X and let $F^{\bullet}\mathcal{L}$ be a decreasing filtration of \mathcal{L} by \mathscr{O}_X -locally free sheaves. Then the flat connection ∇ endows \mathscr{L} with the structure of a left \mathscr{D}_X -module. The *Griffiths transversality*

property $\nabla F^p \mathcal{L} \subset \Omega^1_X \otimes F^{p-1} \mathcal{L}$ for every $p \in \mathbb{Z}$ is equivalent to the property that the corresponding increasing filtration $F_{\bullet}\mathcal{L}$ is an $F\mathscr{D}_X$ -filtration of the \mathscr{D}_X -module \mathcal{L} .

Exercise A.10.

(1) Show that $R_F \mathscr{D}_X$ is naturally filtered by locally free graded $\mathscr{O}_X[z]$ -modules of finite rank by setting (locally)

$$F_k(R_F\mathscr{D}_X) = \sum_{|\alpha| \leqslant k} \mathscr{O}_X[z]\eth_x^{\alpha}.$$

- (2) Show that $\operatorname{gr}^F(R_F \mathscr{D}_X) = \mathbb{C}[z] \otimes_{\mathbb{C}} \operatorname{gr}^F \mathscr{D}_X$ with the tensor product grading.
- (3) For a filtered \mathscr{D}_X -module $(\mathcal{M}, F_{\bullet}\mathcal{M})$, show that, if one defines the filtration

$$F_k(R_F\mathcal{M}) = \sum_{j \leqslant k} F_j\mathcal{M} \otimes_{\mathbb{C}} z^j \mathbb{C}[z],$$

then $F_{\bullet}(R_F\mathcal{M})$ is an $F_{\bullet}(R_F\mathcal{D}_X)$ -filtration and $\operatorname{gr}^F(R_F\mathcal{M})$ can be identified with $\mathbb{C}[z] \otimes_{\mathbb{C}} \operatorname{gr}^F\mathcal{M}$, equipped with the tensor product grading.

Definition A.2.12 (Connection). Let \mathscr{M} be a graded $\widetilde{\mathscr{O}}_X$ -module. A connection on \mathscr{M} is a graded $\widetilde{\mathbb{C}}$ -linear morphism $\widetilde{\nabla} : \mathscr{M} \to \widetilde{\Omega}^1_X \otimes \mathscr{M}$ (of degree zero) which satisfies the Leibniz rule

$$\forall f \in \widetilde{\mathscr{O}}_X, \quad \widetilde{\nabla}(fm) = f \, \widetilde{\nabla}m + \widetilde{\mathrm{d}} \, f \otimes m.$$

Exercise A.11.

(1) Show that $\widetilde{\mathscr{D}}_X$ has a universal connection $\widetilde{\nabla}$ for which $\widetilde{\nabla}(1) = \sum_i \widetilde{\widetilde{d}} x_i \otimes \eth_{x_i}$.

(2) Show the equivalence between graded left $\tilde{\mathscr{D}}_X$ -modules and graded $\tilde{\mathscr{O}}_X$ -modules equipped with an integrable connection.

(3) Extend the properties shown in Exercise A.5 to the present case.

Example A.2.13. The fundamental examples of filtered left and right \mathscr{D}_X -modules are:

- $(\mathscr{O}_X, F_{\bullet}\mathscr{O}_X)$ with $\operatorname{gr}_p^F \mathscr{O}_X = 0$ for $p \neq 0$, so $R_F \mathscr{O}_X = \mathscr{O}_X[z]$,
- $(\omega_X, F_{\bullet}\omega_X)$ with $\operatorname{gr}_p^F \omega_X = 0$ for $p \neq -n$, so $R_F \omega_X = \widetilde{\omega}_X = \widetilde{\Omega}_X^n = z^{-n} \omega_X[z]$.

Convention A.2.14. We will use the following convention.

(i) $\widetilde{\mathscr{O}}_X$ (resp. $\widetilde{\mathscr{C}}_X^{\infty}$) denotes either the sheaf rings \mathscr{O}_X (resp. \mathscr{C}_X^{∞}) or the sheaf of graded rings $\mathscr{O}_X[z] = R_F \mathscr{O}_X$ (resp. $\mathscr{C}_X^{\infty}[z]$), and $\mathsf{Mod}(\widetilde{\mathscr{O}}_X)$ denotes the category of \mathscr{O}_X -modules or that of graded $\mathscr{O}_X[z]$ -modules.

(ii) The notation Θ_X , Ω_X^k , $\wedge^k \Theta_X$ has a similar double meaning.

(iii) Similarly, $\widetilde{\mathscr{D}}_X$ denotes either the sheaf rings \mathscr{D}_X or the sheaf of graded rings $R_F \mathscr{D}_X$, and $\mathsf{Mod}(\widetilde{\mathscr{D}}_X)$ denotes the category of \mathscr{D}_X -modules or that of graded $R_F \mathscr{D}_X$ -modules.

(iv) It will also be convenient to denote by $\widetilde{\mathbb{C}}$ either the field \mathbb{C} or the graded ring $\mathbb{C}[z]$.

(v) In each of the second cases above, we will usually omit the word "graded", although it is always understood.

(vi) One recovers standard results for \mathscr{D}_X -modules by setting z = 1 and $\eth = \partial$.

(vii) The strictness condition that we may consider only refers to the second cases above, it is empty in the first cases.

A.3. Left and right

Considering left or right $\widetilde{\mathscr{D}}_X$ -modules is not completely symmetric. The main reason is that the left \mathscr{D}_X -module \mathscr{O}_X is a sheaf of rings, while its right analogue $\widetilde{\omega}_X := \widetilde{\Omega}_X^n$, is not a sheaf of rings. So for example the behaviour with respect to tensor products over $\widetilde{\mathscr{O}}_X$ is not the same for left and right $\widetilde{\mathscr{D}}$ -modules. Also, the side changing functor defined below sends $\widetilde{\mathscr{D}}_X^{\text{left}}$ to $\widetilde{\omega}_X \otimes_{\widetilde{\mathscr{O}}_X} \widetilde{\mathscr{D}}_X$, and not to $\widetilde{\mathscr{D}}_X$ regarded as a right \mathscr{D}_X -module over itself.

The categories of left (resp. right) $\widetilde{\mathscr{D}}_X$ -modules are denoted by $\mathsf{Mod}^{\mathrm{left}}(\widetilde{\mathscr{D}}_X)$ (resp. $\mathsf{Mod}^{\mathsf{right}}(\widetilde{\mathscr{D}}_X)$ (recall that we consider graded modules and morphisms of degree zero in the case of $\hat{\mathscr{D}} = R_F \mathscr{D}$. We analyze the relations between both categories in this section. Let us first recall the basic lemmas for generating left or right \mathscr{D} -modules.

Exercise A.12 (Generating left $\widetilde{\mathscr{D}}_X$ -modules). Let $\mathscr{M}^{\text{left}}$ be an $\widetilde{\mathscr{O}}_X$ -module and let φ^{left} : $\widetilde{\Theta}_X \otimes_{\widetilde{\mathbb{C}}_Y} \mathscr{M}^{\text{left}} \to \mathscr{M}^{\text{left}}$ be a $\widetilde{\mathbb{C}}$ -linear morphism such that, for any local sections g of $\widetilde{\mathcal{O}}_X, \xi, \eta$ of $\widetilde{\Theta}_X$ and m of $\mathscr{M}^{\text{left}}$, one has

- (1) $\varphi^{\text{left}}(f\xi \otimes m) = f\varphi^{\text{left}}(\xi \otimes m),$
- (2) $\varphi^{\text{left}}(\xi \otimes fm) = f\varphi^{\text{left}}(\xi \otimes m) + \xi(g)m,$ (3) $\varphi^{\text{left}}([\xi,\eta] \otimes m) = \varphi^{\text{left}}(\xi \otimes \varphi^{\text{left}}(\eta \otimes m)) \varphi^{\text{left}}(\eta \otimes \varphi^{\text{left}}(\xi \otimes m)).$

Show that there exists a unique structure of left $\widetilde{\mathscr{D}}_X$ -module on $\mathscr{M}^{\text{left}}$ such that $\xi m = \varphi^{\text{left}}(\xi \otimes m)$ for every ξ, m .

Exercise A.13 (Generating right $\widetilde{\mathscr{D}}_X$ -modules). Let $\mathscr{M}^{\mathrm{right}}$ be an $\widetilde{\mathscr{O}}_X$ -module and let $\varphi^{\text{right}} : \mathscr{M}^{\text{right}} \otimes_{\widetilde{\mathbb{C}}_{Y}} \widetilde{\Theta}_{X} \to \mathscr{M}^{\text{right}}$ be a $\widetilde{\mathbb{C}}$ -linear morphism such that, for any local sections g of $\widetilde{\mathcal{O}}_X, \xi, \eta$ of $\widetilde{\Theta}_X$ and m of $\mathscr{M}^{\mathrm{right}}$, one has

- (1) $\varphi^{\text{right}}(mf \otimes \xi) = \varphi^{\text{right}}(m \otimes f\xi) \ (\varphi^{\text{right}} \text{ is in fact defined on } \mathscr{M}^{\text{right}} \otimes_{\widetilde{\Theta}_{Y}} \widetilde{\Theta}_{X}),$ (2) $\varphi^{\text{right}}(m \otimes f\xi) = \varphi^{\text{right}}(m \otimes \xi)f - m\xi(g),$ (3) $\varphi^{\text{right}}(m \otimes [\xi, \eta]) = \varphi^{\text{right}}(\varphi^{\text{right}}(m \otimes \xi) \otimes \eta) - \varphi^{\text{right}}(\varphi^{\text{right}}(m \otimes \eta) \otimes \xi).$

Show that there exists a unique structure of right $\widetilde{\mathscr{D}}_X$ -module on $\mathscr{M}^{\mathrm{right}}$ such that $m\xi = \varphi^{\text{right}}(m \otimes \xi)$ for every ξ, m .

Example A.3.1 (Most basic examples).

- (1) $\widetilde{\mathscr{D}}_X$ is a left and a right $\widetilde{\mathscr{D}}_X$ -module.
- (2) $\widetilde{\mathscr{O}}_X$ is a left $\widetilde{\mathscr{D}}_X$ -module (Exercise A.14), with grading

$$\widetilde{\mathscr{O}}_{X,p} = \begin{cases} \mathscr{O}_X & \text{if } p \ge 0, \\ 0 & \text{if } p < 0. \end{cases}$$

(3) $\widetilde{\omega}_X := \widetilde{\Omega}_X^{\dim X}$ is a right $\widetilde{\mathscr{D}}_X$ -module (Exercise A.15), with grading

$$\widetilde{\omega}_{X,p} = \begin{cases} \omega_X & \text{if } p \ge -n, \\ 0 & \text{if } p < -n. \end{cases}$$

Exercise A.14 (\mathcal{O}_X is a *simple* left \mathcal{D}_X -module). We consider here the setting of Section A.1.

(1) Use the left action of Θ_X on \mathscr{O}_X to define on \mathscr{O}_X the structure of a left \mathscr{D}_X -module.

(2) Let g be a nonzero holomorphic function on \mathbb{C}^n . Show that there exists a multi-index $\alpha \in \mathbb{N}^n$ such that $(\partial^{\alpha} f)(0) \neq 0$.

(3) Conclude that \mathscr{O}_X is a simple left \mathscr{D}_X -module, i.e., does not contain any proper non trivial \mathscr{D}_X -submodule. Is it simple as a left \mathscr{O}_X -module?

(4) Show that $R_F \mathscr{O}_X$ is not a simple graded $R_F \mathscr{D}_X$ -module. [*Hint*: consider $zR_F \mathscr{O}_X \subset R_F \mathscr{O}_X$.]

Exercise A.15 (ω_X is a *simple* right \mathscr{D}_X -module). Same setting as in Exercise A.14.

(1) Use the right action of Θ_X on ω_X to define on ω_X the structure of a right \mathscr{D}_X -module.

(2) Show that it is simple as a right \mathscr{D}_X -module.

(3) Show that $R_F \omega_X$ is not a simple graded right $R_F \mathscr{D}_X$ -module.

Exercise A.16 (Tensor products over $\widetilde{\mathcal{O}}_X$).

(1) Let $\mathscr{M}^{\text{left}}$ and $\mathscr{N}^{\text{left}}$ be two left $\widetilde{\mathscr{D}}_X$ -modules.

(a) Show that the $\widetilde{\mathscr{O}}_X$ -module $\mathscr{M}^{\text{left}} \otimes_{\widetilde{\mathscr{O}}_X} \mathscr{N}^{\text{left}}$ has the structure of a *left* $\widetilde{\mathscr{D}}_X$ -module when setting, by analogy with the Leibniz rule,

$$\xi \cdot (m \otimes n) = \xi m \otimes n + m \otimes \xi n.$$

(b) If $\mathscr{M}^{\text{left}}$ and $\mathscr{N}^{\text{left}}$ are regarded as $\widetilde{\mathscr{O}}_X$ -modules with connection (Proposition A.1.3 and Exercise A.11), show that the connection on $\mathscr{M}^{\text{left}} \otimes_{\widetilde{\mathscr{O}}_X} \mathscr{N}^{\text{left}}$ coming from the left $\widetilde{\mathscr{D}}_X$ -module structure above is equal to $\widetilde{\nabla} \otimes \text{Id}_{\mathscr{N}} + \text{Id}_{\mathscr{M}} \otimes \widetilde{\nabla}$.

(c) Notice that, in general, $m \otimes n \mapsto (\xi m) \otimes n$ (or $m \otimes n \mapsto m \otimes (\xi n)$) does not define a left $\widetilde{\mathscr{D}}_X$ -action on the $\widetilde{\mathscr{O}}_X$ -module $\mathscr{M} \otimes_{\widetilde{\mathscr{O}}_X} \mathscr{N}$.

(d) Let $\varphi : \mathscr{M} \to \mathscr{M}'$ and $\psi : \mathscr{N} \to \mathscr{N}'$ be $\widetilde{\mathscr{D}}_X$ -linear morphisms. Show that $\varphi \otimes \psi$ is $\widetilde{\mathscr{D}}_X$ -linear.

(e) Show the associativity

$$(\mathscr{M}^{\mathrm{left}} \otimes_{\widetilde{\mathscr{O}}_X} \mathscr{N}^{\mathrm{left}}) \otimes_{\widetilde{\mathscr{O}}_X} \mathscr{P}^{\mathrm{left}} = \mathscr{M}^{\mathrm{left}} \otimes_{\widetilde{\mathscr{O}}_X} (\mathscr{N}^{\mathrm{left}} \otimes_{\widetilde{\mathscr{O}}_X} \mathscr{P}^{\mathrm{left}})$$

(2) Let $\mathscr{M}^{\text{left}}$ be a left $\widetilde{\mathscr{D}}_X$ -module and $\mathscr{N}^{\text{right}}$ be a right $\widetilde{\mathscr{D}}_X$ -module.

(a) Show that $\mathscr{N}^{\mathrm{right}} \otimes_{\widetilde{\mathscr{O}}_X} \mathscr{M}^{\mathrm{left}}$ has the structure of a right $\widetilde{\mathscr{D}}_X$ -module by setting

$$(n \otimes m) \cdot \xi = n\xi \otimes m - n \otimes \xi m$$

Remark: one can define a right $\widetilde{\mathscr{D}}_X$ -module structure on $\mathscr{M}^{\text{left}} \otimes_{\widetilde{\mathscr{O}}_X} \mathscr{N}^{\text{right}}$ by using the natural involution $\mathscr{M}^{\text{left}} \otimes_{\widetilde{\mathscr{O}}_X} \mathscr{N}^{\text{right}} \xrightarrow{\sim} \mathscr{N}^{\text{right}} \otimes_{\widetilde{\mathscr{O}}_X} \mathscr{M}^{\text{left}}$, so this brings no new structure.

(b) Show the associativity

$$(\mathscr{N}^{\mathrm{right}} \otimes_{\widetilde{\mathscr{O}}_X} \mathscr{M}^{\mathrm{left}}) \otimes_{\widetilde{\mathscr{O}}_X} \mathscr{P}^{\mathrm{left}} = \mathscr{N}^{\mathrm{right}} \otimes_{\widetilde{\mathscr{O}}_X} (\mathscr{M}^{\mathrm{left}} \otimes_{\widetilde{\mathscr{O}}_X} \mathscr{P}^{\mathrm{left}})$$

(3) Assume that $\mathscr{M}^{\text{right}}$ and $\mathscr{N}^{\text{right}}$ are right $\widetilde{\mathscr{D}}_X$ -modules. Does there exist a (left or right) $\widetilde{\mathscr{D}}_X$ -module structure on $\mathscr{M}^{\mathrm{right}} \otimes_{\widetilde{\mathscr{O}}_X} \mathscr{N}^{\mathrm{right}}$ defined with analogous formulas?

Exercise A.17 ($\mathscr{H}om$ over $\widetilde{\mathcal{O}}_X$).

(1) Let \mathscr{M}, \mathscr{N} be left $\widetilde{\mathscr{D}}_X$ -modules. Show that $\mathscr{H}om_{\widetilde{\mathscr{O}}_X}(\mathscr{M}, \mathscr{N})$ has a natural structure of left $\widetilde{\mathscr{D}}_X$ -module defined by

$$(\xi \cdot \varphi)(m) = \xi \cdot (\varphi(m)) + \varphi(\xi \cdot m),$$

for any local sections ξ of $\widetilde{\Theta}_X$, m of \mathscr{M} and φ of $\mathscr{H}om_{\widetilde{\mathscr{O}}_X}(\mathscr{M}, \mathscr{N})$.

(2) Similarly, if \mathscr{M}, \mathscr{N} are right $\widetilde{\mathscr{D}}_X$ -modules, then $\mathscr{H}om_{\widetilde{\mathscr{O}}_X}(\mathscr{M}, \mathscr{N})$ has a natural structure of left $\widetilde{\mathscr{D}}_X$ -module defined by

$$(\xi \cdot \varphi)(m) = \varphi(m \cdot \xi) - \varphi(m) \cdot \xi.$$

Exercise A.18 (Tensor product of a left $\widetilde{\mathscr{D}}_X$ -module with $\widetilde{\mathscr{D}}_X$) Let $\mathscr{M}^{\text{left}}$ be a left $\widetilde{\mathscr{D}}_X$ -module. Notice that $\mathscr{M}^{\text{left}} \otimes_{\widetilde{\mathscr{O}}_X} \widetilde{\mathscr{D}}_X$ has two commuting structures of $\widetilde{\mathscr{O}}_X$ -module. Similarly $\widetilde{\mathscr{D}}_X \otimes_{\widetilde{\mathscr{O}}_X} \mathscr{M}^{\text{left}}$ has two such structures. The goal of this exercise is to extend them as $\widetilde{\mathscr{D}}_X$ -structures and examine their relations.

(1) Show that $\mathscr{M}^{\text{left}} \otimes_{\widetilde{\mathscr{O}}_X} \widetilde{\mathscr{D}}_X$ has the structure of a left and of a right $\widetilde{\mathscr{D}}_X$ -module which commute, given by the formulas:

(left)
$$\begin{cases} f \cdot (m \otimes P) = (fm) \otimes P = m \otimes (fP), \\ \xi \cdot (m \otimes P) = (\xi m) \otimes P + m \otimes \xi P, \end{cases}$$

(right)
$$\begin{cases} (m \otimes P) \cdot f = m \otimes (Pf), \\ (m \otimes P) \cdot \xi = m \otimes (P\xi), \end{cases}$$

for any local vector field ξ and any local holomorphic function g. Show that a left $\widetilde{\mathscr{D}}_X$ -linear morphism $\varphi: \mathscr{M}_1^{\text{left}} \to \mathscr{M}_2^{\text{left}}$ extends as a bi- $\widetilde{\mathscr{D}}_X$ -linear morphism $\varphi \otimes 1$: $\mathscr{M}_{1}^{\mathrm{left}} \otimes_{\widetilde{\mathscr{O}}_{X}} \widetilde{\mathscr{D}}_{X} \to \mathscr{M}_{2}^{\mathrm{left}} \otimes_{\widetilde{\mathscr{O}}_{X}} \widetilde{\mathscr{D}}_{X}.$

(2) Similarly, show that $\widetilde{\mathscr{D}}_X \otimes_{\widetilde{\mathscr{O}}_X} \mathscr{M}^{\text{left}}$ also has such structures which commute and are functorial, given by formulas:

(left)
$$\begin{cases} f \cdot (P \otimes m) = (fP) \otimes m, \\ \xi \cdot (P \otimes m) = (\xi P) \otimes m, \end{cases}$$

(right)
$$\begin{cases} (P \otimes m) \cdot f = P \otimes (fm) = (Pf) \otimes m, \\ (P \otimes m) \cdot \xi = P\xi \otimes m - P \otimes \xi m. \end{cases}$$

(3) Show that both morphisms

$$\begin{split} \mathscr{M}^{\mathrm{left}} \otimes_{\widetilde{\mathscr{O}}_X} \widetilde{\mathscr{D}}_X &\longrightarrow \widetilde{\mathscr{D}}_X \otimes_{\widetilde{\mathscr{O}}_X} \mathscr{M}^{\mathrm{left}} & \widetilde{\mathscr{D}}_X \otimes_{\widetilde{\mathscr{O}}_X} \mathscr{M}^{\mathrm{left}} &\longrightarrow \mathscr{M}^{\mathrm{left}} \otimes_{\widetilde{\mathscr{O}}_X} \widetilde{\mathscr{D}}_X \\ & m \otimes P \longmapsto (1 \otimes m) \cdot P & P \otimes m \longmapsto P \cdot (m \otimes 1) \end{split}$$

are left and right $\widetilde{\mathscr{D}}_X$ -linear, induce the identity $\mathscr{M}^{\text{left}} \otimes 1 = 1 \otimes \mathscr{M}^{\text{left}}$, and their composition is the identity of $\mathscr{M}^{\text{left}} \otimes_{\widetilde{\mathscr{O}}_X} \widetilde{\mathscr{D}}_X$ or $\widetilde{\mathscr{D}}_X \otimes_{\widetilde{\mathscr{O}}_X} \mathscr{M}^{\text{left}}$, hence both are reciprocal isomorphisms. Show that this correspondence is functorial (i.e., compatible with morphisms).

(4) Let \mathscr{M} be a left $\widehat{\mathscr{D}}_X$ -module and let \mathscr{L} be an $\widehat{\mathscr{O}}_X$ -module. Justify the following isomorphisms of left $\widehat{\mathscr{D}}_X$ -modules and $\widehat{\mathscr{O}}_X$ -modules for the action on the right:

$$\begin{split} \mathscr{M} \otimes_{\widetilde{\mathscr{O}}_{X}} (\widetilde{\mathscr{D}}_{X} \otimes_{\widetilde{\mathscr{O}}_{X}} \mathscr{L}) &\simeq (\mathscr{M} \otimes_{\widetilde{\mathscr{O}}_{X}} \widetilde{\mathscr{D}}_{X}) \otimes_{\widetilde{\mathscr{O}}_{X}} \mathscr{L} \\ &\simeq (\widetilde{\mathscr{D}}_{X} \otimes_{\widetilde{\mathscr{O}}_{X}} \mathscr{M}) \otimes_{\widetilde{\mathscr{O}}_{X}} \mathscr{L} \simeq \widetilde{\mathscr{D}}_{X} \otimes_{\widetilde{\mathscr{O}}_{X}} (\mathscr{M} \otimes_{\widetilde{\mathscr{O}}_{X}} \mathscr{L}). \end{split}$$

Assume moreover that \mathscr{M} and \mathscr{L} are $\widetilde{\mathscr{O}}_X$ -locally free. Show that $\mathscr{M} \otimes_{\widetilde{\mathscr{O}}_X} (\widetilde{\mathscr{D}}_X \otimes_{\widetilde{\mathscr{O}}_X} \mathscr{L})$ is $\widetilde{\mathscr{D}}_X$ -locally free.

Exercise A.19 (Tensor product of a right $\widetilde{\mathscr{D}}_X$ -module with $\widetilde{\mathscr{D}}_X$)

Let $\mathscr{M}^{\mathrm{right}}$ be a right $\widetilde{\mathscr{D}}_X$ -module.

(1) Show that $\mathscr{M}^{\operatorname{right}} \otimes_{\widetilde{\mathscr{O}}_X} \widetilde{\mathscr{D}}_X$ has two structures of right $\widetilde{\mathscr{D}}_X$ -module denoted triv and tens (tensor; the latter defined by using the left structure on $\widetilde{\mathscr{D}}_X$ and Exercise A.16(2)), given by:

$$(\text{right})_{\text{triv}} \qquad \begin{cases} (m \otimes P) \cdot_{\text{triv}} f = m \otimes (Pf), \\ (m \otimes P) \cdot_{\text{triv}} \xi = m \otimes (P\xi), \end{cases}$$

$$(\text{right})_{\text{tens}} \qquad \begin{cases} (m \otimes P) \cdot_{\text{tens}} f = mf \otimes P = m \otimes fP, \\ (m \otimes P) \cdot_{\text{tens}} \xi = m\xi \otimes P - m \otimes (\xi P). \end{cases}$$

(2) Show that there is a unique involution $\iota : \mathscr{M}^{\mathrm{right}} \otimes_{\widetilde{\mathscr{O}}_X} \xrightarrow{\sim} \mathscr{M}^{\mathrm{right}} \otimes_{\widetilde{\mathscr{O}}_X} \widetilde{\mathscr{D}}_X$ which exchanges both structures and is the identity on $\mathscr{M}^{\mathrm{right}} \otimes 1$, given by $(m \otimes P)_{\mathrm{triv}} \mapsto (m \otimes 1) \cdot_{\mathrm{tens}} P$. [Hint: show first the properties of ι by using local coordinates, and glue the local constructions by uniqueness of ι .]

(3) Let \mathscr{L} be an $\widetilde{\mathscr{O}}_X$ -module and equip $\overline{\mathscr{D}}_X \otimes_{\widetilde{\mathscr{O}}_X} \mathscr{L}$ with its natural structure of left $\widetilde{\mathscr{D}}_X$ -module and that of $\widetilde{\mathscr{O}}_X$ -module where $\widetilde{\mathscr{O}}_X$ acts on \mathscr{L} . Extend the previous involution as an involution of $\widetilde{\mathscr{O}}_X$ -modules, where the $\widetilde{\mathscr{O}}_X$ -action is on \mathscr{L} :

$$\iota:\mathscr{M}^{\mathrm{right}}\otimes_{\widetilde{\mathscr{O}}_{X}}(\widetilde{\mathscr{D}}_{X}\otimes_{\widetilde{\mathscr{O}}_{X}}\mathscr{L})\overset{\sim}{\longrightarrow}(\mathscr{M}^{\mathrm{right}}\otimes_{\widetilde{\mathscr{O}}_{X}}\widetilde{\mathscr{D}}_{X})_{\mathrm{triv}}\otimes_{\widetilde{\mathscr{O}}_{X}}\mathscr{L}.$$

(4) For every $p \ge 0$, consider the *p*th term $F_p \widetilde{\mathscr{D}}_X$ of the filtration of $\widetilde{\mathscr{D}}_X$ by the order (see Exercise A.1.2) with both structures of $\widetilde{\mathscr{O}}_X$ -module (one on the left, one on the right) and equip similarly $\mathscr{M}^{\mathrm{right}} \otimes_{\widetilde{\mathscr{O}}_X} F_p \widetilde{\mathscr{D}}_X$ with two structures tens and triv of $\widetilde{\mathscr{O}}_X$ -modules. Show that, for every p, ι induces an isomorphism of $\widetilde{\mathscr{O}}_X$ -modules $(\mathscr{M}^{\mathrm{right}} \otimes_{\widetilde{\mathscr{O}}_X} F_p \widetilde{\mathscr{D}}_X)_{\mathrm{triv}}$.

Definition A.3.2 (Side-changing of $\widetilde{\mathscr{D}}_X$ -modules). Any left $\widetilde{\mathscr{D}}_X$ -module $\mathscr{M}^{\text{left}}$ gives rise to a right one $\mathscr{M}^{\text{right}}$ by setting $\mathscr{M}^{\text{right}} = \widetilde{\omega}_X \otimes_{\widetilde{\mathscr{O}}_X} \mathscr{M}^{\text{left}}$ and, for any vector field ξ and any function g,

 $(\omega \otimes m) \cdot f = f \omega \otimes m = \omega \otimes f m, \quad (\omega \otimes m) \cdot \xi = \omega \xi \otimes m - \omega \otimes \xi m.$

Conversely, set $\mathscr{M}^{\text{left}} = \mathscr{H}om_{\widetilde{\mathscr{O}}_X}(\widetilde{\omega}_X, \mathscr{M}^{\text{right}})$, which also has in a natural way the structure of a left $\widetilde{\mathscr{D}}_X$ -module (see Exercise A.17(2)). The grading behaves as follows:

$$\mathscr{M}^{\mathrm{right}} = z^{-n} \omega_X \otimes_{\mathscr{O}_X} \mathscr{M}^{\mathrm{left}}(n).$$

Exercise A.20 (Compatibility of side-changing functors). Show that the natural morphisms

$$\mathscr{M}^{\mathrm{left}} \longrightarrow \mathscr{H}\!\mathit{om}_{\widetilde{\mathscr{O}}_{X}}(\widetilde{\omega}_{X}, \widetilde{\omega}_{X} \otimes_{\widetilde{\mathscr{O}}_{X}} \mathscr{M}^{\mathrm{left}}), \qquad \widetilde{\omega}_{X} \otimes_{\widetilde{\mathscr{O}}_{X}} \mathscr{H}\!\mathit{om}_{\widetilde{\mathscr{O}}_{X}}(\widetilde{\omega}_{X}, \mathscr{M}^{\mathrm{right}}) \longrightarrow \mathscr{M}^{\mathrm{right}}$$

are isomorphisms of graded \mathcal{D}_X -modules.

Caveat A.3.3. Let $\widetilde{\omega}_X^{\vee} = \mathscr{H}om_{\widetilde{\mathcal{O}}_X}(\widetilde{\omega}_X, \widetilde{\mathcal{O}}_X)$ as an $\widetilde{\mathcal{O}}_X$ -module. One often finds in the literature the formula $\mathscr{M}^{\text{left}} = \widetilde{\omega}_X^{\vee} \otimes_{\widetilde{\mathcal{O}}_X} \mathscr{M}^{\text{right}}$, which give the $\widetilde{\mathcal{O}}_X$ -module structure of $\mathscr{M}^{\text{left}}$. However, the left $\widetilde{\mathscr{D}}_X$ -module structure is not obtained with a "tensor product formula" as in Exercise A.16, but uses the interpretation as $\mathscr{H}om_{\widetilde{\mathcal{O}}_X}(\widetilde{\omega}_X, \mathscr{M}^{\text{right}})$.

Exercise A.21 (Side-changing on morphisms). To any left $\widetilde{\mathscr{D}}_X$ -linear morphism φ^{left} : $\mathscr{M}_1^{\text{left}} \to \mathscr{M}_2^{\text{left}}$ is associated the $\widetilde{\mathscr{O}}_X$ -linear morphism $\varphi^{\text{right}} = \text{Id}_{\widetilde{\omega}_X} \otimes \varphi^{\text{left}} : \mathscr{M}_1^{\text{right}} \to \mathscr{M}_2^{\text{right}}$.

- (1) Show that φ^{right} is right $\widehat{\mathscr{D}}_X$ -linear.
- (2) Define the reverse correspondence $\varphi^{\text{right}} \mapsto \varphi^{\text{left}}$.

(3) Conclude that the left-right correspondence $\mathsf{Mod}^{\mathsf{left}}(\widetilde{\mathscr{D}}_X) \mapsto \mathsf{Mod}^{\mathsf{right}}(\widetilde{\mathscr{D}}_X)$ is a functor, as well as the right-left correspondence $\mathsf{Mod}^{\mathsf{right}}(\widetilde{\mathscr{D}}_X) \mapsto \mathsf{Mod}^{\mathsf{left}}(\widetilde{\mathscr{D}}_X)$.

The following is now obvious.

Proposition A.3.4. The side-changing functors left-to-right and right-to-left are isomorphisms of between the categories of left and right graded $\widetilde{\mathscr{D}}_X$ -modules, which are inverse one to the other.

Remark A.3.5. The ring $\widetilde{\mathscr{D}}_X$ considered as a right $\widetilde{\mathscr{D}}_X$ -module over itself is not equal to the right $\widetilde{\mathscr{D}}_X$ -module associated with $\widetilde{\mathscr{D}}_X$ regarded as a left $\widetilde{\mathscr{D}}_X$ -module over itself by the side-changing functor.

Exercise A.22 (Compatibility of side-changing functors with tensor product)

Let $\mathscr{M}^{\text{left}}$ and $\mathscr{N}^{\text{left}}$ be two left $\widetilde{\mathscr{D}}_X$ -modules and denote by $\mathscr{M}^{\text{right}}, \mathscr{N}^{\text{right}}$ the corresponding right $\widetilde{\mathscr{D}}_X$ -modules (see Definition A.3.2). Show that there is a natural isomorphism of graded right $\widetilde{\mathscr{D}}_X$ -modules (by using the right structure given in

Exercise A.16(2):

$$\begin{split} \mathscr{N}^{\mathrm{right}} \otimes_{\widetilde{\mathscr{O}}_{X}} \mathscr{M}^{\mathrm{left}} & \xrightarrow{\sim} \mathscr{M}^{\mathrm{right}} \otimes_{\widetilde{\mathscr{O}}_{X}} \mathscr{N}^{\mathrm{left}} \\ (\omega \otimes n) \otimes m \longmapsto (\omega \otimes m) \otimes n \end{split}$$

and that this isomorphism is functorial in $\mathscr{M}^{\text{left}}$ and $\mathscr{N}^{\text{left}}$.

Exercise A.23 (Local expression of the side-changing functors)

Let U be an open set of \mathbb{C}^n .

(1) Show that there exists a unique $\widetilde{\mathbb{C}}$ -linear involution $P \mapsto {}^tP$ from $\widetilde{\mathscr{D}}(U)$ to itself such that

•
$$\forall f \in \widetilde{\mathcal{O}}(U), \, {}^tf = f,$$

- $\forall i \in \{1, \dots, n\}, \ {}^t \eth_{x_i} = -\eth_{x_i},$
- $\forall P, Q \in \widetilde{\mathscr{D}}(U), t(PQ) = {}^{t}Q \cdot {}^{t}P.$

(2) Let \mathscr{M} be a left $\widetilde{\mathscr{D}}_X$ -module and let ${}^t\mathscr{M}$ be \mathscr{M} equipped with the right $\widetilde{\mathscr{D}}_X$ -module structure

$$m \cdot P := {}^{t}Pm.$$

Show that $z^{-n} \overset{t}{\mathscr{M}} \xrightarrow{\sim} \mathscr{M}^{\operatorname{right}}$, that is, $\overset{t}{\mathscr{M}}(n) \xrightarrow{\sim} \mathscr{M}^{\operatorname{right}}$. [*Hint*: use that $F_p \overset{t}{\mathscr{O}}_X = F_{p-n}\omega_X$, hence $R_F \overset{t}{\mathscr{O}}_X = R_{F[n]}\omega_X$, so $\overset{t}{\widetilde{\mathscr{O}}}_X = \widetilde{\omega}_X(-n)$, according to Remark A.2.4(2).] Argue similarly starting with a right $\widetilde{\mathscr{D}}_X$ -module.

A.4. Examples of $\widehat{\mathscr{D}}$ -modules

We list here some classical examples of $\widehat{\mathscr{D}}$ -modules. One can get many other examples by applying various operations on $\widehat{\mathscr{D}}$ -modules.

A.4.1. Let \mathscr{L} be an $\widetilde{\mathscr{O}}_X$ -module. There is a very simple way to get a right $\widetilde{\mathscr{D}}_X$ -module from \mathscr{L} : consider $\mathscr{L} \otimes_{\widetilde{\mathscr{O}}_X} \widetilde{\mathscr{D}}_X$ equipped with the natural right action of $\widetilde{\mathscr{D}}_X$. This is called an *induced* $\widetilde{\mathscr{D}}_X$ -module. Although this construction is very simple, it is also very useful to get cohomological properties of $\widetilde{\mathscr{D}}_X$ -modules. One can also consider the left $\widetilde{\mathscr{D}}_X$ -module $\widetilde{\mathscr{D}}_X \otimes_{\widetilde{\mathscr{O}}_X} \mathscr{L}$ (however, this is not the left $\widetilde{\mathscr{D}}_X$ -module attached to the right one $\mathscr{L} \otimes_{\widetilde{\mathscr{O}}_X} \widetilde{\mathscr{D}}_X$ by the left-right transformation of Definition A.3.2).

A.4.2. One of the main geometrical examples of \mathscr{D}_X -modules are the vector bundles on X equipped with an *integrable* connection. Recall that left \mathscr{D}_X -modules are \mathscr{O}_X modules with an integrable connection. Among them, the coherent \mathscr{D}_X -modules are of particular interest. One can show that such modules are \mathscr{O}_X -locally free, i.e., correspond to holomorphic vector bundles of finite rank on X.

It may happen that, for some X, such a category does not give any interesting geometric object. Indeed, if for instance X has a trivial fundamental group (e.g. $X = \mathbb{P}^1(\mathbb{C})$), then any vector bundle with integrable connection is isomorphic to the trivial bundle \mathscr{O}_X with the connection d. However, on non simply connected Zariski open sets of X, there exist interesting vector bundles with connections. This leads to the notion of meromorphic vector bundle with connection.

Let D be a divisor in X and denote by $\mathscr{O}_X(*D)$ the sheaf of meromorphic functions on X with poles along D at most. This is a sheaf of left \mathscr{D}_X -modules, being an \mathscr{O}_X -module equipped with the natural connection $d: \mathscr{O}_X(*D) \to \Omega^1_X(*D)$.

By definition, a *meromorphic bundle* is a locally free $\mathscr{O}_X(*D)$ module of finite rank. When it is equipped with an integrable connection, it becomes a left \mathscr{D}_X -module.

A.4.3. One can *twist* the previous examples. Assume that ω is a *closed* holomorphic form on X. Define $\nabla : \mathscr{O}_X \to \Omega^1_X$ by the formula $\nabla = d + \omega$. As ω is closed, ∇ is an integrable connection on the trivial bundle \mathscr{O}_X .

Usually, the nonzero closed form on X are meromorphic, with poles on some divisor D. Then ∇ is an integrable connection on $\mathscr{O}_X(*D)$.

If ω is exact, $\omega = df$ for some meromorphic function g on X, then ∇ can be written as $e^{-f} \circ d \circ e^{f}$.

More generally, if \mathcal{M} is any meromorphic bundle with an integrable connection ∇ , then, for any such ω , $\nabla + \omega$ Id defines a new \mathscr{D}_X -module structure on \mathcal{M} .

A.4.4 Denote by \mathfrak{Db}_X the sheaf of distributions on the complex manifold X of dimension n: given any open set U of X, $\mathfrak{Db}_X(U)$ is the space of distributions on U, which is by definition the weak dual of the space of C^{∞} forms with compact support on U, of type (n, n). By Exercise A.15, there is a right action of \mathscr{D}_X on such forms. The left action of \mathscr{D}_X on distributions is defined by adjunction: denote by $\langle \eta, u \rangle$ the natural pairing between a compactly supported C^{∞} -form η and a distribution u on U; let P be a holomorphic differential operator on U; define then $P \cdot u$ such that, for every η , on has

$$\langle \eta, P \cdot u \rangle = \langle \eta \cdot P, u \rangle.$$

Given any distribution u on X, the subsheaf $\mathscr{D}_X \cdot u \subset \mathfrak{Db}_X$ is the \mathscr{D}_X -module generated by this distribution. Saying that a distribution is a solution of a family P_1, \ldots, P_k of differential equation is equivalent to saying that the morphism $\mathscr{D}_X \to \mathscr{D}_X \cdot u$ sending 1 to u induces a surjective morphism $\mathscr{D}_X/(P_1, \ldots, P_k) \to \mathscr{D}_X \cdot u$.

Similarly, the sheaf \mathfrak{C}_X of currents of degree 0 on X is defined in such a way that, for any open set $U \subset X$, $\mathfrak{C}_X(U)$ is dual to $C_c^{\infty}(U)$ with a suitable topology. It is a right \mathscr{D}_X -module.

In local coordinates x_1, \ldots, x_n , a current of degree 0 is nothing but a distribution times the volume form $dx_1 \wedge \cdots \wedge dx_n \wedge d\overline{x}_1 \wedge \cdots \wedge d\overline{x}_n$.

As we are now working with C^{∞} forms or with currents, it is natural not to forget the anti-holomorphic part of these objects. Denote by $\mathscr{O}_{\overline{X}}$ the sheaf of antiholomorphic functions on X and by $\mathscr{D}_{\overline{X}}$ the sheaf of anti-holomorphic differential operators. Then \mathfrak{Db}_X (resp. \mathfrak{C}_X) are similarly left (resp. right) $\mathscr{D}_{\overline{X}}$ -modules. Of course, the \mathscr{D}_X and $\mathscr{D}_{\overline{X}}$ actions do commute, and they coincide when considering multiplication by constants. The conjugation exchanges both structures. For example, if u is a distribution on U, its conjugate \overline{u} is defined by the formula

$$(A.4.1) \qquad \qquad \langle \eta, \overline{u} \rangle := \overline{\langle \overline{\eta}, u \rangle} \qquad (\eta \in \mathscr{E}^{n,n}_c(U)).$$

This is of course compatible with the usual conjugation of L^1_{loc} functions. It is therefore natural to introduce the following sheaves of rings:

$$\mathscr{O}_{X,\overline{X}} := \mathscr{O}_X \otimes_{\mathbb{C}} \mathscr{O}_{\overline{X}}, \qquad \mathscr{D}_{X,\overline{X}} := \mathscr{D}_X \otimes_{\mathbb{C}} \mathscr{D}_{\overline{X}}$$

and consider \mathfrak{Db}_X (resp. \mathfrak{C}_X) as left (resp. right) $\mathscr{D}_X \overline{X}$ -modules.

A.4.5. One can construct new examples from old ones by using various operations.

• Let \mathscr{M} be a left $\widetilde{\mathscr{D}}_X$ -module. Then $\mathscr{H}om_{\widetilde{\mathscr{D}}_X}(\mathscr{M}, \widetilde{\mathscr{D}}_X)$ has a natural structure of right $\widetilde{\mathscr{D}}_X$ -module. Using a resolution \mathscr{N}^{\bullet} of \mathscr{M} by left $\widetilde{\mathscr{D}}_X$ -modules which are acyclic for $\mathscr{H}om_{\widetilde{\mathscr{D}}_X}(\bullet, \widetilde{\mathscr{D}}_X)$, one gets a right $\widetilde{\mathscr{D}}_X$ -module structure on $\mathscr{E}xt^k_{\widetilde{\mathscr{D}}_X}(\mathscr{M}, \widetilde{\mathscr{D}}_X)$ for $k \ge 0$.

• Given two left (resp. a left and a right) $\widetilde{\mathscr{D}}_X$ -modules \mathscr{M} and \mathscr{N} , the same argument enables one to put on the various $\mathscr{T}or_{i,\widetilde{\mathscr{O}}_X}(\mathscr{N},\mathscr{M})$ a left (resp. a right) $\widetilde{\mathscr{D}}_X$ -module structure.

A.5. The de Rham functor

Definition A.5.1 (de Rham). For a left $\widetilde{\mathscr{D}}_X$ -module \mathscr{M} , the de Rham complex DR \mathscr{M} is the bounded complex (with \bullet in degree zero and all nonzero terms in nonnegative degrees)

$$\mathrm{DR}\,\mathscr{M} := \{ 0 \to \mathscr{M} \xrightarrow{\widetilde{\nabla}} \widetilde{\Omega}^1_X \otimes \mathscr{M} \xrightarrow{\widetilde{\nabla}} \cdots \xrightarrow{\widetilde{\nabla}} \widetilde{\Omega}^n_X \otimes \mathscr{M} \to 0 \}.$$

The terms are the $\widetilde{\mathcal{O}}_X$ -modules $\widetilde{\Omega}^{\bullet}_X \otimes_{\widetilde{\mathcal{O}}_X} \mathscr{M}^{\text{left}}$ and the differentials the $\widetilde{\mathbb{C}}$ -linear morphisms $\widetilde{\nabla}$ defined in Exercise A.6 or A.11.

The previous definition produces a *complex* since $\widetilde{\nabla} \circ \widetilde{\nabla} = 0$, according to the integrability condition on $\widetilde{\nabla}$, as remarked in Exercise A.6 or A.11.

Definition A.5.2 (Spencer). The Spencer complex $\operatorname{Sp}(\mathscr{M})$ of a right $\widetilde{\mathscr{D}}_X$ -module \mathscr{M} is the bounded complex (with \bullet in degree zero and all nonzero terms in nonpositive degrees)

$$\operatorname{Sp}(\mathscr{M}) := \{ 0 \to \mathscr{M} \otimes_{\widetilde{\mathscr{O}}_X} \wedge^n \widetilde{\Theta}_X \xrightarrow{\widetilde{\delta}} \cdots \xrightarrow{\widetilde{\delta}} \mathscr{M} \otimes_{\widetilde{\mathscr{O}}_X} \widetilde{\Theta}_X \xrightarrow{\widetilde{\delta}} \mathscr{M} \to 0 \},$$

where the differential $\widetilde{\delta}$ is the \mathbb{C} -linear map given by

$$m \otimes \xi_1 \wedge \dots \wedge \xi_k \xrightarrow{\widetilde{\delta}} \sum_{i=1}^k (-1)^{i-1} m \xi_i \otimes \xi_1 \wedge \dots \wedge \widehat{\xi_i} \wedge \dots \wedge \xi_k + \sum_{i < j} (-1)^{i+j} m \otimes [\xi_i, \xi_j] \wedge \xi_1 \wedge \dots \wedge \widehat{\xi_i} \wedge \dots \wedge \widehat{\xi_j} \wedge \dots \wedge \xi_k.$$

Exercise A.24. Check that $\operatorname{Sp}(\mathscr{M})$ is indeed a complex, i.e., that $\widetilde{\delta} \circ \widetilde{\delta} = 0$.

Of special interest will be, of course, the de Rham or Spencer complex of the ring $\widetilde{\mathscr{D}}_X$, considered as a left or right $\widetilde{\mathscr{D}}_X$ -module. Notice that in $\mathrm{DR}(\widetilde{\mathscr{D}}_X)$ the differentials are right $\widetilde{\mathscr{D}}_X$ -linear, and in $\mathrm{Sp}(\widetilde{\mathscr{D}}_X)$ they are left $\widetilde{\mathscr{D}}_X$ -linear.

Exercise A.25 (Sp($\widetilde{\mathscr{D}}_X$) is a resolution of $\widetilde{\mathscr{O}}_X$ as a left $\widetilde{\mathscr{D}}_X$ -module)

The natural surjective morphism $\widetilde{\mathscr{D}}_X \to \widetilde{\mathscr{O}}_X$ of left $\widetilde{\mathscr{D}}_X$ -modules has kernel the image of $\widetilde{\mathscr{D}}_X \otimes \widetilde{\Theta}_X \to \widetilde{\mathscr{D}}_X$. In other words, we have a morphism of complexes of left $\widetilde{\mathscr{D}}_X$ -modules

$$\operatorname{Sp}(\widetilde{\mathscr{D}}_X) \longrightarrow \widetilde{\mathscr{O}}_X$$

(where $\tilde{\mathscr{O}}_X$ is regarded as a complex with a nonzero term in degree zero only), which induces an isomorphism

$$\mathscr{H}^0$$
 Sp $(\widetilde{\mathscr{D}}_X) \xrightarrow{\sim} \widetilde{\mathscr{O}}_X.$

In this exercise, one proves that $\mathscr{H}^k(\operatorname{Sp}(\widetilde{\mathscr{D}}_X)) = 0$ for $k \neq 0$, so that the morphism above is a quasi-isomorphism.

Let $F_{\bullet} \widetilde{\mathscr{D}}_X$ be the filtration of $\widetilde{\mathscr{D}}_X$ by the order of differential operators. Filter the Spencer complex $\operatorname{Sp}(\widetilde{\mathscr{D}}_X)$ by the subcomplexes $F_p(\operatorname{Sp}(\widetilde{\mathscr{D}}_X))$ defined as

$$\cdot \stackrel{\widetilde{\delta}}{\longrightarrow} F_{p-k} \widetilde{\mathscr{D}}_X \otimes \wedge^k \widetilde{\Theta}_X \stackrel{\widetilde{\delta}}{\longrightarrow} F_{p-k+1} \widetilde{\mathscr{D}}_X \otimes \wedge^{k-1} \widetilde{\Theta}_X \stackrel{\widetilde{\delta}}{\longrightarrow} \cdots$$

(1) Show that, locally on X, using coordinates x_1, \ldots, x_n , the graded complex $\operatorname{gr}^F \operatorname{Sp}(\widetilde{\mathscr{D}}_X) := \bigoplus_p \operatorname{gr}_p^F \operatorname{Sp}(\widetilde{\mathscr{D}}_X)$ is equal to the Koszul complex of the ring $\widetilde{\mathscr{O}}_X[\xi_1, \ldots, \xi_n]$ with respect to the regular sequence ξ_1, \ldots, ξ_n .

(2) Conclude that $\operatorname{gr}^F \operatorname{Sp}(\widetilde{\mathscr{D}}_X)$ is a resolution of $\widetilde{\mathscr{O}}_X$.

(3) Check that $F_p \operatorname{Sp}(\widetilde{\mathscr{D}}_X) = 0$ for p < 0, $F_0 \operatorname{Sp}(\widetilde{\mathscr{D}}_X) = \operatorname{gr}_0^F \operatorname{Sp}(\widetilde{\mathscr{D}}_X)$ is isomorphic to $\widetilde{\mathscr{O}}_X$ and deduce that the complex

$$\operatorname{gr}_p^F \operatorname{Sp}(\widetilde{\mathscr{D}}_X) := \{ \cdots \xrightarrow{\widetilde{\delta}} \operatorname{gr}_{p-k}^F \widetilde{\mathscr{D}}_X \otimes \wedge^k \widetilde{\Theta}_X \xrightarrow{\widetilde{\delta}} \operatorname{gr}_{p-k+1}^F \widetilde{\mathscr{D}}_X \otimes \wedge^{k-1} \widetilde{\Theta}_X \xrightarrow{\widetilde{\delta}} \cdots \}$$

is acyclic (i.e., quasi-isomorphic to 0) for p > 0.

(4) Show that the inclusion $F_0 \operatorname{Sp}(\widetilde{\mathscr{D}}_X) \hookrightarrow F_p \operatorname{Sp}(\widetilde{\mathscr{D}}_X)$ is a quasi-isomorphism for every $p \ge 0$ and deduce, by passing to the inductive limit, that the Spencer complex $\operatorname{Sp}(\widetilde{\mathscr{D}}_X)$ is a resolution of $\widetilde{\mathscr{O}}_X$ as a left $\widetilde{\mathscr{D}}_X$ -module by locally free left $\widetilde{\mathscr{D}}_X$ -modules.

Exercise A.26 (DR($\widetilde{\mathscr{D}}_X$)[n] is a resolution of $\widetilde{\omega}_X$ as a right $\widetilde{\mathscr{D}}_X$ -module)

(1) Similarly, the natural morphism of right \mathscr{D}_X -modules

$$\widetilde{\omega}_X \otimes_{\widetilde{\mathscr{O}}_X} \widetilde{\mathscr{D}}_X \longrightarrow \widetilde{\omega}_X$$

extends as a morphism of complexes of right $\widetilde{\mathscr{D}}_X$ -modules

$$\mathrm{DR}(\mathscr{D}_X)[n] \longrightarrow \widetilde{\omega}_X.$$

Show that $\mathscr{H}^k(\mathrm{DR}\,\widetilde{\mathscr{D}}_X) = 0$ for $k \neq n$, so that the shifted complex $\mathrm{DR}(\widetilde{\mathscr{D}}_X)[n]$ is a resolution of $\widetilde{\omega}_X$ as a right $\widetilde{\mathscr{D}}_X$ -module by locally free right $\widetilde{\mathscr{D}}_X$ -modules.

(2) Let \mathscr{M} be a left $\widetilde{\mathscr{D}}_X$ -module. Show that the complex

$$\mathrm{DR}(\mathscr{D}_X \otimes_{\widetilde{\mathscr{O}}_X} \mathscr{M})[n]$$

is a resolution of $\mathscr{M}^{\mathrm{right}} = \widetilde{\omega}_X \otimes_{\widetilde{\mathscr{O}}_X} \mathscr{M}$ by right $\widetilde{\mathscr{D}}_X$ -modules, where the left and right structures of $\widetilde{\mathscr{D}}_X \otimes_{\widetilde{\mathscr{O}}_X} \mathscr{M}$ are those of Exercise A.18(2), and the left one is used to compute the de Rham complex.

Exercise A.27. Let $\mathscr{M}^{\mathrm{right}}$ be a right $\widetilde{\mathscr{D}}_X$ -module

(1) Show that the natural morphism

$$\mathscr{M}^{\mathrm{right}} \otimes_{\widetilde{\mathscr{D}}_X} (\widetilde{\mathscr{D}}_X \otimes_{\widetilde{\mathscr{O}}_X} \wedge^k \widetilde{\Theta}_X) \longrightarrow \mathscr{M}^{\mathrm{right}} \otimes_{\widetilde{\mathscr{O}}_X} \wedge^k \widetilde{\Theta}_X$$

defined by $m \otimes P \otimes \xi \mapsto mP \otimes \xi$ induces an isomorphism of complexes

$$\mathscr{M}^{\mathrm{right}} \otimes_{\widetilde{\mathscr{D}}_X} \mathrm{Sp}(\widetilde{\mathscr{D}}_X) \xrightarrow{\sim} \mathrm{Sp}(\mathscr{M}^{\mathrm{right}}).$$

(2) Similar question for $\mathrm{DR}(\widetilde{\mathscr{D}}_X) \otimes_{\widetilde{\mathscr{D}}_X} \mathscr{M}^{\mathrm{left}} \to \mathrm{DR}(\mathscr{M}^{\mathrm{left}}).$

Let \mathscr{M} be a left $\widetilde{\mathscr{D}}_X$ -module and let $\mathscr{M}^{\text{right}}$ the associated right module. We will now compare $\text{DR}_X(\mathscr{M})$ and $\text{Sp}(\mathscr{M}^{\text{right}})$. Given any $k \ge 0$, the *contraction* is the morphism

Lemma A.5.4. There exists a unique isomorphism of complexes of right $\widetilde{\mathscr{D}}_X$ -modules (i.e., is compatible with the differentials of these complexes)

$$\iota: \widetilde{\omega}_X \otimes_{\widetilde{\mathscr{Q}}_X} \operatorname{Sp}(\widetilde{\mathscr{D}}_X) \xrightarrow{\sim} \operatorname{DR}(\widetilde{\mathscr{D}}_X)[n]$$

which induces the identity

$$\widetilde{\omega}_X \otimes_{\widetilde{\mathscr{O}}_X} \operatorname{Sp}^0(\widetilde{\mathscr{D}}_X) = \widetilde{\omega}_X \otimes_{\widetilde{\mathscr{O}}_X} \widetilde{\mathscr{D}}_X = \operatorname{DR}^n \widetilde{\mathscr{D}}_X.$$

It is induced by the isomorphisms of right $\widetilde{\mathscr{D}}_X$ -modules (see Notation 2.3.7 for the notation $\varepsilon(\bullet)$)

$$\widetilde{\omega}_X \otimes_{\widetilde{\mathscr{O}}_X} \left(\widetilde{\mathscr{D}}_X \otimes_{\widetilde{\mathscr{O}}_X} \wedge^k \widetilde{\Theta}_X \right) \xrightarrow{\iota} \widetilde{\Omega}_X^{n-k} \otimes_{\widetilde{\mathscr{O}}_X} \widetilde{\mathscr{D}}_X \\ \left[\omega \otimes (1 \otimes \xi) \right] \cdot P \longmapsto \left(\varepsilon (k+1) \omega (\xi \wedge \bullet) \right) \otimes P$$

(where the right structure of the right-hand term is the natural one and that of the left-hand term is nothing but that induced by the left structure after going from left to right).

Proof. From the properties of the function ε (see Notation 2.3.7) and the value $\iota^0 = \text{Id}$, it is enough to prove that the diagram

$$\begin{split} \widetilde{\omega}_X \otimes_{\widetilde{\mathscr{O}}_X} \left(\widetilde{\mathscr{D}}_X \otimes_{\widetilde{\mathscr{O}}_X} \wedge^k \widetilde{\Theta}_X \right) & \longrightarrow \widetilde{\Omega}_X^{n-k} \otimes_{\widetilde{\mathscr{O}}_X} \widetilde{\mathscr{D}}_X \\ \delta \\ \delta \\ \widetilde{\omega}_X \otimes_{\widetilde{\mathscr{O}}_X} \left(\widetilde{\mathscr{D}}_X \otimes_{\widetilde{\mathscr{O}}_X} \wedge^{k-1} \widetilde{\Theta}_X \right) & \longrightarrow \widetilde{\Omega}_X^{n-k+1} \otimes_{\widetilde{\mathscr{O}}_X} \widetilde{\mathscr{D}}_X \end{split}$$

commutes up to the sign $(-1)^k$. It is also enough to check this locally, and, in local coordinates (x_1, \ldots, x_n) , we are reduced to checking this on sections of the form $(\tilde{\mathbf{d}} \boldsymbol{x} \otimes P) \otimes (\tilde{\mathbf{d}}_{x_1} \wedge \cdots \wedge \tilde{\mathbf{d}}_{x_k})$. For $i = 1, \ldots, k$, let us set $\widehat{\vartheta_{x_i}} := \vartheta_{x_1} \wedge \cdots \wedge \vartheta_{x_i} \wedge \cdots \wedge \vartheta_{x_k}$ for simplicity.

On the one hand, since $\widetilde{d} \boldsymbol{x} \cdot \widetilde{d}_{x_i} = 0$ and $\widehat{\mathfrak{d}}_{x_i} \, \, \, \exists \, \widetilde{\mathbf{x}} = (-1)^{k-i} \widetilde{d}_x \wedge \widetilde{d}_{k+1} \wedge \cdots \wedge \widetilde{d}_x$, we have

$$\widetilde{\delta} \left[(\widetilde{\mathbf{d}} \boldsymbol{x} \otimes P) \otimes (\mathfrak{d}_{x_1} \wedge \dots \wedge \mathfrak{d}_{x_k}) \right] = \sum_{i=1}^k (-1)^{i-1} (\widetilde{\mathbf{d}} \boldsymbol{x} \otimes P) \mathfrak{d}_{x_i} \otimes (\widehat{\mathfrak{d}}_{x_i})$$
$$= \sum_{i=1}^k (-1)^i (\widetilde{\mathbf{d}} \boldsymbol{x} \otimes \mathfrak{d}_{x_i} P) \otimes (\widehat{\mathfrak{d}}_{x_i})$$
$$\stackrel{\bot}{\longrightarrow} (-1)^k \sum_{i=1}^k (\widetilde{\mathbf{d}} x_i \wedge \widetilde{\mathbf{d}} x_{k+1} \wedge \dots \wedge \widetilde{\mathbf{d}} x_n) \otimes \mathfrak{d}_{x_i} P.$$

On the other hand,

$$\widetilde{\nabla} \left[(\widetilde{\partial}_{x_1} \wedge \dots \wedge \widetilde{\partial}_{x_k}) \, \sqcup \, (\widetilde{\mathrm{d}} \boldsymbol{x} \otimes P) \right] = \widetilde{\nabla} \left[(\widetilde{\mathrm{d}} x_{k+1} \wedge \dots \wedge \widetilde{\mathrm{d}} x_n) \otimes P \right]$$
$$= \sum_{i=1}^k (\widetilde{\mathrm{d}} x_i \wedge \widetilde{\mathrm{d}} x_{k+1} \wedge \dots \wedge \widetilde{\mathrm{d}} x_n) \otimes \widetilde{\partial}_{x_i} P. \qquad \Box$$

Exercise A.28.

(1) If \mathscr{M} is any left $\widetilde{\mathscr{D}}_X$ -module and $\mathscr{M}^{\mathrm{right}} = \widetilde{\omega}_X \otimes_{\widetilde{\mathscr{O}}_X} \mathscr{M}$ is the associated right $\widetilde{\mathscr{D}}_X$ -module, show that ι induces an isomorphism

$$\mathscr{M}^{\mathrm{right}} \otimes_{\widetilde{\mathscr{D}}_X} \mathrm{Sp}(\widetilde{\mathscr{D}}_X) \xrightarrow{\sim} \mathrm{DR}(\widetilde{\mathscr{D}}_X)[n] \otimes_{\widetilde{\mathscr{D}}_X} \mathscr{M}$$

which is termwise $\widetilde{\mathcal{O}}_X$ -linear.

(2) Use Exercise A.27 and the involution of Exercises A.18(4) and A.19(3) to show that the $\widetilde{\mathcal{O}}_X$ -linear isomorphism

$$\widetilde{\omega}_X \otimes_{\widetilde{\mathcal{O}}_X} \mathscr{M} \otimes_{\widetilde{\mathcal{O}}_X} \wedge^k \widetilde{\Theta}_X \xrightarrow{\sim} \widetilde{\Omega}_X^{n-k} \otimes_{\widetilde{\mathcal{O}}_X} \mathscr{M}$$

given on $\widetilde{\omega}_X \otimes_{\widetilde{\mathscr{O}}_X} \mathscr{M} \otimes_{\widetilde{\mathscr{O}}_X} \wedge^k \widetilde{\Theta}_X$ by

$$\omega \otimes m \otimes \xi \longmapsto \varepsilon(k+1)\omega(\xi \wedge \bullet) \otimes m$$

induces a functorial isomorphism $\operatorname{Sp}(\mathscr{M}^{\operatorname{right}}) \xrightarrow{\sim} \operatorname{DR}(\mathscr{M})[n]$ for any left $\widetilde{\mathscr{D}}_X$ -module \mathscr{M} , which is termwise $\widetilde{\mathscr{O}}_X$ -linear. (Other solution: direct computation between $\widetilde{\delta}$ and $\widetilde{\nabla}$ through this isomorphism.)

Notation A.5.5 (^PDR). We will denote by ^PDR($\mathscr{M}^{\text{right}}$) the Spencer complex Sp($\mathscr{M}^{\text{right}}$) and we keep the notation DR($\mathscr{M}^{\text{left}}$) for the de Rham complex of a left $\widetilde{\mathscr{D}}_X$ -module. The previous exercise gives an isomorphism

$$(A.5.5*) \xrightarrow{p} DR(\mathscr{M}^{right}) \xrightarrow{\sim} DR(\mathscr{M}^{left})[n].$$

We will use this notation below. Exercise A.27 clearly shows that ^PDR is a functor from the category of right (resp. left) $\widetilde{\mathscr{D}}_X$ -modules to the category of complexes of sheaves of $\widetilde{\mathbb{C}}$ -modules. It can be extended to a functor between the corresponding derived categories.

Definition A.5.6 (Contraction by a one-form). The contraction morphism

$$\widetilde{\Omega}^1_X \otimes \wedge^k \widetilde{\Theta}_X \xrightarrow{\ \ \, } \wedge^{k-1} \widetilde{\Theta}_X$$

is the unique morphism such that the following diagram commutes:

where the vertical morphisms are induced by (A.5.3).

Notice that if k = 1 and ξ (resp. η) are local sections of Θ_X (resp. $\tilde{\Omega}^1_X$), then $\eta \perp \xi = \xi \perp \eta$.

Remark A.5.7 (Action of a closed one-form on the de Rham complex)

Let η be a *closed* holomorphic one-form on X. Then the exterior product by η induces a morphism

$$\eta \wedge \bullet : \mathrm{DR}(\mathscr{M}^{\mathrm{left}}) \longrightarrow \mathrm{DR}(\mathscr{M}^{\mathrm{left}})[1].$$

Indeed, for a local section m of \mathcal{M} and a k-form ω , we have

$$\widetilde{
abla}(\eta\wedge\omega\otimes m)=\widetilde{\mathrm{d}}\eta\wedge\omega\otimes m+\eta\wedge\widetilde{
abla}(\omega\otimes m)=\eta\wedge\widetilde{
abla}(\omega\otimes m),$$

so that the morphism $\eta \wedge$ commutes with $\overline{\nabla}$.

It follows then from Lemma A.5.4 and Exercise A.28 that the contraction by a *closed* one-form η induces a morphism of complexes

$$\eta \sqcup \bullet : \mathrm{DR}(\mathscr{M}^{\mathrm{right}}) \longrightarrow \mathrm{DR}(\mathscr{M}^{\mathrm{right}})[1].$$

Note that, if $\eta = \tilde{d}f$ is *exact*, then the induced morphism

$$\eta \wedge \, : \mathscr{H}^{i} \operatorname{DR}(\mathscr{M}^{\operatorname{left}}) \longrightarrow \mathscr{H}^{i+1} \operatorname{DR}(\mathscr{M}^{\operatorname{left}})$$

is zero. Indeed, if a local section μ of $\widetilde{\Omega}_X^k \otimes \mathscr{M}^{\text{left}}$ satisfies $\widetilde{\nabla}\mu = 0$, then $\widetilde{d}f \wedge \mu = \widetilde{\nabla}(f\mu)$. In other words, the morphism $\eta \wedge$ on the cohomology only depends on the class of η in $H^1\Gamma(X, (\widetilde{\Omega}_X^{\bullet}, \widetilde{d}))$. The same result holds when we make η acting on the complex $\Gamma(X, \text{DR}(\mathscr{M}^{\text{left}}))$, and a similar result holds for the action $\eta \sqcup \bullet$ on $\text{DR}(\mathscr{M}^{\text{right}})$. *Remark A.5.8* (C^{∞} de Rham and Spencer complexes). Denote by $(\widetilde{\mathscr{E}}_X^{(\bullet,0)}, \widetilde{\mathbf{d}}')$ the complex $\mathscr{C}_X^{\infty} \otimes_{\widetilde{\mathscr{O}}_X} \widetilde{\Omega}_X^{\bullet}$ with the differential induced by $\widetilde{\mathbf{d}}$ (here, we assume $\bullet \ge 0$). More generally, set

$$\widetilde{\mathscr{E}}_X^{(p,q)} = \widetilde{\Omega}_X^p \wedge \widetilde{\mathscr{E}}_X^{(0,q)} = \widetilde{\mathscr{E}}_X^{(p,0)} \wedge \widetilde{\mathscr{E}}_X^{(0,q)}$$

and let d" be the (usual) anti-holomorphic differential. For every p, the complex $(\widetilde{\mathscr{E}}_X^{(p,\bullet)}, \mathbf{d}'')$ is a resolution of $\widetilde{\Omega}_X^p$. We therefore have a complex $(\widetilde{\mathscr{E}}_X^{\bullet}, \widetilde{\mathbf{d}})$, which is the single complex associated to the double complex $(\widetilde{\mathscr{E}}_X^{(\bullet,\bullet)}, \widetilde{\mathbf{d}}', \mathbf{d}'')$.

In particular, we have a natural quasi-isomorphism of complexes of right $\mathcal{D}_X\text{-}\mathrm{mod-}$ ules:

$$(\widetilde{\Omega}^{\bullet}_X \otimes_{\widetilde{\mathscr{O}}_X} \widetilde{\mathscr{D}}_X, \widetilde{\nabla}) \overset{\sim}{\longrightarrow} (\widetilde{\mathscr{E}}^{\bullet}_X \otimes_{\widetilde{\mathscr{O}}_X} \widetilde{\mathscr{D}}_X, \widetilde{\nabla})$$

by sending holomorphic k-forms to (k, 0)-forms. Remark that the terms of these complexes are flat over $\widetilde{\mathscr{O}}_X$ and are fine sheaves.

A.6. Induced $\widetilde{\mathscr{D}}$ -modules

A subcategory of $\mathsf{Mod}(\widetilde{\mathscr{D}}_X)$ proves very useful in many places, namely that of induced right $\widetilde{\mathscr{D}}_X$ -modules.

Exercise A.29. Let \mathscr{L} be an $\widetilde{\mathscr{O}}_X$ -module.

(1) Show that, for every k, we have a (termwise) exact sequence of complexes

 $0 \to \mathscr{L} \otimes_{\widetilde{\mathscr{O}}_X} F_{k-1}(\mathrm{Sp}(\widetilde{\mathscr{D}}_X)) \to \mathscr{L} \otimes_{\widetilde{\mathscr{O}}_X} F_k(\mathrm{Sp}(\widetilde{\mathscr{D}}_X)) \to \mathscr{L} \otimes_{\widetilde{\mathscr{O}}_X} \mathrm{gr}_k^F(\mathrm{Sp}(\widetilde{\mathscr{D}}_X)) \to 0.$

[*Hint*: use that the terms of the complexes $F_j(\operatorname{Sp}(\widetilde{\mathscr{D}}_X))$ and $\operatorname{gr}_k^F(\operatorname{Sp}(\widetilde{\mathscr{D}}_X))$ are $\widetilde{\mathscr{O}}_X$ -locally free.]

- (2) Show that $\mathscr{L} \otimes_{\widetilde{\mathscr{O}}_X} \operatorname{gr}^F \operatorname{Sp}(\widetilde{\mathscr{D}}_X)$ is a resolution of \mathscr{L} as an $\widetilde{\mathscr{O}}_X$ -module.
- (3) Show that $\mathscr{L} \otimes_{\widetilde{\mathscr{O}}_{X}} \operatorname{Sp}(\widetilde{\mathscr{D}}_{X})$ is a resolution of \mathscr{L} as an $\widetilde{\mathscr{O}}_{X}$ -module.

Remark A.6.1. Let \mathscr{L} be an $\widetilde{\mathscr{O}}_X$ -module. It induces a right $\widetilde{\mathscr{D}}_X$ -module $\mathscr{L} \otimes_{\widetilde{\mathscr{O}}_X} \widetilde{\mathscr{D}}_X$. We note that $\mathscr{L} \otimes_{\widetilde{\mathscr{O}}_X} \widetilde{\mathscr{D}}_X$ has two structures of $\widetilde{\mathscr{O}}_X$ -module, one coming from the action on \mathscr{L} and the other one from the right $\widetilde{\mathscr{D}}_X$ -structure, and they do not coincide. We will mainly use the right one. The "left" $\widetilde{\mathscr{O}}_X$ -module structure on $\mathscr{L} \otimes_{\widetilde{\mathscr{O}}_X} \widetilde{\mathscr{D}}_X$ will only be used when noticing that some naturally defined sheaves of $\widetilde{\mathbb{C}}$ -vector spaces are in fact sheaves of $\widetilde{\mathscr{O}}_X$ -modules. On the other hand, $\mathscr{L} \otimes_{\widetilde{\mathscr{O}}_X} \widetilde{\mathscr{D}}_X$ has a canonical structure of right $\widetilde{\mathscr{D}}_X$ -module.

The category $\mathsf{Mod}_i(\widetilde{\mathscr{D}}_X)$ of right induced differential modules is the full subcategory of $\mathsf{Mod}(\widetilde{\mathscr{D}}_X)$ consisting of induced $\widetilde{\mathscr{D}}_X$ -modules (i.e., we consider as morphisms all $\widetilde{\mathscr{D}}_X$ -linear morphisms). It is an additive category (but not an abelian category).

Let \mathscr{M} be a right $\widehat{\mathscr{D}}_X$ -module and let \mathscr{L} be an $\widehat{\mathscr{O}}_X$ -module. Considering the natural $\widetilde{\mathscr{O}}_X$ -module structure on $\mathscr{M} \otimes_{\widetilde{\mathscr{O}}_X} \mathscr{L}$, we define an induced right $\widetilde{\mathscr{D}}_X$ -module $(\mathscr{M} \otimes_{\widetilde{\mathscr{O}}_X} \mathscr{L}) \otimes_{\widetilde{\mathscr{O}}_X} \widetilde{\mathscr{D}}_X$. Here, the $\widetilde{\mathscr{D}}_X$ -module structure on \mathscr{M} is not used.

On the other hand, considering the canonical left $\widetilde{\mathscr{D}}_X$ -module structure on $\widetilde{\mathscr{D}}_X \otimes_{\widetilde{\mathscr{O}}_X} \mathscr{L}$ and using Exercise A.16(2), we obtain a right $\widetilde{\mathscr{D}}_X$ -module structure on $\mathscr{M} \otimes_{\widetilde{\mathscr{O}}_X} (\widetilde{\mathscr{D}}_X \otimes_{\widetilde{\mathscr{O}}_X} \mathscr{L})$. Here, the $\widetilde{\mathscr{D}}_X$ -module structure on \mathscr{M} is used in an essential way.

Exercise A.30. Prove that the canonical $\widetilde{\mathscr{O}}_X$ -linear morphism

$$\begin{split} \mathscr{M} \otimes_{\widetilde{\mathscr{O}}_X} \mathscr{L} & \longrightarrow \mathscr{M} \otimes_{\widetilde{\mathscr{O}}_X} (\widetilde{\mathscr{D}}_X \otimes_{\widetilde{\mathscr{O}}_X} \mathscr{L}) \\ & m \otimes \ell \longmapsto m \otimes 1 \otimes \ell \end{split}$$

induces in a unique way a $\widetilde{\mathscr{D}}_X$ -linear morphism

 $(\mathscr{M} \otimes_{\widetilde{\mathscr{O}}_{X}} \mathscr{L}) \otimes_{\widetilde{\mathscr{O}}_{X}} \widetilde{\mathscr{D}}_{X} \longrightarrow \mathscr{M} \otimes_{\widetilde{\mathscr{O}}_{X}} (\widetilde{\mathscr{D}}_{X} \otimes_{\widetilde{\mathscr{O}}_{X}} \mathscr{L})$

which is an isomorphism. [*Hint*: argue as in Exercise A.19.]

Proposition A.6.2 (The canonical resolution by induced $\widehat{\mathscr{D}}_X$ -modules)

Let \mathscr{M} be a right $\widetilde{\mathscr{D}}_X$ -module. Then the complex $\mathscr{M} \otimes_{\widetilde{\mathscr{O}}_X} \operatorname{Sp}(\widetilde{\mathscr{D}}_X)$ is isomorphic to a complex of right induced $\widetilde{\mathscr{D}}_X$ -modules which is a resolution of \mathscr{M} as such.

One should not confuse $\mathscr{M} \otimes_{\widetilde{\mathscr{O}}_X} \operatorname{Sp}(\widetilde{\mathscr{D}}_X)$ with $\mathscr{M} \otimes_{\widetilde{\mathscr{D}}_X} \operatorname{Sp}(\widetilde{\mathscr{D}}_X) \simeq \operatorname{Sp}(\mathscr{M})$ as in Exercise A.27(1), where a tensor product over $\widetilde{\mathscr{D}}_X$ is considered.

Proof. That the terms of the complex are induced $\widetilde{\mathscr{D}}_X$ -modules follows from Exercise A.30 applied to $\mathscr{L} = \wedge^k \widetilde{\Theta}_X$. Since $\operatorname{Sp}(\widetilde{\mathscr{D}}_X)$ is a resolution of $\widetilde{\mathscr{O}}_X$ as a $\widetilde{\mathscr{D}}_X$ -module, hence as an $\widetilde{\mathscr{O}}_X$ -module, and since the terms of $\operatorname{Sp}(\widetilde{\mathscr{D}}_X)$ are $\widetilde{\mathscr{O}}_X$ -locally free, we conclude that $\mathscr{M} \otimes_{\widetilde{\mathscr{O}}_X} \operatorname{Sp}(\widetilde{\mathscr{D}}_X)$ is a resolution of \mathscr{M} .

Let $C_i^*(\widetilde{\mathscr{D}}_X)$ the category of \star -bounded complexes of the additive category $\mathsf{Mod}_i(\widetilde{\mathscr{D}}_X)$ and let $\mathsf{K}_i^*(\widetilde{\mathscr{D}}_X)$ be the corresponding homotopy category. Since $\operatorname{Sp} \widetilde{\mathscr{D}}_X$ is a complex of locally free $\widetilde{\mathscr{O}}_X$ -modules, the functor $\mathscr{M}^\bullet \to \mathscr{M}^\bullet \otimes_{\widetilde{\mathscr{O}}_X} \operatorname{Sp} \widetilde{\mathscr{D}}_X$ is a functor of triangulated categories, and sends acyclic complexes to acyclic complexes according to the previous proposition. It induces therefore a functor $\mathsf{D}^*(\widetilde{\mathscr{D}}_X) \to \mathsf{D}_i^*(\widetilde{\mathscr{D}}_X)$.

Corollary A.6.3 (Equivalence of $\mathsf{D}^{\star}(\widetilde{\mathscr{D}}_{X})$ with $\mathsf{D}_{i}^{\star}(\widetilde{\mathscr{D}}_{X})$). The natural functor $\mathsf{D}_{i}^{\star}(\widetilde{\mathscr{D}}_{X}) \to \mathsf{D}^{\star}(\widetilde{\mathscr{D}}_{X})$ is an equivalence of categories, and the functor $\mathsf{D}^{\star}(\widetilde{\mathscr{D}}_{X}) \to \mathsf{D}_{i}^{\star}(\widetilde{\mathscr{D}}_{X})$ induced by $\mathscr{M}^{\bullet} \mapsto \mathscr{M}^{\bullet} \otimes_{\widetilde{\mathscr{O}}_{X}} \operatorname{Sp} \widetilde{\mathscr{D}}_{X}$ is a quasi-inverse functor.

A.7. Pullback of left $\widetilde{\mathscr{D}}$ -modules

Let us begin with some relative complements to Section A.3. Let $f: X \to Y$ be a holomorphic map between analytic manifolds. For any local section ξ of the sheaf $\widetilde{\Theta}_X$ of z-vector fields on X, $Tf(\xi)$ is a local section of $\widetilde{\mathscr{O}}_X \otimes_{f^{-1}\widetilde{\mathscr{O}}_Y} f^{-1}\widetilde{\Theta}_Y$. We hence have an $\widetilde{\mathscr{O}}_X$ -linear map

$$Tf: \widetilde{\Theta}_X \longrightarrow \widetilde{\mathscr{O}}_X \otimes_{f^{-1}\widetilde{\mathscr{O}}_Y} f^{-1}\widetilde{\Theta}_Y$$

and dually

$$T^*f: \widetilde{\mathscr{O}}_X \otimes_{f^{-1}\widetilde{\mathscr{O}}_Y} \widetilde{\Omega}^1_Y \longrightarrow \widetilde{\Omega}^1_X$$

Therefore, if \mathscr{N} is any left $\widetilde{\mathscr{D}}_Y$ -module, the connection $\widetilde{\nabla}^Y$ on \mathscr{N} can be lifted as a connection

$$\widetilde{\nabla}^X : f^* \mathcal{N} := \widetilde{\mathcal{O}}_X \otimes_{f^{-1} \widetilde{\mathcal{O}}_Y} f^{-1} \mathcal{N} \longrightarrow \widetilde{\Omega}^1_X \otimes_{f^{-1} \widetilde{\mathcal{O}}_Y} f^{-1} \mathcal{N} = \widetilde{\Omega}^1_X \otimes_{\widetilde{\mathcal{O}}_X} f^* \mathcal{N}$$

by setting

(A.7.1)
$$\widetilde{\nabla}^X = \widetilde{\mathbf{d}} \otimes \mathrm{Id} + (T^* f \otimes \mathrm{Id}_{\mathscr{N}}) \circ (1 \otimes \widetilde{\nabla}^Y)$$

Exercise A.31 (Definition of the pullback of a left $\widetilde{\mathscr{D}}_X$ -module)

(1) Show that the connection $\widetilde{\nabla}^X$ on $f^*\mathscr{N} := \widetilde{\mathscr{O}}_X \otimes_{f^{-1}\widetilde{\mathscr{O}}_Y} f^{-1}\mathscr{N}$ is integrable and defines the structure of a left $\widetilde{\mathscr{D}}_X$ -module on $f^*\mathscr{N}$. The corresponding $\widetilde{\mathscr{D}}_X$ -module is denoted by ${}_{\mathrm{D}}f^*\mathscr{N}$.

(2) Show that, if \mathscr{N} also has a right $\widetilde{\mathscr{D}}_Y$ -module structure commuting with the left one, then $\widetilde{\nabla}^X$ is right $f^{-1}\widetilde{\mathscr{D}}_Y$ -linear, and ${}_{\mathrm{D}}f^*\mathscr{N}$ is a right $f^{-1}\widetilde{\mathscr{D}}_Y$ -module.

Exercise A.32.

(1) Express the previous connection in local coordinates on X and Y.

(2) Show that, if $\mathscr{M}^{\text{left}}$ is any left $\widetilde{\mathscr{D}}_X$ -module and \mathscr{N} any left $f^{-1}\widetilde{\mathscr{D}}_Y$ -module, then $\mathscr{M}^{\text{left}} \otimes_{f^{-1}\widetilde{\mathscr{O}}_Y} f^{-1}\mathscr{N}$ may be equipped with a left $\widetilde{\mathscr{D}}_X$ -module structure: if ξ is a local z-vector field on X, i.e., a local section of $\widetilde{\Theta}_X$, set

$$\xi \cdot (m \otimes n) = (\xi m) \otimes n + Tf(\xi)(m \otimes n).$$

[*Hint*: identify $\mathscr{M}^{\text{left}} \otimes_{f^{-1}\widetilde{\mathscr{O}}_{Y}} f^{-1}\mathscr{N}$ with $\mathscr{M}^{\text{left}} \otimes_{\widetilde{\mathscr{O}}_{X}} {}_{\mathrm{D}}f^{*}\mathscr{N}$ and use Exercise A.31.]

Definition A.7.2 (Transfer modules).

(1) The sheaf $\widetilde{\mathscr{D}}_{X \to Y} = \widetilde{\mathscr{O}}_X \otimes_{f^{-1}\widetilde{\mathscr{O}}_Y} f^{-1}\widetilde{\mathscr{D}}_Y = {}_{\mathrm{D}}f^*\widetilde{\mathscr{D}}_Y$ is a left-right $(\widetilde{\mathscr{D}}_X, f^{-1}\widetilde{\mathscr{D}}_Y)$ bimodule when using the natural right $f^{-1}\widetilde{\mathscr{D}}_Y$ -module structure and the left $\widetilde{\mathscr{D}}_X$ -module introduced above.

(2) The sheaf $\widetilde{\mathscr{D}}_{Y \leftarrow X}$ is obtained from $\widetilde{\mathscr{D}}_{X \to Y}$ by using the usual left-right transformation on both sides:

$$\widetilde{\mathscr{D}}_{Y \leftarrow X} = \mathscr{H}om_{f^{-1}\widetilde{\mathscr{O}}_Y} \big(\widetilde{\omega}_Y, \widetilde{\omega}_X \otimes_{\widetilde{\mathscr{O}}_X} \widetilde{\mathscr{D}}_{X \to Y} \big).$$

Exercise A.33 (Local computation of $\widetilde{\mathscr{D}}_{X \to Y}$).

(1) Show that ${}_{\mathrm{D}}f^*\widetilde{\mathscr{D}}_Y$ is a locally free $\widetilde{\mathscr{O}}_X$ -module. [*Hint*: Use that $\widetilde{\mathscr{D}}_Y$ is a locally free $\widetilde{\mathscr{O}}_Y$ -module.]

(2) Choose local coordinates x_1, \ldots, x_n on X and y_1, \ldots, y_m on Y. Show that $\widetilde{\mathscr{D}}_{X \to Y} = \widetilde{\mathscr{O}}_X[\mathfrak{d}_{y_1}, \ldots, \mathfrak{d}_{y_m}]$ and, with this identification, the left $\widetilde{\mathscr{D}}_X$ -structure is given by

$$\eth_{x_i} \cdot \sum_{\alpha} a_{\alpha}(x) \eth_y^{\alpha} = \sum_{\alpha} \left(z \frac{\partial a_{\alpha}}{\partial x_i} + \sum_{j=1}^m a_{\alpha}(x) \frac{\partial f_j}{\partial x_i} \eth_{y_j} \right) \eth_y^{\alpha}.$$

Exercise A.34 ($\widetilde{\mathscr{D}}_{X \to Y}$ for a closed embedding). Assume that X is a complex submanifold of Y of codimension d, defined by $g_1 = \cdots = g_d = 0$, where the g_i are holomorphic functions on Y. Show that

$$\widetilde{\mathscr{D}}_{X \to Y} = \widetilde{\mathscr{D}}_Y / \sum_{i=1}^d g_i \widetilde{\mathscr{D}}_Y$$

with its natural right $\widetilde{\mathscr{D}}_Y$ structure. In local coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_d)$ such that $g_i = y_i$, show that $\widetilde{\mathscr{D}}_{X \to Y} = \widetilde{\mathscr{D}}_X[\eth_{y_1}, \ldots, \eth_{y_d}].$

Conclude that, if f is an embedding, the sheaves $\widetilde{\mathscr{D}}_{X \to Y}$ and $\widetilde{\mathscr{D}}_{Y \leftarrow X}$ are locally free over $\widetilde{\mathscr{D}}_X$.

Exercise A.35 (The chain rule). Consider holomorphic maps $f : X \to Y$ and $f' : Y \to Z$.

(1) Give an canonical isomorphism $\widetilde{\mathscr{D}}_{X \to Y} \otimes_{f^{-1} \widetilde{\mathscr{D}}_Y} f^{-1} \widetilde{\mathscr{D}}_{Y \to Z} \xrightarrow{\sim} \widetilde{\mathscr{D}}_{X \to Z}$ as right $(f' \circ f)^{-1} \widetilde{\mathscr{D}}_Z$ -modules.

(2) Use the chain rule to show that this isomorphism is left $\widetilde{\mathscr{D}}_X$ -linear.

We can now give a better definition of the pullback of a left $\widetilde{\mathscr{D}}_Y$ -module \mathscr{N} , better in the sense that it is defined inside of the category of $\widetilde{\mathscr{D}}$ -modules. It also enables one to give a definition of a derived inverse image.

Definition A.7.3 (of the pullback of a left $\widetilde{\mathscr{D}}_Y$ -module). Let \mathscr{N} be a left $\widetilde{\mathscr{D}}_Y$ -module. The pullback ${}_{\mathrm{D}}f^*\mathscr{N}$ is the left $\widetilde{\mathscr{D}}_X$ -module $\widetilde{\mathscr{D}}_{X\to Y} \otimes_{f^{-1}\widetilde{\mathscr{D}}_Y} \mathscr{N}$.

Exercise A.36 (Restriction to z = 1). Show that

$$({}_{\mathrm{D}}f^{*}\mathcal{N})/(z-1)_{\mathrm{D}}f^{*}\mathcal{N} = {}_{\mathrm{D}}f^{*}(\mathcal{N}/(z-1)\mathcal{N}).$$

Exercise A.37.

(1) Show that the provious definition coincides with that of Exercise A.31(1).

(2) Let $f : X \to Y$, $f' : Y \to Z$ be holomorphic maps and let \mathscr{N} be a left $\widetilde{\mathscr{D}}_Z$ -module. Show that ${}_{\mathrm{D}}(f' \circ f)^* \mathscr{N} \simeq {}_{\mathrm{D}}f^*({}_{\mathrm{D}}f'^* \mathscr{N})$.

The derived pullback $\boldsymbol{L}_{\mathrm{D}}f^*\mathscr{N}$ is now defined by the usual method, i.e., by taking a flat resolution of \mathscr{N} as a left $\widetilde{\mathscr{D}}_Y$ -module, or by taking a right $f^{-1}\widetilde{\mathscr{D}}_Y$ -flat resolution of $\widetilde{\mathscr{D}}_{X \to Y}$ by $(\widetilde{\mathscr{D}}_X, f^{-1}\widetilde{\mathscr{D}}_Y)$ -bimodules. The cohomology modules $\boldsymbol{L}^j{}_{\mathrm{D}}f^*\mathscr{N} :=$ $\mathscr{T}or_i^{f^{-1}\widetilde{\mathscr{D}}_Y}(\widetilde{\mathscr{D}}_{X \to Y}, f^{-1}\mathscr{N})$ are left $\widetilde{\mathscr{D}}_X$ -modules.

Remark A.7.4 (Side-changing and pullback). The pullback for a right $\widetilde{\mathscr{D}}_Y$ -module $\mathscr{N}^{\text{right}}$ is obtained by applying the side-changing functor at the source and the target. Let $\mathscr{N}^{\text{left}}$ be the left $\widetilde{\mathscr{D}}_Y$ -module associated with $\mathscr{N}^{\text{right}}$, so that $\mathscr{N}^{\text{right}} = \widetilde{\omega}_Y \otimes \mathscr{N}^{\text{left}}$. Then we set

$$_{\mathrm{D}}f^{*}\mathscr{N}^{\mathrm{right}}:=\widetilde{\omega}_{X}\otimes{}_{\mathrm{D}}f^{*}\mathscr{N}^{\mathrm{left}},$$

and similarly with $L_{\rm D} f^*$. Notice the change of grading by dim $Y - \dim X$, due to the grading of $\widetilde{\omega}_X \otimes f^{-1} \widetilde{\omega}_Y^{-1}$.

A.8. Pushforward of $\widetilde{\mathscr{D}}$ -modules

Let $f: X \to Y$ be a holomorphic map between complex manifolds. The pullback of a C^{∞} function on Y is easy to define and, by adjunction, the pushforward of a current of degree 0 is easily defined provided that f is proper. On the other hand, the pullback of a form of maximal degree on Y is usually not of maximal degree on X, so the pushforward of a distribution is not defined in an easy way. This example is an instance of the fact that the pushforward of $\widetilde{\mathscr{D}}_X$ -modules by a proper holomorphic map should be defined in a simple way for right $\widetilde{\mathscr{D}}_X$ -modules, while for left $\widetilde{\mathscr{D}}_X$ -modules one should use the side-changing functors. Let us start with two simple and natural examples of pushforward of $\widetilde{\mathscr{D}}_X$ -modules.

Example A.8.1 (Pushforward of a right $\widetilde{\mathscr{D}}$ -module by a closed embedding)

If f is a closed embedding, it is proper, so the ordinary pushforward and the pushforward with proper support will be the same. After Exercise A.34, it is natural to set

$${}_{\mathrm{D}}f_{!}\mathscr{M} = {}_{\mathrm{D}}f_{*}\mathscr{M} = f_{*}(\mathscr{M} \otimes_{\widetilde{\mathscr{D}}_{X}} \mathscr{D}_{X \to Y}),$$

so that ${}_{\mathrm{D}}f_*, {}_{\mathrm{D}}f_!$ are functors $\mathsf{Mod}(\widetilde{\mathscr{D}}_X) \mapsto \mathsf{Mod}(\widetilde{\mathscr{D}}_Y)$.

Example A.8.2 (Pushforward of a left $\widetilde{\mathscr{D}}$ -module by a projection)

If $X = Y \times T$ and f is the projection $Y \times T \to Y$, let us denote by $\widetilde{\Omega}_{X/Y}^1$ the sheaf of *relative* differential forms, i.e., which do not contain $\widetilde{d}y_j$ in their local expression in coordinates adapted to the product $Y \times T$. If \mathscr{M} is a left $\widetilde{\mathscr{D}}_X$ -module, we can form the relative de Rham complex ${}^{\mathrm{P}}\mathrm{DR}_{X/Y}\mathscr{M}$ by mimicking Definition A.5.1 and by using the relative connection $\widetilde{\nabla}_{X/Y}$. On the other hand, there remains an action of $\widetilde{\nabla}_Y$ on \mathscr{M} . Due to the integrability property of $\widetilde{\nabla}$ on \mathscr{M} , both connections $\widetilde{\nabla}_{X/Y}$ and $\widetilde{\nabla}_Y$ commute, so that the relative de Rham complex ${}^{\mathrm{P}}\mathrm{DR}_{X/Y}\mathscr{M}$ is naturally equipped with a $f^{-1}\widetilde{\mathscr{O}}_Y$ -connection $\widetilde{\nabla}_Y$. We can then set, for $\star = *$ or $\star = !$,

$${}_{\mathrm{D}}f_{\star}\mathscr{M} = (\mathbf{R}f_{\star}{}^{\mathrm{P}}\mathrm{DR}_{X/Y}\mathscr{M}, \nabla_{Y}).$$

To make this definition more precise, we can replace the holomorphic relative de Rham complex $(\widetilde{\Omega}^{\bullet}_{X/Y}, \widetilde{d}_{X/Y})$ with the C^{∞} relative de Rham complex $(\widetilde{\mathscr{E}}^{\bullet}_{X/Y}, \widetilde{d}_{X/Y})$ defined as in Remark A.5.8. In such a way, we obtain

$${}_{\mathrm{D}}f_{\star}\mathscr{M}\simeq \left(f_{\star}(\widetilde{\mathscr{E}}_{X/Y}^{\bullet}\otimes\mathscr{M}),\widetilde{\nabla}_{Y}\right) \quad (\star=*,!).$$

Each term of the complex, hence each cohomology sheaf $\mathscr{H}^k f_{\star}(\widetilde{\mathscr{E}}_{X/Y}^{\bullet} \otimes \mathscr{M})$, is thus endowed with a flat connection $\widetilde{\nabla}_Y$, that is, with a left $\widetilde{\mathscr{D}}_Y$ -module structure.

Since any morphism can be decomposed as a closed embedding followed be a projection, through the graph embedding, we could simply say that the pushforward by a closed embedding (resp. a projection) of a left (resp. right) morphism is obtained by side-changing at the source and the target from the definitions above, and define the pushforward by any holomorphic map f by composing the pushforward functors in these simple cases. Nevertheless, in order to check various other properties, it is useful to have an intrinsic definition for any holomorphic mapping f. The case of right $\widetilde{\mathscr{D}}_X$ -modules is simpler to define first, as explained above.

A.8.a. Pushforward of right $\widetilde{\mathscr{D}}$ -modules

Definition A.8.3 (Pushforward of a right $\widetilde{\mathscr{D}}$ -module). Setting $\star = *$ or $\star = !$, the *direct* image $_{D}f_{\star}$ is the functor from $Mod(\widetilde{\mathscr{D}}_{X})$ to $D^{+}(\widetilde{\mathscr{D}}_{Y})$ defined ⁽²⁾ by

$${}_{\mathrm{D}}f_{\star}\mathscr{M} := \mathbf{R}f_{\star}\big(\mathscr{M}\otimes^{\mathbf{L}}_{\widetilde{\mathscr{D}}_{X}}\widetilde{\mathscr{D}}_{X\to Y}\big).$$

Remarks A.8.4.

(1) If f is proper, or proper on the support of \mathscr{M} , we have an isomorphism in the category $\mathsf{D}^+(\widetilde{\mathscr{D}}_Y)$:

$$_{\mathrm{D}}f_{!}\mathscr{M} \xrightarrow{\sim} {}_{\mathrm{D}}f_{*}\mathscr{M}.$$

(2) If f is a closed embedding, we recover Example A.8.1.

(3) If \mathscr{F} is any sheaf on X, we have $R^j f_* \mathscr{F} = 0$ and $R^j f_! \mathscr{F} = 0$ for $j \notin [0, 2 \dim X]$. Therefore, taking into account the length dim X of the relative Spencer complex, we find that $\mathscr{H}^j{}_{\mathrm{D}}f_*\mathscr{M}$ and $\mathscr{H}^j{}_{\mathrm{D}}f_!\mathscr{M}$ are zero for $j \notin [-\dim X, 2 \dim X]$: we say that ${}_{\mathrm{D}}f_*\mathscr{M}, {}_{\mathrm{D}}f_!\mathscr{M}$ have bounded amplitude.

We will now give an explicit representative of the pushforward. Recall that the Spencer complex $\operatorname{Sp}(\widetilde{\mathscr{D}}_X)$, which was defined in A.5.2, is a complex of locally free left $\widetilde{\mathscr{D}}_X$ -modules (hence locally free $\widetilde{\mathscr{O}}_X$ -modules). Denote by $\operatorname{Sp}_{X \to Y}(\widetilde{\mathscr{D}}_X)$ the complex $\operatorname{Sp}(\widetilde{\mathscr{D}}_X) \otimes_{\widetilde{\mathscr{O}}_X} \widetilde{\mathscr{D}}_{X \to Y}$ (the left $\widetilde{\mathscr{O}}_X$ -structure on each factor is used for the tensor product). It is a complex of $(\widetilde{\mathscr{D}}_X, f^{-1}\widetilde{\mathscr{D}}_Y)$ -bimodules: the right $f^{-1}\widetilde{\mathscr{D}}_Y$ structure is the trivial one; the left $\widetilde{\mathscr{D}}_X$ -structure is that defined by Exercise A.16(1).

Exercise A.38 (The relative Spencer complex).

(1) Show that $\operatorname{Sp}_{X \to Y}(\widetilde{\mathscr{D}}_X)$ is a resolution of $\widetilde{\mathscr{D}}_{X \to Y}$ as a bimodule.

(2) Show that the terms of the complex $\operatorname{Sp}_{X \to Y}(\widetilde{\mathscr{D}}_X)$ are locally free left $\widetilde{\mathscr{D}}_X$ -modules. [*Hint*: use Exercise A.18(4).]

(3) Show that $\operatorname{gr}^F \mathscr{D}_{X \to Y} = R_F \mathscr{D}_{X \to Y} / z R_F \mathscr{D}_{X \to Y}$ is identified with $\pi^* \operatorname{Sym} \Theta_Y$ as a graded (Sym Θ_X)-module (see Exercise A.4). For example, if $Y = \operatorname{pt}$, so that $\mathscr{D}_{X \to Y} = \mathscr{O}_X$, $\operatorname{gr}^F \mathscr{O}_X = \mathscr{O}_X$ is regarded as a (Sym Θ_X)-module: in local coordinates, we have Sym $\Theta_X = \mathbb{C}\{x_1, \ldots, x_n\}[\xi_1, \ldots, \xi_n]$ and

 $\mathbb{C}\{x_1,\ldots,x_n\}=\mathbb{C}\{x_1,\ldots,x_n\}[\xi_1,\ldots,\xi_n]/(\xi_1,\ldots,\xi_n).$

(4) For $f = \text{Id} : X \to X$, the complex $\text{Sp}_{X \to X}(\mathscr{D}_X) = \text{Sp}(\mathscr{D}_X) \otimes_{\mathscr{O}_X} \mathscr{D}_X$ is a resolution of $\mathscr{D}_{X \to X} = \mathscr{D}_X$ as a left and right \mathscr{D}_X -module (notice that the left structure of \mathscr{D}_X is used for the tensor product).

(5) For $f: X \to \text{pt}$, the complex $\text{Sp}_{X \to \text{pt}}(\mathscr{D}_X) = \text{Sp}_X(\mathscr{D}_X)$ is a resolution of $\mathscr{D}_{X \to \text{pt}} = \mathscr{O}_X$.

^{2.} Recall that, if $\widetilde{\mathscr{D}}_X = R_F \mathscr{D}_X$, then $\mathsf{Mod}(\widetilde{\mathscr{D}}_X) := \mathsf{Modgr}(R_F \mathscr{D}_X)$.

Corollary A.8.5. We have

$${}_{\mathrm{D}}f_*\mathscr{M} = \mathbf{R}f_*\big(\mathscr{M} \otimes_{\widetilde{\mathscr{D}}_X} \mathrm{Sp}_{X \to Y}(\widetilde{\mathscr{D}}_X)\big), \quad {}_{\mathrm{D}}f_!\mathscr{M} = \mathbf{R}f_!\big(\mathscr{M} \otimes_{\widetilde{\mathscr{D}}_X} \mathrm{Sp}_{X \to Y}(\widetilde{\mathscr{D}}_X)\big). \quad \Box$$

Example A.8.6 (Pushforward by a constant map). For Y = pt, and \mathscr{M} a right $\widetilde{\mathscr{D}}_X$ -module, we have

$${}_{\mathrm{D}}f_*\mathscr{M} = \mathbf{R}\Gamma(X, \mathrm{DR}\,\mathscr{M}), \quad {}_{\mathrm{D}}f_!\mathscr{M} = \mathbf{R}\Gamma_c(X, \mathrm{DR}\,\mathscr{M}).$$

Example A.8.7 (Pushforward by a projection). If $X = Y \times T$ and f is the projection $Y \times T \to Y$, denote by $\Theta_{X/Y}$ the sheaf of *relative* tangent vector fields, i.e., which do not contain \eth_{y_j} in their local expression in coordinates adapted to the product $Y \times T$.

The relative Spencer complex $\widetilde{\mathscr{D}}_{X/Y} \otimes_{\widetilde{\mathscr{O}}_X} \wedge^{-\bullet} \widetilde{\Theta}_{X/Y}$ is defined in the same way as its absolute analogue, and is a resolution of $\widetilde{\mathscr{O}}_X$ as a left $\widetilde{\mathscr{D}}_{X/Y}$ -module. As a consequence, $\widetilde{\mathscr{D}}_{X/Y} \otimes_{\widetilde{\mathscr{O}}_X} \wedge^{-\bullet} \widetilde{\Theta}_{X/Y} \otimes_{f^{-1}\widetilde{\mathscr{O}}_Y} f^{-1} \widetilde{\mathscr{D}}_Y$ is also a resolution of $\widetilde{\mathscr{O}}_X \otimes_{f^{-1}\widetilde{\mathscr{O}}_Y} f^{-1} \widetilde{\mathscr{D}}_Y = \widetilde{\mathscr{D}}_{X \to Y}$ as a bimodule by locally free left $\widetilde{\mathscr{D}}_X$ -modules. By identifying $\widetilde{\mathscr{D}}_X$ with $\widetilde{\mathscr{D}}_{X/Y} \otimes_{f^{-1}\widetilde{\mathscr{O}}_Y} f^{-1} \widetilde{\mathscr{D}}_Y$, we can also write this resolution as $\widetilde{\mathscr{D}}_X \otimes_{\widetilde{\mathscr{O}}_X} \wedge^{-\bullet} \widetilde{\Theta}_{X/Y}$. We moreover have a canonical quasi-isomorphism as bimodules

$$\begin{aligned} \operatorname{Sp}_{X \to Y}(\widetilde{\mathscr{D}}_X) &= \left(\widetilde{\mathscr{D}}_X \otimes_{\widetilde{\mathscr{O}}_X} \wedge^{-\bullet} \widetilde{\Theta}_{X/Y}\right) \otimes_{f^{-1}\widetilde{\mathscr{O}}_Y} f^{-1} \left(\wedge^{-\bullet} \widetilde{\Theta}_Y \otimes_{\widetilde{\mathscr{O}}_Y} \widetilde{\mathscr{D}}_Y \right) \\ &= \left(\widetilde{\mathscr{D}}_X \otimes_{\widetilde{\mathscr{O}}_X} \wedge^{-\bullet} \widetilde{\Theta}_{X/Y}\right) \otimes_{f^{-1}\widetilde{\mathscr{D}}_Y} f^{-1} \left(\operatorname{Sp}_Y(\widetilde{\mathscr{D}}_Y) \otimes_{\widetilde{\mathscr{O}}_Y} \widetilde{\mathscr{D}}_Y\right) \\ &\stackrel{\sim}{\longrightarrow} \left(\widetilde{\mathscr{D}}_X \otimes_{\widetilde{\mathscr{O}}_X} \wedge^{-\bullet} \widetilde{\Theta}_{X/Y}\right) \otimes_{f^{-1}\widetilde{\mathscr{D}}_Y} f^{-1} \widetilde{\mathscr{D}}_{Y \to Y} \\ &= \widetilde{\mathscr{D}}_X \otimes_{\widetilde{\mathscr{O}}_X} \wedge^{-\bullet} \widetilde{\Theta}_{X/Y}. \end{aligned}$$

Corollary A.8.5 now reads

(A.8.7*)
$${}_{\mathrm{D}}f_*\mathscr{M} = \mathbf{R}f_*(\mathscr{M} \otimes_{\widetilde{\mathscr{O}}_X} \wedge^{-\bullet} \widetilde{\Theta}_{X/Y}), {}_{\mathrm{D}}f_!\mathscr{M} = \mathbf{R}f_!(\mathscr{M} \otimes_{\widetilde{\mathscr{O}}_X} \wedge^{-\bullet} \widetilde{\Theta}_{X/Y}),$$

where the right $\widetilde{\mathscr{D}}_Y$ structure is naturally induced from that of $f^{-1}\widetilde{\mathscr{D}}_Y$ on \mathscr{M} .

Example A.8.8 (Pushforward by a graph inclusion). Let $g: X \to \mathbb{C}$ be a holomorphic function and let $\iota_g: X \to X \times \mathbb{C}$ denotes the graph embedding of g, with coordinate t on the factor \mathbb{C} . Let \mathscr{M} be a right $\widetilde{\mathscr{D}}_X$ -module. Then ${}_{\mathsf{D}}\iota_{g*}\mathscr{M} \simeq \iota_{g*}\mathscr{M}[\eth_t]$ with the right $\widetilde{\mathscr{D}}_{X\times\mathbb{C}}$ -action defined locally be the following formulas (recall that for a holomorphic function h(x, t, z), the bracket $[\eth_t^k, h]$ can be written as $\sum_{j < k} a_{h,j}(x, t, z) \eth_t^j$):

$$(m \otimes \eth_t^k) \cdot \eth_{x_i} = m\eth_{x_i} \otimes \eth_t^k - m \frac{\partial g}{\partial x_i} \otimes \eth_t^{k+1}$$

(A.8.8*)
$$(m \otimes \eth_t^k) \cdot \eth_t = m \otimes \eth_t^{k+1},$$

$$(m \otimes \eth_t^k) \cdot h(\boldsymbol{x}, t, z) = \sum_{j < k} ma_{h,j}(\boldsymbol{x}, g, z) \otimes \eth_t^j.$$

Exercise A.39. Extend ${}_{\mathrm{D}}f_*$ and ${}_{\mathrm{D}}f_!$ as functors from $\mathsf{D}^+(\widetilde{\mathscr{D}}_X)$ (or $\mathsf{D}^{\mathrm{b}}(\widetilde{\mathscr{D}}_X)$) to $\mathsf{D}^+(\widetilde{\mathscr{D}}_Y)$. [*Hint*: replace first $\mathscr{M}^{\bullet} \otimes_{\widetilde{\mathscr{D}}_X} \operatorname{Sp}_{X \to Y}(\widetilde{\mathscr{D}}_X)$ with the associated single complex.]

As in Remark A.8.4(3), show that if \mathscr{M}^{\bullet} has bounded amplitude, then so has ${}_{\mathrm{D}}f_{!}\mathscr{M}^{\bullet}$.

Recall that the flabby sheaves are injective with respect to the functor f_* (direct image) in the category of sheaves (of modules over a ring) and, being *c*-soft, are injective with respect to the functor $f_!$ (direct image with proper support). The Godement canonical resolution is an explicit functorial flabby resolution for any sheaf.

Definition A.8.9 (Godement resolution).

(1) The Godement functor \mathbb{C}^0 (see [God64, p. 167]) associates to any sheaf \mathscr{L} the flabby sheaf $\mathbb{C}^0(\mathscr{L})$ of its discontinuous sections and to any morphism the corresponding family of germs of morphisms. Then there is a canonical injection $\mathscr{L} \to \mathbb{C}^0(\mathscr{L})$.

(2) Set inductively (see [God64, p. 168]) $\mathcal{Z}^{0}(\mathscr{L}) = \mathscr{L}, \mathcal{Z}^{k+1}(\mathscr{L}) = \mathbb{C}^{k}(\mathscr{L})/\mathcal{Z}^{k}(\mathscr{L}),$ $\mathbb{C}^{k+1}(\mathscr{L}) = \mathbb{C}^{0}(\mathcal{Z}^{k+1}(\mathscr{L}))$ and define $\delta : \mathbb{C}^{k}(\mathscr{L}) \to \mathbb{C}^{k+1}(\mathscr{L})$ as the composition $\mathbb{C}^{k}(\mathscr{L}) \to \mathcal{Z}^{k+1}(\mathscr{L}) \to \mathbb{C}^{0}(\mathcal{Z}^{k+1}(\mathscr{L})).$ This defines a complex $(\mathbb{C}^{\bullet}(\mathscr{L}), \delta)$, that we will denote as $(\text{God}^{\bullet}\mathscr{L}, \delta).$

(3) Given any sheaf \mathscr{L} , $(\operatorname{God}^{\bullet} \mathscr{L}, \delta)$ is a resolution of \mathscr{L} by flabby sheaves. For a complex $(\mathscr{L}^{\bullet}, d)$, we regard $\operatorname{God}^{\bullet} \mathscr{L}^{\bullet}$ as a double complex ordered as written, i.e., with differential $(\delta_i, (-1)^i d_j)$ on $\operatorname{God}^i \mathscr{L}^j$, and therefore also as the associated simple complex.

Corollary A.8.10. We have, by taking the single complex associated to the double complex, and for $\star = *$ or $\star = !$,

$${}_{\mathsf{D}}f_{\star}\mathscr{M} = f_{\star}\operatorname{God}^{\bullet}(\mathscr{M} \otimes_{\widetilde{\mathscr{Q}}_{Y}}\operatorname{Sp}_{X \to Y}(\widetilde{\mathscr{D}}_{X})).$$

Exercise A.40 (Compatibility with the Godement functor). (1) Show by induction on k that, for every $k \ge 0$, the functor God^k is exact (see [God64, p. 168]). Given an exact sequence $0 \to \mathscr{L}' \to \mathscr{L} \to \mathscr{L}'' \to 0$ of sheaves, show that we have an exact sequence of complexes

$$0 \longrightarrow \operatorname{God}^{\bullet} \mathscr{L}' \longrightarrow \operatorname{God}^{\bullet} \mathscr{L} \longrightarrow \operatorname{God}^{\bullet} \mathscr{L}'' \longrightarrow 0.$$

Similarly, show that the functors $f_{\star} \operatorname{God}^k$ are exact (with $\star = * \text{ or } \star = !$) and deduce an exact sequence of complexes

$$0 \longrightarrow f_{\star} \operatorname{God}^{\bullet} \mathscr{L}' \longrightarrow f_{\star} \operatorname{God}^{\bullet} \mathscr{L} \longrightarrow f_{\star} \operatorname{God}^{\bullet} \mathscr{L}'' \longrightarrow 0.$$

Deduce also that, for every $k \ge 0$ and a complex \mathscr{L}^{\bullet} , we have

$$\mathscr{H}^i(f_\star \operatorname{God}^k \mathscr{L}^{\bullet}) \simeq f_\star \operatorname{God}^k \mathscr{H}^i \mathscr{L}^{\bullet}.$$

(2) Show that, if \mathscr{L} and \mathscr{F} are $\widetilde{\mathscr{O}}_X$ -modules and if \mathscr{F} is locally free, then we have a natural inclusion $\mathcal{C}^0(\mathscr{L}) \otimes_{\widetilde{\mathscr{O}}_X} \mathscr{F} \hookrightarrow \mathcal{C}^0(\mathscr{L} \otimes_{\widetilde{\mathscr{O}}_X} \mathscr{F})$, which is an equality if \mathscr{F} has finite rank. More generally, show by induction that we have a natural morphism $\mathcal{C}^k(\mathscr{L}) \otimes_{\widetilde{\mathscr{O}}_X} \mathscr{F} \to \mathcal{C}^k(\mathscr{L} \otimes_{\widetilde{\mathscr{O}}_X} \mathscr{F})$, which is an equality if \mathscr{F} has finite rank.

(3) With the same assumptions, show that both complexes $\operatorname{God}^{\bullet}(\mathscr{L}) \otimes_{\widetilde{\mathcal{O}}_{X}} \mathscr{F}$ and $\operatorname{God}^{\bullet}(\mathscr{L} \otimes_{\widetilde{\mathcal{O}}_{X}} \mathscr{F})$ are resolutions of $\mathscr{L} \otimes_{\widetilde{\mathcal{O}}_{X}} \mathscr{F}$. Conclude that the natural morphism of complexes $\operatorname{God}^{\bullet}(\mathscr{L}) \otimes_{\widetilde{\mathcal{O}}_{X}} \mathscr{F} \to \operatorname{God}^{\bullet}(\mathscr{L} \otimes_{\widetilde{\mathcal{O}}_{X}} \mathscr{F})$ is a quasi-isomorphism, and an equality if \mathscr{F} has finite rank.

(4) Let \mathscr{M} be a right $\widetilde{\mathscr{D}}_X$ -module. Show that the natural morphism of complex

 $(\mathrm{God}^{\bullet}\mathscr{M})\otimes_{\widetilde{\mathscr{O}}_{X}}\mathrm{Sp}\,\widetilde{\mathscr{D}}_{X}\longrightarrow\mathrm{God}^{\bullet}(\mathscr{M}\otimes_{\widetilde{\mathscr{O}}_{X}}\mathrm{Sp}\,\widetilde{\mathscr{D}}_{X})$

is a quasi-isomorphism.

(5) Let \mathscr{M} be a right \mathscr{D}_X -module. Show that

$$\operatorname{Sp}(\operatorname{God}^{\bullet} \mathscr{M}) = \operatorname{God}^{\bullet} \operatorname{Sp} \mathscr{M}.$$

(6) If $f: X = Y \times T \to Y$ is the projection, show that, for $\star = *, !$,

$${}_{\mathrm{D}}f_{\star}\mathscr{M} = f_{\star}\operatorname{God}^{\bullet}\left(\mathscr{M}\otimes_{\widetilde{\mathscr{O}}_{Y}}\wedge^{-\bullet}\widetilde{\Theta}_{X/Y}\right).$$

[*Hint*: use Example A.8.7.]

Exercise A.41 (Restriction to z = 1).

(1) Show that the Godement functor applied to sheaves of \mathbb{C} -modules restricts, for z = 1, to the Godement functor applied to sheaves of \mathbb{C} -vector spaces.

(2) Show that $\operatorname{Sp}_{X \to Y}(\mathscr{D}_X) = \operatorname{Sp}_{X \to Y}(\widetilde{\mathscr{D}}_X)/(z-1)\operatorname{Sp}_{X \to Y}(\widetilde{\mathscr{D}}_X).$

(3) Conclude that ${}_{\mathrm{D}}f_{\star}\mathscr{M}/(z-1){}_{\mathrm{D}}f_{\star}\mathscr{M} = {}_{\mathrm{D}}f_{\star}(\mathscr{M}/(z-1)\mathscr{M})$ and, for every i, $\mathscr{H}^{i}{}_{\mathrm{D}}f_{\star}\mathscr{M}/(z-1)\mathscr{H}^{i}{}_{\mathrm{D}}f_{\star}\mathscr{M} = \mathscr{H}^{i}{}_{\mathrm{D}}f_{\star}(\mathscr{M}/(z-1)\mathscr{M}) \ (\star = *, !).$

Exercise A.42 (Computation of the pushforward with differential forms)

Let $f: X \to Y$ be a holomorphic map, let \mathscr{M} be a right $\widetilde{\mathscr{D}}_X$ -module and let $\mathscr{M}^{\text{left}}$ be the associated left $\widetilde{\mathscr{D}}_X$ -module. As $\widetilde{\mathscr{D}}_{X \to Y}$ is a left $\widetilde{\mathscr{D}}_X$ -module,

$$\mathscr{M}^{\text{left}} \otimes_{\widetilde{\mathscr{O}}_X} \widetilde{\mathscr{D}}_{X \to Y} = \mathscr{M} \otimes_{f^{-1}\widetilde{\mathscr{O}}_Y} f^{-1} \widetilde{\mathscr{D}}_Y$$

has a natural structure of left $\widetilde{\mathscr{D}}_X$ -module (by setting $\xi(\mu \otimes \mathbf{1}_Y) = \xi\mu \otimes \mathbf{1}_Y + \mu \otimes Tf(\xi)$, see Exercise A.16(2)) and of course a compatible structure of right $f^{-1}\widetilde{\mathscr{D}}_Y$ -module. It is often convenient to compute the pushforward ${}_{\mathrm{D}}f_{\star}\mathscr{M}$ with a complex of differential forms (de Rham) and not a complex with poly-vector fields (Spencer). This exercise gives such a formula.

(1) Show that the de Rham complex

$$\widetilde{\Omega}_{X}^{\bullet}(\mathscr{M}^{\mathrm{left}} \otimes_{\widetilde{\mathscr{O}}_{X}} \widetilde{\mathscr{D}}_{X \to Y}) = \widetilde{\Omega}_{X}^{\bullet}(\mathscr{M}^{\mathrm{left}} \otimes_{\widetilde{\mathscr{O}}_{X}} f^{*} \widetilde{\mathscr{D}}_{Y}) = \widetilde{\Omega}_{X}^{\bullet}(\mathscr{M}^{\mathrm{left}} \otimes_{f^{-1} \widetilde{\mathscr{O}}_{Y}} f^{-1} \widetilde{\mathscr{D}}_{Y})$$

is isomorphic to $\mathscr{M} \otimes_{\widetilde{\mathscr{D}}_X} \operatorname{Sp}_{X \to Y}(\widetilde{\mathscr{D}}_X)[-n]$, as a complex of right $f^{-1}\widetilde{\mathscr{D}}_Y$ -modules, by using the isomorphism (see Lemma A.5.4)

$$\omega \otimes \mu \otimes \xi \otimes \mathbf{1}_Y \longmapsto \varepsilon(k+1)\omega(\xi \wedge \bullet) \otimes \mu \otimes \mathbf{1}_Y \quad (\xi \in \wedge^{n-k} \widetilde{\Theta}_X).$$

[*Hint*: see Exercise A.28.]

(2) Check that the connection induced by the left $\widetilde{\mathscr{D}}_X$ -module structure on $\mathscr{M}^{\text{left}} \otimes_{\widetilde{\mathscr{D}}_X} f^* \widetilde{\mathscr{D}}_Y$ is $\widetilde{\nabla} \otimes \text{Id} + \text{Id}_{\mathscr{M}^{\text{left}}} \otimes \widetilde{\nabla}^X$, where $\widetilde{\nabla}^X$ is obtained from the universal connection $\widetilde{\nabla}^Y$ on $\widetilde{\mathscr{D}}_Y$ by the formula A.7.1.

(3) Conclude that, for $\star = *, !,$

$${}_{\mathrm{D}}f_{\star}\mathscr{M} = \mathbf{R}f_{\star}\widetilde{\Omega}^{\bullet}_{X}(\mathscr{M}^{\mathrm{left}} \otimes_{f^{-1}\widetilde{\mathscr{O}}_{Y}} f^{-1}\widetilde{\mathscr{D}}_{Y})[n]$$

This is the complex of left $\widetilde{\mathscr{D}}_Y$ -modules associated to the double complex

$$f_{\star} \operatorname{God}^{\bullet} \widetilde{\Omega}^{\bullet}_{X}(\mathscr{M}^{\operatorname{left}} \otimes_{f^{-1} \widetilde{\mathscr{O}}_{Y}} f^{-1} \widetilde{\mathscr{D}}_{Y})[n].$$

Show that this complex is quasi-isomorphic to the complex

$$f_{\star} \widetilde{\Omega}^{\bullet}_{X}(\operatorname{God}^{\bullet} \mathscr{M}^{\operatorname{left}} \otimes_{f^{-1} \widetilde{\mathscr{O}}_{Y}} f^{-1} \widetilde{\mathscr{D}}_{Y})[n].$$

[*Hint*: use Exercise A.40.]

(4) Show that the latter complex is the single complex associated with the double complex having terms $f_{\star}(\widetilde{\Omega}_{X}^{n+i} \otimes \operatorname{God}^{j} \mathscr{M}^{\operatorname{left}}) \otimes_{\widetilde{\mathscr{O}}_{Y}} \widetilde{\mathscr{D}}_{Y}$ and first differential $f_{\star}(\widetilde{\nabla} \otimes \operatorname{Id} + \operatorname{Id}_{\mathscr{M}^{\operatorname{left}}} \otimes \widetilde{\nabla}^{X})$ (the second differential is induced by the Godement differential).

Remark A.8.11 (C^{∞} computation of the pushforward). In the setting of the previous exercise, it is often more convenient to replace the Godement resolution by a Dolbeault resolution. We then have

$${}_{\mathrm{D}}f_{\star}\mathscr{M} = f_{\star}\widetilde{\mathscr{E}}_{X}^{\bullet}(\mathscr{M}^{\mathrm{left}} \otimes_{f^{-1}\widetilde{\mathscr{O}}_{Y}} f^{-1}\widetilde{\mathscr{D}}_{Y})[n],$$

where the differential in the latter complex is obtained in the usual way from the holomorphic differential of Exercise A.42(2) and the anti-holomorphic differential d".

Example A.8.12 (C^{∞} computation of the pushforward by a projection)

As in Example A.8.7, let $f: X = Y \times T \to Y$ be the projection. Setting $p = \dim T$, the relative version of Exercise A.28(2) gives an isomorphism, for $\mathscr{M} = \mathscr{M}^{\text{right}}$,

$$\mathscr{M} \otimes \wedge^k \widetilde{\Theta}_{X/Y} \simeq f^{-1} \widetilde{\omega}_Y \otimes (\widetilde{\omega}_{X/Y} \otimes \mathscr{M}^{\text{left}} \otimes \wedge^k \widetilde{\Theta}_{X/Y}) \xrightarrow{\sim} f^{-1} \widetilde{\omega}_Y \otimes \widetilde{\Omega}_{X/Y}^{p-k} \otimes \mathscr{M}^{\text{left}},$$

an denoting by $\widetilde{\mathscr{E}}^{\bullet}_{X/Y}$ the complex of relative C^{∞} differential forms (with relative differentials \widetilde{d}' and d''), (A.8.7*) reads

$$(A.8.12*) \qquad {}_{\scriptscriptstyle D}f_{\star}\mathscr{M} = \widetilde{\omega}_Y \otimes_{\widetilde{\mathscr{O}}_Y} f_{\star}(\widetilde{\mathscr{E}}_{X/Y}^{\bullet} \otimes_{\widetilde{\mathscr{O}}_X} \mathscr{M}^{\rm left})[p] \quad \star = * \text{ or } !.$$

The Lefschetz morphism. As a consequence of Remark A.8.11, given a (1,1)-form $\tilde{\eta} \in \Gamma(X, \widetilde{\mathscr{E}}_X^{(1,1)})$ which \tilde{d} -closed (equivalently, \tilde{d}' and d''-closed), there is a well-defined morphism ($\star = *$ or $\star = !$)

$$\widetilde{\eta} \wedge : {}_{\mathrm{D}} f_{\star} \mathscr{M} \longrightarrow {}_{\mathrm{D}} f_{\star} \mathscr{M}[2],$$

induced by $\tilde{\eta} \wedge : f_{\star} \tilde{\mathscr{E}}_{X}^{\bullet} \to f_{\star} \tilde{\mathscr{E}}_{X}^{\bullet}[2]$. It is clearly functorial with respect to \mathscr{M} , that is, given any morphism $\varphi : \mathscr{M}_{1} \to \mathscr{M}_{2}$, the following diagram commutes (where \star is either for \star or for !):

Definition A.8.13 (The Lefschetz morphism attached to a closed (1, 1)-form)

The Lefschetz morphism associated to a (usual) closed (1, 1)-form η on X is the morphism

$$\mathcal{L}_{\eta} := \frac{1}{z} \eta \wedge : {}_{\mathrm{D}} f_{\star} \mathscr{M} \longrightarrow {}_{\mathrm{D}} f_{\star} \mathscr{M}[2](1).$$

It is functorial with respect to \mathcal{M}

Remark A.8.14 (The Lefschetz morphism attached to a line bundle)

Let $f: X \to Y$ be any morphism between complex manifolds and let \mathscr{L} be a line bundle on X, with Chern class $c_1(\mathscr{L}) \in H^2(X,\mathbb{Z})$. We will define a Lefschetz morphism

$$\mathcal{L}_{\mathscr{L}}: {}_{\mathsf{D}}f_{\star}\mathscr{M} \longrightarrow {}_{\mathsf{D}}f_{\star}\mathscr{M}[2](1).$$

We can choose a closed (1, 1)-form η on X whose class in $H^2(X, \mathbb{C})$ is equal to the complexified class $c_1(\mathscr{L})_{\mathbb{C}}$. We regard η as a closed relative (1, 1)-form with respect to the projection. As noticed in Remark A.5.7, namely by using a similar argument, the action of L_{η} given in Definition A.8.13 only depends on the class of η in $H^2(X, \mathbb{C})$. Notice also that, since η has degree two, wedging (or contracting) with η on the left or on the right gives the same result.

We thus define $L_{\mathscr{L}}$ as L_{η} . This operator only depends on $c_1(\mathscr{L})_{\mathbb{C}}$. It is functorial with respect to \mathscr{M} .

Remark A.8.15 (Restriction to z = 1 of the Lefschetz morphism)

It is obvious that the restriction to z = 1 of the morphism $L_{\mathscr{L}}$ is the morphism

$$\mathcal{L}_{\mathscr{L}}: {}_{\mathcal{D}}f_{\star}\mathcal{M} \longrightarrow {}_{\mathcal{D}}f_{\star}\mathcal{M}[2].$$

Other properties of the pushforward functor

Exercise A.43 (Pushforward of induced $\widetilde{\mathscr{D}}$ -modules). Let \mathscr{L} be an $\widetilde{\mathscr{O}}_X$ -module and let $\mathscr{M} = \mathscr{L} \otimes_{\widetilde{\mathscr{O}}_X} \widetilde{\mathscr{D}}_X$ be the associated induced right $\widetilde{\mathscr{D}}_X$ -module. Let $f: X \to Y$ be a holomorphic map.

(1) Show that $\mathscr{L} \otimes_{\widetilde{\mathscr{O}}_X} \operatorname{Sp}_{X \to Y}(\widetilde{\mathscr{D}}_X) \to \mathscr{L} \otimes_{\widetilde{\mathscr{O}}_X} \widetilde{\mathscr{D}}_{X \to Y}$ is a quasi-isomorphism. [*Hint*: use that $\widetilde{\mathscr{D}}_X$ is $\widetilde{\mathscr{O}}_X$ -locally free.]

(2) Deduce that

$$\mathscr{M} \otimes_{\widetilde{\mathscr{D}}_X} \operatorname{Sp}_{X \to Y}(\widetilde{\mathscr{D}}_X) = \mathscr{M} \otimes_{\widetilde{\mathscr{D}}_X} \widetilde{\mathscr{D}}_{X \to Y} = \mathscr{L} \otimes_{f^{-1} \widetilde{\mathscr{O}}_Y} f^{-1} \widetilde{\mathscr{D}}_Y.$$

(3) Show that ${}_{\mathrm{D}}f_!(\mathscr{L} \otimes_{\widetilde{\mathscr{O}}_X} \widetilde{\mathscr{D}}_X)$ is quasi-isomorphic to $(\mathbf{R}f_!\mathscr{L}) \otimes_{\widetilde{\mathscr{O}}_Y} \widetilde{\mathscr{D}}_Y$. [*Hint*: use the projection formula.]

Exercise A.44 (Pushforward of $\widehat{\mathscr{D}}$ -modules and pushforward of $\widehat{\mathscr{O}}$ -modules)

Let $f: X \to Y$ be a holomorphic map and let \mathscr{M} be a right $\widehat{\mathscr{O}}_X$ -module. It is also an $\widetilde{\mathscr{O}}_X$ -module. The goal of this exercise is to exhibit natural $\widetilde{\mathscr{O}}_Y$ -linear morphisms $(\star = *, !)$

$$R^{i}f_{\star}\mathscr{M}\longrightarrow \mathscr{H}^{i}{}_{\mathrm{D}}f_{\star}\mathscr{M}.$$

(1) Show that $\widetilde{\mathscr{D}}_X \otimes_{f^{-1}\widetilde{\mathscr{O}}_Y} f^{-1}\widetilde{\mathscr{D}}_Y$ has a natural global section 1.

(2) Show that there is a natural $f^{-1}\widetilde{\mathcal{O}}_Y$ -linear morphism of complexes

 $\mathcal{M} \longrightarrow \mathcal{M} \otimes_{\widetilde{\mathscr{Q}}_{X}} \operatorname{Sp}_{X \to Y}(\widetilde{\mathscr{Q}}_{X}), \quad m \longmapsto m \otimes \mathbf{1},$

where \mathscr{M} is considered as a complex with \mathscr{M} in degree 0 and all other terms equal to 0, so the differential are all equal to 0. [*Hint*: use Exercise A.18(3) to identify $\operatorname{Sp}_{X\to Y}^{0}(\widetilde{\mathscr{D}}_{X}) = \widetilde{\mathscr{D}}_{X} \otimes_{\widetilde{\mathscr{O}}_{X}} \widetilde{\mathscr{D}}_{X\to Y}$ with its twisted left $\widetilde{\mathscr{D}}_{X}$ -structure (denoted by $\widetilde{\mathscr{D}}_{X\to Y} \otimes_{\widetilde{\mathscr{O}}_{X}} \widetilde{\mathscr{D}}_{X}$ in loc. cit.) with $\widetilde{\mathscr{D}}_{X} \otimes_{\widetilde{\mathscr{O}}_{X}} \widetilde{\mathscr{D}}_{X\to Y}$, where the tensor product uses the right $\widetilde{\mathscr{O}}_{X}$ -structure on $\widetilde{\mathscr{D}}_{X}$ and the left $\widetilde{\mathscr{D}}_{X}$ structure is the trivial one, and then with $\widetilde{\mathscr{D}}_{X} \otimes_{f^{-1}\widetilde{\mathscr{O}}_{Y}} f^{-1}\widetilde{\mathscr{O}}_{Y}$ with trivial left $\widetilde{\mathscr{D}}_{X}$ -structure and tensor product using the right $\widetilde{\mathscr{O}}_{X}$ -structure of $\widetilde{\mathscr{D}}_{X}$. Identify then $\mathscr{M} \otimes_{\widetilde{\mathscr{D}}_{X}} (\widetilde{\mathscr{D}}_{X} \otimes_{\widetilde{\mathscr{O}}_{X}} \widetilde{\mathscr{D}}_{X\to Y})$ with $\mathscr{M} \otimes_{f^{-1}\widetilde{\mathscr{O}}_{Y}} f^{-1}\widetilde{\mathscr{D}}_{Y}$.] (3) Conclude with the existence of the desired morphisms.

Exercise A.45 (Grading and pushforward, right case). Let $(\mathcal{M}, F_{\bullet}\mathcal{M})$ be a filtered right \mathscr{D}_X -module. Set $\mathscr{M} = R_F \mathscr{M}$, so that $\operatorname{gr}_{\mathcal{M}}^F = \mathscr{M}/z \mathscr{M}$.

(1) Show that

$$(\mathscr{M} \otimes_{R_F \mathscr{D}_X} \operatorname{Sp} R_F \mathscr{D}_{X \to Y}) \otimes_{\mathbb{C}[z]} \mathbb{C}[z] / z \mathbb{C}[z] \simeq \operatorname{gr}^F \mathscr{M} \otimes^{\boldsymbol{L}}_{\operatorname{Sym} \Theta_X} f^* \operatorname{Sym} \Theta_Y.$$

[*Hint*: Use the associativity of \otimes and Exercise A.38(3).]

(2) Assume that ${}_{\mathrm{D}}f_{\star}\mathcal{M}$ is *strict* (i.e., the complex of Corollary A.8.10 is strict in the sense of Definition A.2.5 or 8.2.2). Show that, for every *i*, we have, as graded modules

$$\operatorname{gr}^{F}\mathscr{H}^{i}{}_{{}_{\mathrm{D}}}f_{\star}\mathfrak{M}\simeq\mathscr{H}^{i}\boldsymbol{R}f_{\star}(\operatorname{gr}^{F}\mathfrak{M}\otimes^{\boldsymbol{L}}_{\operatorname{Sym}\Theta_{X}}f^{*}\operatorname{Sym}\Theta_{Y}).$$

A.8.b. Composition of direct images and the Leray spectral sequence for right $\widetilde{\mathscr{D}}$ -modules

We compare the result of the pushforward functor by the composition of two maps with the pushforward by the second map of the pushforward by the first map. We find an isomorphism at the level of derived categories, that we will translate as a spectral sequence, which is the $\tilde{\mathscr{D}}$ -module analogue of the Leray spectral sequence (see Section A.11.c).

Theorem A.8.16 (Composition of direct images). Let

$$f: X \longrightarrow Y \quad and \quad f': Y \longrightarrow Z$$

be two holomorphic maps. There is a functorial canonical isomorphism of functors

$${}_{\mathrm{D}}(f' \circ f)_! = {}_{\mathrm{D}}f'_! {}_{\mathrm{D}}f_!.$$

If f is proper, we also have

$${}_{\mathrm{D}}(f' \circ f)_* = {}_{\mathrm{D}}f'_*{}_{\mathrm{D}}f_*.$$

Proof. We have a natural morphism of complexes

$$\operatorname{Sp}_{X \to Y}(\widetilde{\mathscr{D}}_X) \otimes_{f^{-1}\widetilde{\mathscr{D}}_Y} f^{-1} \operatorname{Sp}_{Y \to Z}(\widetilde{\mathscr{D}}_Y) \longrightarrow \operatorname{Sp}_{X \to Y}(\widetilde{\mathscr{D}}_X) \otimes_{f^{-1}\widetilde{\mathscr{D}}_Y} f^{-1}\widetilde{\mathscr{D}}_{Y \to Z},$$

lifting the identity morphism of $\widetilde{\mathscr{D}}_{X\to Y} \otimes_{f^{-1}\widetilde{\mathscr{D}}_Y} f^{-1}\widetilde{\mathscr{D}}_{Y\to Z}$, obtained by using the augmentation morphism $\operatorname{Sp}_{Y\to Z}(\widetilde{\mathscr{D}}_Y) \to \widetilde{\mathscr{D}}_{Y\to Z}$.

On the one hand, the left-hand term is a resolution (in the category of $(\widetilde{\mathscr{D}}_X, (f' \circ f)^{-1} \widetilde{\mathscr{D}}_Z)$ -bimodules) of $\widetilde{\mathscr{D}}_{X \to Z}$ by locally free $\widetilde{\mathscr{D}}_X$ -modules. Indeed, remark that, as $\operatorname{Sp}_{Y \to Z}(\widetilde{\mathscr{D}}_Y)$ is $\widetilde{\mathscr{D}}_Y$ locally free, one has

$$\begin{split} \operatorname{Sp}_{X \to Y}(\widetilde{\mathscr{D}}_X) \otimes_{f^{-1}\widetilde{\mathscr{D}}_Y} f^{-1} \operatorname{Sp}_{Y \to Z}(\widetilde{\mathscr{D}}_Y) &\xrightarrow{\sim} \widetilde{\mathscr{D}}_{X \to Y} \otimes_{f^{-1}\widetilde{\mathscr{D}}_Y} f^{-1} \operatorname{Sp}_{Y \to Z}(\widetilde{\mathscr{D}}_Y) \\ &= \widetilde{\mathscr{O}}_X \otimes_{f^{-1}\widetilde{\mathscr{O}}_Y} f^{-1} \operatorname{Sp}_{Y \to Z}(\widetilde{\mathscr{D}}_Y) \\ &= \widetilde{\mathscr{O}}_X \otimes_{f^{-1}\widetilde{\mathscr{O}}_Y} f^{-1} \widetilde{\mathscr{D}}_{Y \to Z} \\ &= \widetilde{\mathscr{O}}_X \otimes_{f'^{-1}f^{-1}\widetilde{\mathscr{O}}_Z} f'^{-1} f^{-1} \widetilde{\mathscr{D}}_Z \quad (\widetilde{\mathscr{D}}_{Y \to Z} \text{ is } \widetilde{\mathscr{O}}_Y \text{ locally free}) \\ &= \widetilde{\mathscr{D}}_{X \to Z}. \end{split}$$

On the other hand, there is a natural morphism

$$\operatorname{Sp}_{X \to Y}(\widetilde{\mathscr{D}}_X) \otimes_{f^{-1}\widetilde{\mathscr{D}}_Y} f^{-1}\widetilde{\mathscr{D}}_{Y \to Z} \longrightarrow \operatorname{Sp}_{X \to Z}(\widetilde{\mathscr{D}}_X).$$

Indeed, we have a natural morphism

$$\left[\operatorname{Sp}^{\bullet}(\widetilde{\mathscr{D}}_{X})\otimes_{\widetilde{\mathscr{O}}_{X}}\widetilde{\mathscr{D}}_{X\to Y}\right]\otimes_{f^{-1}\widetilde{\mathscr{D}}_{Y}}f^{-1}\widetilde{\mathscr{D}}_{Y\to Z}\xrightarrow{\sim}\operatorname{Sp}^{\bullet}(\widetilde{\mathscr{D}}_{X})\otimes_{\widetilde{\mathscr{O}}_{X}}\widetilde{\mathscr{D}}_{X\to Z},$$

which is an isomorphism of $(\widetilde{\mathscr{D}}_X, (f' \circ f)^{-1} \widetilde{\mathscr{D}}_Z)$ -bimodules, according to the chain rule (Exercise A.35).

We have found a morphism, lifting the identity,

$$\operatorname{Sp}_{X \to Y}(\widetilde{\mathscr{D}}_X) \otimes_{f^{-1}\widetilde{\mathscr{D}}_Y} f^{-1} \operatorname{Sp}_{Y \to Z}(\widetilde{\mathscr{D}}_Y) \longrightarrow \operatorname{Sp}_{X \to Z}(\widetilde{\mathscr{D}}_X),$$

between two resolutions (in the category of $(\widetilde{\mathscr{D}}_X, (f' \circ f)^{-1} \widetilde{\mathscr{D}}_Z)$ -bimodules) of $\widetilde{\mathscr{D}}_{X \to Z}$ by locally free $\widetilde{\mathscr{D}}_X$ -modules. This morphism is therefore a quasi-isomorphism. We now have, for an object \mathscr{M} of $\mathsf{Mod}(\widetilde{\mathscr{D}}_X)$ or of $\mathsf{D}^+(\widetilde{\mathscr{D}}_X)$ (see Remark A.8.17 for details):

$$\begin{split} {}_{\mathrm{D}}(f'\circ f)_{!}\mathscr{M} &= \mathbf{R}(f'\circ f)_{!} \big(\mathscr{M}\otimes_{\widetilde{\mathscr{D}}_{X}} \operatorname{Sp}_{X\to Z}(\widehat{\mathscr{D}}_{X})\big) \\ &\simeq \mathbf{R}(f'\circ f)_{!} \big(\mathscr{M}\otimes_{\widetilde{\mathscr{D}}_{X}} \operatorname{Sp}_{X\to Y}(\widetilde{\mathscr{D}}_{X})\otimes_{f^{-1}\widetilde{\mathscr{D}}_{Y}} f^{-1}\operatorname{Sp}_{Y\to Z}(\widetilde{\mathscr{D}}_{Y})\big) \\ &\simeq \mathbf{R}f'_{!}\mathbf{R}f_{!} \big(\mathscr{M}\otimes_{\widetilde{\mathscr{D}}_{X}} \operatorname{Sp}_{X\to Y}(\widetilde{\mathscr{D}}_{X})\otimes_{f^{-1}\widetilde{\mathscr{D}}_{Y}} f^{-1}\operatorname{Sp}_{Y\to Z}(\widetilde{\mathscr{D}}_{Y})\big) \\ &\simeq \mathbf{R}f'_{!} \Big[\mathbf{R}f_{!} \big(\mathscr{M}\otimes_{\widetilde{\mathscr{D}}_{X}} \operatorname{Sp}_{X\to Y}(\widetilde{\mathscr{D}}_{X})\big)\otimes_{\widetilde{\mathscr{D}}_{Y}} \operatorname{Sp}_{Y\to Z}(\widetilde{\mathscr{D}}_{Y})\Big] \\ &= {}_{\mathrm{D}}f'_{!} \Big[\mathbf{R}f_{!} \big(\mathscr{M}\otimes_{\widetilde{\mathscr{D}}_{X}} \operatorname{Sp}_{X\to Y}(\widetilde{\mathscr{D}}_{X})\big)\otimes_{\widetilde{\mathscr{D}}_{Y}} \operatorname{Sp}_{Y\to Z}(\widetilde{\mathscr{D}}_{Y})\Big] \end{split}$$

Remark that the analogous result holds with ${}_{\mathrm{D}}f_*$ if f is proper on the support of \mathscr{M} .

This theorem reduces the computation of the direct image by any morphism $f: X \to Y$ by decomposing it as $f = p \circ \iota_f$, where $\iota_f: X \hookrightarrow X \times Y$ denotes the graph inclusion $x \mapsto (x, f(x))$. As ι_f is an embedding, it is proper, so we have ${}_{\mathrm{D}}f_* = {}_{\mathrm{D}}p_{*\mathrm{D}}\iota_{f*}$.

Remark A.8.17. We can make explicit the isomorphism of Theorem A.8.16(6) by using the Godement resolution as follows. We have, for an object \mathscr{M} of $\mathsf{Mod}(\widetilde{\mathscr{D}}_X)$ or of $\mathsf{D}^+(\widetilde{\mathscr{D}}_X)$:

$${}_{\mathrm{D}}(f'\circ f)_{!}\mathscr{M} = (f'\circ f)_{!}\operatorname{God}^{\bullet}\left(\mathscr{M}\otimes_{\widetilde{\mathscr{D}}_{X}}\operatorname{Sp}_{X\to Z}(\widetilde{\mathscr{D}}_{X})\right)$$

$$\simeq (f'\circ f)_{!}\operatorname{God}^{\bullet}\left(\mathscr{M}\otimes_{\widetilde{\mathscr{D}}_{X}}\operatorname{Sp}_{X\to Y}(\widetilde{\mathscr{D}}_{X})\otimes_{f^{-1}\widetilde{\mathscr{D}}_{Y}}f^{-1}\operatorname{Sp}_{Y\to Z}(\widetilde{\mathscr{D}}_{Y})\right)$$

$$(f\circ f)_{!} = (f'\circ f)_{!}\operatorname{God}^{\bullet}\left(\mathscr{M}\otimes_{\widetilde{\mathscr{D}}_{X}}\operatorname{Sp}_{X\to Y}(\widetilde{\mathscr{D}}_{X})\otimes_{f^{-1}\widetilde{\mathscr{D}}_{Y}}f^{-1}\operatorname{Sp}_{Y\to Z}(\widetilde{\mathscr{D}}_{Y})\right)$$

(A.8.17.a)
$$\simeq f'_! \operatorname{God}^{\bullet} f_! \operatorname{God}^{\bullet} (\mathscr{M} \otimes_{\widetilde{\mathscr{D}}_X} \operatorname{Sp}_{X \to Y} (\mathscr{D}_X) \otimes_{f^{-1} \widetilde{\mathscr{D}}_Y} f^{-1} \operatorname{Sp}_{Y \to Z} (\mathscr{D}_Y))$$

(A.8.17.b)
$$\simeq f'_! \operatorname{God}^{\bullet} f_! \left[\operatorname{God}^{\bullet} \left(\mathscr{M} \otimes_{\widetilde{\mathscr{D}}_X} \operatorname{Sp}_{X \to Y}(\mathscr{D}_X) \right) \otimes_{f^{-1} \widetilde{\mathscr{D}}_Y} f^{-1} \operatorname{Sp}_{Y \to Z}(\mathscr{D}_Y) \right]$$

(A.8.17.c)
$$\simeq f'_! \operatorname{God}^{\bullet} \left[f_! \operatorname{God}^{\bullet} \left(\mathscr{M} \otimes_{\widetilde{\mathscr{D}}_X} \operatorname{Sp}_{X \to Y}(\widetilde{\mathscr{D}}_X) \right) \otimes_{\widetilde{\mathscr{D}}_Y} \operatorname{Sp}_{Y \to Z}(\widetilde{\mathscr{D}}_Y) \right]$$

= ${}_{\mathrm{D}} f'_! {}_{\mathrm{D}} f_! \mathscr{M}.$

Indeed, (A.8.17.a), as $f_! \operatorname{God}^{\bullet}$ is *c*-soft, it is acyclic for $f'_!$, hence the natural morphism $f'_! f_! \operatorname{God}^{\bullet} \to f'_! \operatorname{God}^{\bullet} f_! \operatorname{God}^{\bullet}$ is an isomorphism. Next, (A.8.17.b) follows from Exercise A.40, as the terms of $\operatorname{Sp}_{Y\to Z}(\widetilde{\mathscr{D}}_Y)$ are $\widetilde{\mathscr{D}}_Y$ -locally free (see A.38(2)). Lastly, (A.8.17.c) follows from the projection formula for $f_!$ (e.g. [**KS90**, Prop. 2.5.13]).

If f is proper, then $f_! = f_*$ and $f_! \operatorname{God}^{\bullet}$ is flabby, so (A.8.17.a) still holds with f'_* , and the same reasoning gives ${}_{\mathrm{D}}(f' \circ f)_* = {}_{\mathrm{D}}f'_*{}_{\mathrm{D}}f_*$.

If f is not proper, we cannot assert in general that $_{\mathrm{D}}(f' \circ f)_* = {}_{\mathrm{D}}f'_* f_*$. However, such an identity still holds when applied to suitable subcategories of $\mathsf{D}^+(\widetilde{\mathscr{D}}_X)$, the main examples being:

- the restriction of f to the support of \mathcal{M} is proper,
- \mathscr{M} has $\widetilde{\mathscr{D}}_X$ -coherent cohomology.

In such cases, the natural morphism coming in the projection formula for f_* is a quasi-isomorphism (see [**MN93**, §II.5.4] for the coherent case).

Remark A.8.18 (Behaviour of the Spencer complex by pushforward)

In the proof of Theorem A.8.16, let us set Z = pt, so that $\text{Sp}_{Y \to Z}(\widehat{\mathscr{D}}_Y) = \text{Sp}_Y(\widehat{\mathscr{D}}_Y)$. By the same argument, but not applying the functor Rf'_1 , we obtain

$$\operatorname{Sp}_Y({}_{\mathrm{D}}f_!\mathscr{M})\simeq \mathbf{R}f_!\operatorname{Sp}_X(\mathscr{M})$$

Remark A.8.19 (The Leray spectral sequence). Let us consider the expression (A.8.17.c). Firstly, $f_! \operatorname{God}^{\bullet} (\mathscr{M} \otimes_{\widetilde{\mathscr{D}}_X} \operatorname{Sp}_{X \to Y}(\widetilde{\mathscr{D}}_X))$ is a bi-complex, that we replace with its associated single complex (K'^{\bullet}, δ') , having cohomology ${}_{\mathrm{D}}f_!^{i}\mathscr{M}$. Similarly, $f'_! \operatorname{God}^{\bullet} [K'^{\bullet} \otimes_{\widetilde{\mathscr{D}}_Y} \operatorname{Sp}_{Y \to Z}(\widetilde{\mathscr{D}}_Y)]$ is a triple complex, from which we consider the associated double complex $(K^{\bullet, \bullet}, \delta', \delta'')$ by grouping the terms corresponding to $f'_! \operatorname{God}^{\bullet}$ and $\operatorname{Sp}_{Y \to Z}(\widetilde{\mathscr{D}}_Y)$. The single complex attached to $(K^{\bullet, \bullet}, \delta', \delta'')$ has cohomology ${}_{\mathrm{D}}(f' \circ f)_!^k\mathscr{M}$, according to our previous computation.

Since the terms of the complex $\text{Sp}_{Y \to Z}$ are \mathscr{D}_Y -locally free (see Exercise A.38(2)), we have

$$\mathscr{H}^{i}_{\delta'}(K'^{\bullet} \otimes_{\widetilde{\mathscr{D}}_{Y}} \operatorname{Sp}_{Y \to Z}(\widetilde{\mathscr{D}}_{Y})) = {}_{\mathrm{D}}f^{i}_{!}\mathscr{M} \otimes_{\widetilde{\mathscr{D}}_{Y}} \operatorname{Sp}_{Y \to Z}(\widetilde{\mathscr{D}}_{Y}).$$

Similarly, according to Exercise A.40(1), we have

 $\mathscr{H}^{i}_{\delta'}(f'_{!}\operatorname{God}^{j}(K'^{\bullet}\otimes_{\widetilde{\mathscr{D}}_{Y}}\operatorname{Sp}_{Y\to Z}(\widetilde{\mathscr{D}}_{Y})))=f'_{!}\operatorname{God}^{j}({}_{\scriptscriptstyle \mathrm{D}}f^{i}_{!}\mathscr{M}\otimes_{\widetilde{\mathscr{D}}_{Y}}\operatorname{Sp}_{Y\to Z}(\widetilde{\mathscr{D}}_{Y})).$

We deduce that, with respect to the spectral sequence attached to the double complex $(K^{\bullet,\bullet}, \delta', \delta'')$, we have

$$E_2^{i,j} = H^j_{\delta''}(H^i_{\delta'}(K^{\bullet,\bullet})) = {}_{\mathrm{D}}f'^j_!({}_{\mathrm{D}}f^i_!\mathscr{M}).$$

We call this spectral sequence the *Leray spectral sequence* for the composition $f' \circ f$. Since the functors $f_!$ and $f'_!$ have finite cohomological dimension, we can truncate the complexes above at a finite order and the spectral sequence degenerates at a finite step. In such a way, the abutment $_{D}(f' \circ f)^{k}_{!}\mathcal{M}$ comes equipped with a natural filtration, that we call the *Leray filtration*, such that

$$E^{i,j}_{\infty} = \operatorname{gr}^{i}_{\operatorname{Ler} \mathsf{D}}(f' \circ f)^{i+j}_{!} \mathscr{M}.$$

We have a similar result for $_{D}(f' \circ f)^{k}_{*}\mathcal{M}$ if f is proper.

By using Exercise A.41, we note that the restriction to z = 1 of the Leray spectral sequence is the Leray spectral sequence for \mathscr{D}_X -modules.

A.8.c. Pushforward of left $\widetilde{\mathscr{D}}$ -modules. We make explicit the effect of sidechanging with respect to the pushforward functor. Our definition is intended to be similar to the standard convention for the constant map $X \to \text{pt}$ (see Caveat A.8.21). It will also be shown to coincide, after taking cohomology, with the notion of Gauss-Manin connection, in the case of a proper smooth morphism, see Theorem A.11.23.

Definition A.8.20 (Pushforward of a left $\widetilde{\mathscr{D}}_X$ -module). If \mathscr{M} is a left $\widetilde{\mathscr{D}}_X$ -module (recall that $n = \dim X, m = \dim Y$), one *defines* the pushforward functor by side-changing:

$${}_{\mathrm{D}}f_{\star}\mathscr{M} = ({}_{\mathrm{D}}f_{\star}\mathscr{M}^{\mathrm{right}})^{\mathrm{left}}[m-n] \quad (\star = *, !),$$
$$= \widetilde{\omega}_{Y}^{\vee} \otimes ({}_{\mathrm{D}}f_{\star}(\widetilde{\omega}_{X} \otimes \mathscr{M}))[m-n] \quad (\mathrm{see \ Caveat \ A.3.3})$$

Caveat A.8.21. The standard definition of the pushforward of a left $\widetilde{\mathscr{D}}$ -module does not introduce the shift [m-n]. We introduce it here in order that, when f is the constant map to a point, we get $\mathscr{H}^k_{\ D} f_* \mathscr{M}^{\text{left}} = H^k(X, \text{DR } \mathscr{M}^{\text{left}})$ according to (A.5.5*), a convention which is commonly used for vector bundles with flat connection. Similarly, according to Remark A.8.18, we have

$$\mathrm{DR}(_{\mathrm{D}}f_{!}\mathscr{M})\simeq \mathbf{R}f_{!}\,\mathrm{DR}\,\mathscr{M}.$$

Indeed,

$$DR({}_{D}f_{!}\mathscr{M}) \simeq Sp_{Y}(({}_{D}f_{!}\mathscr{M})^{right}[-m]) \quad (due \text{ to } (A.5.5*))$$
$$\simeq Sp_{Y}({}_{D}f_{!}(\mathscr{M}^{right})[-n]) \quad (Definition A.8.20)$$
$$\simeq \mathbf{R}f_{!} Sp_{X}(\mathscr{M}^{right})[-n] \quad (Remark A.8.18)$$
$$\simeq \mathbf{R}f_{!} DR\mathscr{M} \quad (due \text{ to } (A.5.5*)). \qquad \Box$$

Exercise A.46 (Computation of the pushforward of a left $\widehat{\mathscr{D}}_X$ -module)

(1) We consider the setting of Exercise A.42 with a left $\widetilde{\mathscr{D}}_X$ -module \mathscr{M} . By using the results of this exercise, show

$$({}_{\mathrm{D}}f_{\star}\mathscr{M})^{\mathrm{right}} = \mathbf{R}f_{\star}\widetilde{\Omega}^{\bullet}_{X}(\mathscr{M} \otimes_{f^{-1}\widetilde{\mathscr{O}}_{Y}} f^{-1}\widetilde{\mathscr{D}}_{Y})[m].$$

(2) We now compute the pushforward by the graph inclusion $\iota_f : X \hookrightarrow X \times Y$ in local coordinates y_1, \ldots, y_m on Y. Set $f_j = y_j \circ f$. Show that $\mathscr{H}^j{}_{\mathrm{D}}\iota_{f*}\mathscr{M} = 0$ for $j \neq -m$, and $\mathscr{H}^{-m}{}_{\mathrm{D}}\iota_{f*}\mathscr{M} \simeq \iota_{f*}\mathscr{M}[\eth_{y_1}, \ldots, \eth_{y_m}](-m)$ with left $\widetilde{\mathscr{D}}_{X \times Y}$ structure given locally by

$$\begin{aligned} \eth_{y_j} \cdot \mu \eth_y^\alpha &= \mu \eth_y^{\alpha+\mathbf{1}_j}, \\ \eth_{x_i} \cdot \mu \eth_y^\alpha &= (\eth_{x_i} \mu) \eth_y^\alpha - \sum_{j=1}^m \frac{\partial f_j}{\partial x_i} \mu \eth_y^{\alpha+\mathbf{1}_j}. \end{aligned}$$

[*Hint*: for the shift (-m) of the grading, use Exercise A.23(2).]

(3) Show that, when $f : X \to Y$ is a projection, we recover the definition of Example A.8.2.

(4) Let η be a closed (1, 1)-form on X. Show that the Lefschetz morphism

$$L_{\eta}: \frac{1}{2}\eta \wedge : {}_{\mathrm{D}}f_{\star}\mathscr{M} \longrightarrow {}_{\mathrm{D}}f_{\star}\mathscr{M}[2](1)$$

is functorial in \mathcal{M} and compatible with the side-changing functor.

(5) As in Remark A.8.14, define the Lefschetz morphism

$$\mathcal{L}_{\mathscr{L}}: {}_{\mathsf{D}}f_{\star}\mathscr{M} \longrightarrow {}_{\mathsf{D}}f_{\star}\mathscr{M}[2](1)$$

attached to a line bundle on X, for any morphism $f: X \to Y$.

Exercise A.47 (Grading and pushforward, left case). With the assumptions as in Exercise A.45(2), but assuming that \mathcal{M} is a left \mathscr{D}_X -module, show that

$$\mathrm{gr}^{F}\mathscr{H}^{i}{}_{\mathrm{D}}f_{\star}\mathfrak{M}\simeq\mathscr{H}^{i+n-m}\boldsymbol{R}f_{\star}\big(\omega_{X/Y}\otimes_{\mathscr{O}_{X}}\mathrm{gr}_{\bullet+n-m}^{F}\mathfrak{M}\otimes_{\mathrm{Sym}\,\Theta_{X}}^{\boldsymbol{L}}f^{*}\operatorname{Sym}\Theta_{Y}\big),$$

where $\omega_{X/Y} := \omega_X \otimes_{\mathscr{O}_X} f^* \omega_Y^{\otimes -1}$, and we have set $n = \dim X$, $m = \dim Y$. For example, if Y = pt, deduce that

$$\operatorname{gr}^{F} \boldsymbol{H}^{i}(X, \operatorname{DR} \mathcal{M}) \simeq \boldsymbol{H}^{i+n} (X, \omega_{X} \otimes (\operatorname{gr}_{\bullet+n}^{F} \mathcal{M} \otimes_{\operatorname{Sym} \Theta_{X}}^{L} \mathscr{O}_{X})).$$

A.8.d. A morphism of adjunction. There are various adjunction morphisms for $\widetilde{\mathscr{D}}$ -modules in the literature (see [Kas03, HTT08]). We will give here a simple one, in the case where the source and target of the *proper holomorphic map* $f: X \to Y$ have the same dimension. In such a case, the cotangent map T^*f induces a morphism

$$f^{-1}\widetilde{\Omega}^k_Y \longrightarrow \widetilde{\Omega}^k_X$$

for every k, which is compatible with the differential d, and similarly for C^{∞} forms.

Proposition A.8.22. Under this assumption, if \mathscr{M} is a left $\widetilde{\mathscr{D}}_Y$ -module, there is a functorial morphism

$$\mathscr{M} \longrightarrow {}_{\mathrm{D}} f^0_* {}_{\mathrm{D}} f^* \mathscr{M}.$$

Proof. Set $n = \dim X = \dim Y$. By Exercise A.42 we have

$$({}_{\mathrm{D}}f_{*\mathrm{D}}f^{*}\mathscr{M})^{\mathrm{right}} \simeq f_{*}\Big(\widetilde{\mathscr{E}}_{X}^{\bullet} \otimes ({}_{\mathrm{D}}f^{*}\mathscr{M} \otimes_{f^{-1}\widetilde{\mathscr{O}}_{Y}} f^{-1}\widetilde{\mathscr{D}}_{Y})[n]\Big)$$
$$\simeq f_{*}\Big(\widetilde{\mathscr{E}}_{X}^{\bullet} \otimes_{f^{-1}\widetilde{\mathscr{O}}_{Y}} f^{-1}(\mathscr{M} \otimes_{\widetilde{\mathscr{O}}_{Y}} \widetilde{\mathscr{D}}_{Y})[n]\Big)$$
$$\simeq f_{*}\widetilde{\mathscr{E}}_{X}^{\bullet} \otimes_{\widetilde{\mathscr{O}}_{Y}} (\mathscr{M} \otimes_{\widetilde{\mathscr{O}}_{Y}} \widetilde{\mathscr{D}}_{Y})[n],$$

where the last isomorphism is the sheaf-theoretic projection formula for a proper map. By using the isomorphism of Exercise A.18(3), we finally obtain

$$({}_{\mathrm{D}}f_{*\mathrm{D}}f^{*}\mathscr{M})^{\mathrm{right}} \simeq f_{*}\widetilde{\mathscr{E}}_{X}^{\bullet} \otimes_{\widetilde{\mathscr{O}}_{Y}} (\widetilde{\mathscr{D}}_{Y} \otimes_{\widetilde{\mathscr{O}}_{Y}} \mathscr{M})[n].$$

The cotangent map $f^{-1}\widetilde{\mathscr{E}}_Y^k \to \widetilde{\mathscr{E}}_X^k$ induces, by using the sheaf-theoretic adjunction $f_*f^{-1} \to \mathrm{Id}$, a morphism $\widetilde{\mathscr{E}}_Y^k \to f_*\widetilde{\mathscr{E}}_X^k$ compatible with differentials, hence a morphism $\widetilde{\mathscr{E}}_Y^{\bullet} \otimes (\widetilde{\mathscr{D}}_Y \otimes_{\widetilde{\mathscr{O}}_Y} \mathscr{M})[n] \longrightarrow ({}_{\mathrm{D}}f_*\mathscr{M})^{\mathrm{right}}.$

Lastly, by using Exercise A.26(2), we find

$$\widetilde{\mathscr{E}}_{Y}^{\bullet} \otimes (\widetilde{\mathscr{D}}_{Y} \otimes_{\widetilde{\mathscr{O}}_{Y}} \mathscr{M})[n] \xleftarrow{\sim} \widetilde{\Omega}_{Y}^{\bullet} \otimes (\widetilde{\mathscr{D}}_{Y} \otimes_{\widetilde{\mathscr{O}}_{Y}} \mathscr{M})[n] \xrightarrow{\sim} \mathscr{M}^{\mathrm{right}}.$$

A.9. Coherence of $\widetilde{\mathscr{D}}_X$

Let us begin by recalling the definition of coherence. Let \mathscr{A} be a sheaf of rings on a space X.

Definition A.9.1.

(1) A sheaf of \mathscr{A} -modules \mathscr{F} is said to be \mathscr{A} -coherent if it is locally of finite type:

$$\forall x \in X, \ \exists U_x, \ \exists q, \quad \exists \mathscr{A}^q_{|U_x} \twoheadrightarrow \mathscr{F}_{|U_x},$$

and if, for any open set U of X and any \mathscr{A} -linear morphism $\varphi : \mathscr{A}_{|U}^r \to \mathscr{F}_{|U}$, the kernel of φ is locally of finite type.

(2) The sheaf \mathscr{A} is a coherent sheaf of rings if it is coherent as a (left and right) module over itself.

Lemma A.9.2. Assume \mathscr{A} coherent. Let \mathscr{F} be a sheaf of \mathscr{A} -module. Then \mathscr{F} is \mathscr{A} coherent if and only if \mathscr{F} is locally of finite presentation: $\forall x \in X, \exists U_x, \exists p, q \text{ and}$ an exact sequence

$$\mathscr{A}^p_{|U_x} \longrightarrow \mathscr{A}^q_{|U_x} \longrightarrow \mathscr{F}_{|U_x} \longrightarrow 0.$$

Classical theorems of Cartan and Oka claim the *coherence of* $\widetilde{\mathscr{O}}_X$, and a theorem of Frisch asserts that, if K is a compact polycylinder, $\widetilde{\mathscr{O}}_X(K)$ is a Noetherian ring. It follows that $\operatorname{gr}^F \widetilde{\mathscr{D}}_X(K)$ is a Noetherian ring, and one deduces that $\widetilde{\mathscr{D}}_X(K)$ is left and right Noetherian. From this one concludes that the sheaf of rings $\widetilde{\mathscr{D}}_X$ is coherent (see [**GM93, Kas03**] for details).

Exercise A.48.

(1) Prove similarly the coherence of the sheaf of rings $\mathrm{gr}^F \widehat{\mathscr{D}}_X$.

(2) Let $D \subset X$ be a hypersurface and let $\widetilde{\mathscr{O}}_X(*D)$ be the sheaf of meromorphic functions on X with poles on D at most (with arbitrary order). Prove similarly that $\widetilde{\mathscr{O}}_X(*D)$ is a coherent sheaf of rings.

(3) Prove that $\widetilde{\mathscr{D}}_X(*D) := \widetilde{\mathscr{O}}_X(*D) \otimes_{\widetilde{\mathscr{O}}_X} \widetilde{\mathscr{D}}_X$ is a coherent sheaf of rings.

(4) Let $\iota : Y \hookrightarrow X$ denote the inclusion of a smooth submanifold. Show that $i^* \widetilde{\mathscr{D}}_X := \widetilde{\mathscr{O}}_Y \otimes_{\widetilde{\mathscr{O}}_Y} \widetilde{\mathscr{D}}_X$ is a coherent sheaf of rings on Y.

(5) Let $Y \subset X$ be a smooth hypersurface of X. Show that $V_0 \widetilde{\mathscr{D}}_X$ (see Section 7.2) is a coherent sheaf of rings.

A.10. Coherent $\widetilde{\mathscr{D}}_X$ -modules and coherent filtrations

Let \mathscr{M} be a $\widetilde{\mathscr{D}}_X$ -module. From the preliminary reminder on coherence, we know that \mathscr{M} is $\widetilde{\mathscr{D}}_X$ -coherent if it is locally finitely presented, i.e., if for any $x \in X$ there exists an open neighbourhood U_x of x an an exact sequence $\widetilde{\mathscr{D}}_{X|U_x}^q \to \widetilde{\mathscr{D}}_{X|U_x}^p \to \mathscr{M}_{|U_x}$.

Exercise A.49.

(1) Let $\mathscr{M} \subset \mathscr{N}$ be a $\widetilde{\mathscr{D}}_X$ -submodule of a coherent $\widetilde{\mathscr{D}}_X$ -module \mathscr{N} . Show that, if \mathscr{M} is locally finitely generated, then it is coherent.

(2) Let $\phi : \mathscr{M} \to \mathscr{N}$ be a morphism between coherent $\widetilde{\mathscr{D}}_X$ -modules. Show that Ker ϕ and Coker ϕ are coherent.

A.10.a. Coherent filtrations

Definition A.10.1 (Coherent filtrations). Let $F_{\bullet}\mathscr{M}$ be a filtration of \mathscr{M} (see Section A.2). We say that the filtration is *coherent* if the Rees module $R_F\mathscr{M}$ is coherent over the coherent sheaf $R_F \widetilde{\mathscr{D}}_X$ (i.e., locally finitely presented).

It is useful to have various criteria for a filtration to be coherent.

Exercise A.50 (Characterization of coherent filtrations).

(1) Show that the following properties are equivalent:

(a) $F_{\bullet}\mathcal{M}$ is a coherent filtration;

(b) for every $k \in \mathbb{Z}$, $F_k \mathscr{M}$ is $\widehat{\mathscr{O}}_X$ -coherent, and, for every $x \in X$, there exists a neighbourhood U of x and $k_0 \in \mathbb{Z}$ such that, for every $k \ge 0$, $F_k \widetilde{\mathscr{D}}_{X|U} \cdot F_{k_0} \mathscr{M}_{|U} = F_{k+k_0} \mathscr{M}_{|U}$;

(c) the graded module $\operatorname{gr}^F \mathscr{M}$ is $\operatorname{gr}^F \widetilde{\mathscr{D}}_X$ -coherent.

(2) Conclude that, if $F_{\bullet}\mathcal{M}, G_{\bullet}\mathcal{M}$ are two coherent filtrations of \mathcal{M} , then, locally on X, there exists k_0 such that, for every k, we have

$$F_{k-k_0}\mathscr{M} \subset G_k\mathscr{M} \subset F_{k+k_0}\mathscr{M}.$$

Proposition A.10.2 (Local existence of coherent filtrations). If \mathscr{M} is $\widetilde{\mathscr{D}}_X$ -coherent, then it admits locally on X a coherent filtration.

Proof. Exercise A.51.

397

Exercise A.51 (Local existence of coherent filtrations).

(1) Write $R_F \mathscr{M} = \bigoplus_k F_k \mathscr{M} \zeta^k$ and show that, if \mathscr{M} has a coherent filtration, then it is $\widetilde{\mathscr{D}}_X$ -coherent. [*Hint*: use that the tensor product $\mathbb{C}[\zeta]/(\zeta - 1) \otimes_{\mathbb{C}[z]} \bullet$ is right exact.]

(2) Conversely, show that any coherent $\widehat{\mathscr{D}}_X$ -module admits locally a coherent filtration. [*Hint*: choose a local presentation $\widetilde{\mathscr{D}}_{X|U}^q \xrightarrow{\varphi} \widetilde{\mathscr{D}}_{X|U}^p \to \mathscr{M}_{|U} \to 0$, and show that the filtration induced on $\mathscr{M}_{|U}$ by $F_{\bullet} \widetilde{\mathscr{D}}_{X|U}^p$ is coherent by using Exercise A.50: Set $\mathscr{K} = \operatorname{Im} \varphi$ and reduce the assertion to showing that $F_j \widetilde{\mathscr{D}}_X \cap \mathscr{K}$ is $\widetilde{\mathscr{O}}_X$ -coherent; prove that, up to shrinking U, there exists $k_o \in \mathbb{N}$ such that $\varphi(F_k \widetilde{\mathscr{D}}_{X|U}^q) \subset F_{k+k_o} \widetilde{\mathscr{D}}_{X|U}^p$ for every k; deduce that $\varphi(F_k \widetilde{\mathscr{D}}_X^q|_U)$, being locally of finite type and contained in a coherent $\widetilde{\mathscr{O}}_X$ -module, is $\widetilde{\mathscr{O}}_X$ -coherent for every k; conclude by using the fact that an increasing sequence of coherent $\widetilde{\mathscr{O}}_X$ -modules in a coherent $\widetilde{\mathscr{O}}_X$ -module is locally stationary.]

(3) Show that a coherent filtration $F_{\bullet}\mathcal{M}$ satisfies $F_p\mathcal{M} = 0$ for $p \ll 0$ locally [*Hint*: use that this holds for the filtration constructed in (2) and apply Exercise A.50(2).]

(4) Show that, locally, any coherent $\widetilde{\mathscr{D}}_X$ -module is generated over $\widetilde{\mathscr{D}}_X$ by a coherent $\widetilde{\mathscr{O}}_X$ -submodule.

(5) Let \mathscr{M} be a coherent $\widetilde{\mathscr{D}}_X$ -module and let \mathscr{F} be an $\widetilde{\mathscr{O}}_X$ -submodule which is locally finitely generated. Show that \mathscr{F} is $\widetilde{\mathscr{O}}_X$ -coherent. [*Hint*: choose a coherent filtration $F_{\bullet}\mathscr{M}$ and show that, locally, $\mathscr{F} \subset F_k\mathscr{M}$ for some k; apply then the analogue of Exercise A.49(1) for $\widetilde{\mathscr{O}}_X$ -modules.]

The notion of a coherent filtration is the main tool to obtain results on coherent $\widetilde{\mathscr{D}}_X$ -modules from theorems on coherent $\widetilde{\mathscr{O}}_X$ -modules, and the main results concerning coherent $\widetilde{\mathscr{D}}_X$ -modules are obtained from the theorems of Cartan and Oka for $\widetilde{\mathscr{O}}_X$ -modules.

Theorem A.10.3 (Theorems of Cartan-Oka for $\widetilde{\mathscr{D}}_X$ -modules)

Let \mathscr{M} be a left \mathscr{D}_X -module and let K be a compact polycylinder contained in an open subset U of X, such that \mathscr{M} has a coherent filtration on U. Then,

- (1) $\Gamma(K, \mathcal{M})$ generates $\mathcal{M}_{|K}$ as an \mathcal{O}_{K} -module,
- (2) For every $i \ge 1$, $H^i(K, \mathscr{M}) = 0$.

Proof. This is easily obtained from the theorems A and B for $\widetilde{\mathcal{O}}_X$ -modules, by using inductive limits (for A it is obvious and, for B, see [God64, Th. 4.12.1]).

Theorem A.10.4 (Characterization of coherence for $\widetilde{\mathscr{D}}_X$ -modules, see [GM93])

(1) Let \mathscr{M} be a left $\widetilde{\mathscr{D}}_X$ -module. Then, for any small enough compact polycylinder K, we have the following properties:

- (a) $\mathscr{M}(K)$ is a finite type $\widetilde{\mathscr{D}}(K)$ -module,
- (b) For every $x \in K$, $\widetilde{\mathcal{O}}_x \otimes_{\widetilde{\mathcal{O}}(K)} \mathscr{M}(K) \to \mathscr{M}_x$ is an isomorphism.

(2) Conversely, if there exists a covering $\{K_{\alpha}\}$ by polycylinders K_{α} such that X is the union of the interiors of the K_{α} and that on any K_{α} the properties (1a) and (1b) are fulfilled, then \mathscr{M} is $\widetilde{\mathscr{D}}_X$ -coherent.

Exercise A.52.

(1) Show similar statements for $R_F \widetilde{\mathscr{D}}_X$ -modules, $\mathrm{gr}^F \widetilde{\mathscr{D}}_X$ -modules, $\widetilde{\mathscr{O}}_X(*D)$ -modules and $i^* \widetilde{\mathscr{D}}_X$ -modules (see Exercise A.48).

(2) Let \mathscr{M} be a coherent $\widetilde{\mathscr{D}}_X$ -module. Show that $\widetilde{\mathscr{D}}_X(*D) \otimes_{\widetilde{\mathscr{D}}_X} \mathscr{M}$ is $\widetilde{\mathscr{D}}_X(*D)$ coherent and that $i^*\mathscr{M}$ is $i^*\widetilde{\mathscr{D}}_X$ -coherent.

Exercise A.53 (External product).

(1) Let A, B be two Noetherian \mathbb{C} -algebras. Show that $A \otimes_{\widetilde{\mathbb{C}}} B$ is Noetherian.

(2) Let X, Y be two complex manifolds and let p_X, p_Y be the projections from $X \times Y$ to X and Y respectively. For any pair of sheaves of \mathbb{C} -vector spaces $\mathscr{F}_X, \mathscr{F}_Y$ on X and Y respectively, set $\mathscr{F}_X \boxtimes_{\mathbb{C}} \mathscr{F}_Y := p_X^{-1} \mathscr{F}_X \otimes_{\mathbb{C}} p_Y^{-1} \mathscr{F}_Y$. Show that $\widetilde{\mathcal{O}}_X \boxtimes_{\mathbb{C}} \widetilde{\mathcal{O}}_Y$ is a coherent sheaf of rings on $X \times Y$. [*Hint*: Use an analogue of Theorem A.10.4(2).] (3) Prove similar properties for $\widetilde{\mathscr{D}}_X \boxtimes_{\mathbb{C}} \widetilde{\mathscr{D}}_Y$.

(4) Show that $\widetilde{\mathcal{O}}_{X \times Y}$ is faithfully flat over $\widetilde{\mathcal{O}}_X \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{O}}_Y$. [*Hint*: Use [Ser56, Prop. 28].]

(5) Show that

$$\widetilde{\mathscr{D}}_{X\times Y} = \widetilde{\mathscr{O}}_{X\times Y} \otimes_{(\widetilde{\mathscr{O}}_X\boxtimes_{\widetilde{\mathbb{C}}}\widetilde{\mathscr{O}}_Y)} (\widetilde{\mathscr{D}}_X\boxtimes_{\widetilde{\mathbb{C}}}\widetilde{\mathscr{D}}_Y) = (\widetilde{\mathscr{D}}_X\boxtimes_{\widetilde{\mathbb{C}}}\widetilde{\mathscr{D}}_Y) \otimes_{(\widetilde{\mathscr{O}}_X\boxtimes_{\widetilde{\mathbb{C}}}\widetilde{\mathscr{O}}_Y)} \widetilde{\mathscr{O}}_{X\times Y}.$$

(6) For an $\widetilde{\mathscr{O}}_X$ -module \mathscr{L}_X (resp. a $\widetilde{\mathscr{D}}_X$ -module \mathscr{M}_X) and an $\widetilde{\mathscr{O}}_Y$ -module \mathscr{L}_Y (resp. a $\widetilde{\mathscr{D}}_Y$ -module \mathscr{M}_Y), set

$$\begin{aligned} \mathscr{L}_X \boxtimes_{\widetilde{\mathscr{O}}} \mathscr{L}_Y &= (\mathscr{L}_X \boxtimes_{\widetilde{\mathbb{C}}} \mathscr{L}_Y) \otimes_{\widetilde{\mathscr{O}}_X \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathscr{O}}_Y} \widetilde{\mathscr{O}}_{X \times Y} \\ \mathscr{M}_X \boxtimes_{\widetilde{\mathscr{O}}} \mathscr{M}_Y &= (\mathscr{M}_X \boxtimes_{\widetilde{\mathbb{C}}} \mathscr{M}_Y) \otimes_{\widetilde{\mathscr{O}}_X \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathscr{O}}_Y} \widetilde{\mathscr{O}}_{X \times Y} \\ &= (\mathscr{M}_X \boxtimes_{\widetilde{\mathbb{C}}} \mathscr{M}_Y) \otimes_{\widetilde{\mathscr{O}}_X \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathscr{O}}_Y} \widetilde{\mathscr{O}}_{X \times Y}. \end{aligned}$$

Show that if $\mathscr{L}_X, \mathscr{L}_Y$ are $\widetilde{\mathscr{O}}$ -coherent (resp. $\mathscr{M}_X, \mathscr{M}_Y$ are $\widetilde{\mathscr{O}}$ -coherent), then $\mathscr{L}_X \boxtimes_{\widetilde{\mathscr{O}}} \mathscr{L}_Y$ is $\widetilde{\mathscr{O}}_{X \times Y}$ -coherent (resp. $\mathscr{M}_X \boxtimes_{\widetilde{\mathscr{O}}} \mathscr{M}_Y$ is $\widetilde{\mathscr{O}}_{X \times Y}$ -coherent).

(7) Show that, if $F_{\bullet}\mathscr{M}_X, F_{\bullet}\mathscr{M}_Y$ are coherent filtrations, then $F_j(\mathscr{M}_X \boxtimes_{\widetilde{\mathscr{D}}} \mathscr{M}_Y) := \sum_{k+\ell=j} F_k \mathscr{M}_X \boxtimes_{\widetilde{\mathscr{D}}} F_\ell \mathscr{M}_Y$ is a coherent filtration of $\mathscr{M}_X \boxtimes_{\widetilde{\mathscr{D}}} \mathscr{M}_Y$ for which

$$\operatorname{gr}^{F}(\mathscr{M}_{X}\boxtimes_{\widetilde{\mathscr{D}}}\mathscr{M}_{Y}) = \operatorname{gr}^{F}\mathscr{M}_{X}\boxtimes_{\operatorname{gr}^{F}\widetilde{\mathscr{D}}}\operatorname{gr}^{F}\mathscr{M}_{Y}.$$

[*Hint*: See [Kas03, §4.3].]

A first application of Theorem A.10.4 is a variant of the classical Artin-Rees lemma:

Corollary A.10.5. Let \mathscr{M} be a $\widetilde{\mathscr{D}}_X$ -module with a coherent filtration $F_{\bullet}\mathscr{M}$ and let \mathscr{N} be a coherent $\widetilde{\mathscr{D}}_X$ -submodule of \mathscr{M} . Then the filtration $F_{\bullet}\mathscr{N} := \mathscr{N} \cap F_{\bullet}\mathscr{M}$ is coherent.

Proof. Let K be a small compact polycylinder for $R_F \mathscr{M}$. Then $\Gamma(K, R_F \mathscr{M})$ is finitely generated, hence so is $\Gamma(K, R_F \mathscr{N})$, as $\Gamma(K, R_F \widetilde{\mathscr{D}}_X)$ is Noetherian. It remains to be proved that, for any $x \in K$ and any k, the natural morphism

$$\widetilde{\mathscr{O}}_x \otimes_{\widetilde{\mathscr{O}}(K)} (F_k \mathscr{M}(K) \cap \mathscr{N}(K)) \longrightarrow F_k \mathscr{M}_x \cap \mathscr{N}_x$$

is an isomorphism. This follows from the flatness of $\widetilde{\mathscr{O}}_x$ over $\widetilde{\mathscr{O}}(K)$ (see [Fri67]). \Box

Exercise A.54. Similarly, prove that if $\varphi : \mathscr{M} \to \mathscr{N}$ is a surjective morphism of coherent $\widetilde{\mathscr{D}}_X$ -modules and if $F_{\bullet}\mathscr{M}$ is coherent, then $F_{\bullet}\mathscr{N} := \varphi(F_{\bullet}\mathscr{M})$ is coherent as well.

A.10.b. Support and characteristic variety. Let \mathscr{M} be a coherent $\widehat{\mathscr{D}}_X$ -module. Being a sheaf on X, \mathscr{M} has a support Supp \mathscr{M} , which is the closed subset complement to the set of $x \in X$ in the neighbourhood of which \mathscr{M} is zero.

Lemma A.10.6. The support of a coherent $\widetilde{\mathcal{O}}_X$ -module is a closed analytic subset of X.

Proof. This is standard if $\widetilde{\mathscr{O}}_X = \mathscr{O}_X$. On the other hand, if $\widetilde{\mathscr{O}}_X = R_F \mathscr{O}_X$, let $\widetilde{\mathscr{I}}$ be a graded ideal of $\widetilde{\mathscr{O}}_X$, locally generated by functions $f_j z^j$ with $f_j \in \mathscr{O}_X$. Then the support of $\widetilde{\mathscr{O}}_X/\widetilde{\mathscr{I}}$ is that of $\mathscr{O}_X/(f_j)_j$.

Such a property extends to coherent $\widetilde{\mathscr{D}}_X$ -modules:

Proposition A.10.7. The support Supp \mathscr{M} of a coherent $\widetilde{\mathscr{D}}_X$ -module \mathscr{M} is a closed analytic subset of X.

Proof. The property of being an analytic subset being local, we may assume that \mathscr{M} is generated over $\widetilde{\mathscr{D}}_X$ by a coherent $\widetilde{\mathscr{O}}_X$ -submodule \mathscr{F} (see Exercise A.51(4)). Then the support of \mathscr{M} is equal to the support of \mathscr{F} .

The support is usually not the right geometric object attached to a $\widetilde{\mathscr{D}}_X$ -module \mathscr{M} , as it does not provide enough information on \mathscr{M} . A finer object is the *characteristic* variety. Using the convention A.2.14, we set $\widetilde{T}^*X = T^*X$ or $\widetilde{T}^*X = T^*X \times \mathbb{C}_z$.

Definition A.10.8 (Characteristic variety). Let \mathscr{M} be a coherent $\widetilde{\mathscr{D}}_X$ -module. The characteristic variety Char \mathscr{M} is the subset of the cotangent space \widetilde{T}^*X defined locally as the support of $\operatorname{gr}^F \mathscr{M}$ for some (or any) local coherent filtration of \mathscr{M} .

Exercise A.55. Let $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ be an exact sequence of $\widetilde{\mathscr{D}}_X$ -modules. Show that Char $\mathcal{M} = \operatorname{Char} \mathcal{M}' \cup \operatorname{Char} \mathcal{M}''$. [*Hint*: take a coherent filtration on \mathcal{M} and induce it on \mathcal{M}' and \mathcal{M}'' .]

Exercise A.56 (Coherent $\widetilde{\mathscr{D}}_X$ -modules with characteristic variety T_X^*X)

Assume that \mathscr{M} is coherent with characteristic variety contained in $T_X^*X \times \mathbb{C}_z$.

(1) Show that, for any local coherent filtration $F_{\bullet}\mathcal{M}$, the graded module $\operatorname{gr}^{F}\mathcal{M}$ is locally of finite type, hence coherent (see Exercise A.51(5)) over $\widetilde{\mathcal{O}}_{X}$.

(2) Deduce that, locally on X, there exists p_o such that $\operatorname{gr}_p^F \mathscr{M} = 0$ for $p \ge p_o$.

- (3) Conclude that \mathscr{M} is $\widetilde{\mathscr{O}}_X$ -coherent.
- (4) For a \mathscr{D}_X -module \mathcal{M} , deduce that \mathcal{M} is locally free of finite rank.

Exercise A.57 (Coherent \mathscr{D}_X -modules with characteristic variety contained in $T^*_Y X$)

In this exercise, we switch to the case of \mathscr{D}_X -modules. Let $\iota: Y \hookrightarrow X$ be the inclusion of a smooth codimension p closed submanifold. Define the p-th algebraic local cohomology with support in Y by $R^p\Gamma_{[Y]}\mathscr{O}_X = \lim_{K} \mathscr{E}xt^p(\mathscr{O}_X/\mathscr{I}_Y^k, \mathscr{O}_X)$, where \mathscr{I}_Y is the ideal defining Y. $R^p\Gamma_{[Y]}\mathscr{O}_X$ has a natural structure of \mathscr{D}_X -module. In local coordinates (x_1, \ldots, x_n) where Y is defined by $x_1 = \cdots = x_p = 0$, we have

$$R^{p}\Gamma_{[Y]}\mathcal{O}_{X} \simeq \frac{\mathcal{O}_{\mathbb{C}^{n}}[1/x_{1}\cdots x_{n}]}{\sum_{i=1}^{p}\mathcal{O}_{\mathbb{C}^{n}}(x_{i}/x_{1}\cdots x_{n})}.$$

Denote this \mathscr{D}_X -module by $\mathscr{B}_Y X$.

(1) Show that $\mathcal{B}_Y X$ has support contained in Y and characteristic variety equal to $T_V^* X$.

(2) Identify $\mathcal{B}_Y X$ with ${}_{\mathrm{D}}\iota_* \mathcal{O}_Y$.

(3) Let \mathcal{M} be a coherent \mathscr{D}_X -module with characteristic variety equal to T_Y^*X . Show that \mathcal{M} is locally isomorphic to $(\mathcal{B}_Y X)^d$ for some d.

A.10.c. Holonomic $\tilde{\mathscr{D}}_X$ -modules and duality

Definition A.10.9 (Holonomic $\widetilde{\mathcal{D}}_X$ -modules). A coherent $\widetilde{\mathcal{D}}_X$ -module \mathscr{M} is said to be holonomic if $\operatorname{Char} \mathscr{M} \subset \Lambda \times \widetilde{\mathbb{C}}_z$, where Λ is a Lagrangian closed subvariety of T^*X .

Such a Lagrangian subvariety is the union of its irreducible components, each of which is usually written as T_Z^*X , where Z is a closed irreducible subvariety of X and T_Z^*X means the closure, in the cotangent space T^*X of the conormal bundle $T_{Z^o}^*X$ of the smooth part Z^o of Z. It is also known that, due to the existence of stratifications satisfying Whitney condition (a), there exist a locally finite family $(Z_i^o)_{i \in I}$ of locally closed sub-manifolds Z_i^o of Z, with analytic closure and one of them being Z^o , such that $T_Z^*X \subset \bigsqcup_i T_{Z^o}^*X$.

For example, a coherent $\widehat{\mathscr{D}}_X$ -module as in Exercise A.56 is holonomic. The case of \mathscr{D}_X -modules is the most useful. We will recall some fundamental results.

Proposition A.10.10. Let \mathcal{M} be a coherent \mathcal{D}_X -module. We have

$$\mathscr{E}xt^{i}_{\mathscr{D}_{X}}(\mathcal{M},\mathscr{D}_{X}) = 0 \quad for \ i \ge n+1.$$

Theorem A.10.11. Let \mathcal{M} be a coherent \mathcal{D}_X -module and $x \in \text{Supp }\mathcal{M}$. Then

$$2n - \dim_x \operatorname{Char} \mathcal{M} = \inf\{i \in \mathbb{N} \mid \mathscr{E}xt^i_{\mathscr{D}_{X,x}}(\mathcal{M}_x, \mathscr{D}_{X,x}) = 0\}.$$

Corollary A.10.12. Let \mathcal{M} be a coherent \mathscr{D}_X -module. Then \mathcal{M} is holonomic if and only if $\mathscr{E}xt^i_{\mathscr{D}_X}(\mathcal{M}, \mathscr{D}_X) = 0$ for $i \neq \dim X$. The \mathscr{D}_X -module $\mathscr{E}xt^{\dim X}_{\mathscr{D}_X}(\mathcal{M}, \mathscr{D}_X)$ (see Section A.4.5), after having applied the suitable side changing functor to it, is called the dual of \mathcal{M} , and denoted by $\mathbf{D}\mathcal{M}$. **Theorem A.10.13 (Bi-duality, see [Kas76]).** Let \mathcal{M} be a holonomic \mathscr{D}_X -module. Then its dual module $\mathbf{D}\mathcal{M}$ is holonomic and the natural functorial morphism from \mathcal{M} to its bi-dual module $\mathbf{D}\mathbf{D}\mathcal{M}$ is an isomorphism.

Let us now consider holonomicity and duality for strict coherent $R_F \mathscr{D}_X$ -modules.

Exercise A.58. Let \mathcal{M} be a coherent \mathscr{D}_X -module equipped with a coherent filtration $F_{\bullet}\mathcal{M}$.

(1) Show that $\operatorname{Char}(R_F\mathcal{M}) = (\operatorname{Char}\mathcal{M}) \times \mathbb{C}_z$, so that \mathscr{M} is holonomic (in the sense of Definition A.10.9) if and only if \mathcal{M} is holonomic. (In other words, for a strict coherent $\widetilde{\mathscr{D}}_X$ -module $\mathscr{M}, \mathscr{M}/(z-1)\mathscr{M}$ is holonomic if and only if \mathscr{M} itself is holonomic.)

(2) In such a case, show that $\mathscr{E}xt^{i}_{R_{F}\mathscr{D}_{X}}(R_{F}\mathcal{M}, R_{F}\mathscr{D}_{X})$ consists of z-torsion if $i \neq \dim X$.

Definition A.10.14. We say that $R_F \mathcal{M}$ is *strict holonomic* if $R_F \mathcal{M}$ is holonomic and $\mathscr{E}xt^i_{R_F\mathscr{D}_X}(R_F\mathcal{M}, R_F\mathscr{D}_X)$ is a strict $R_F\mathscr{D}_X$ -module for every *i* (and as a consequence, $\mathscr{E}xt^i_{R_F\mathscr{D}_X}(R_F\mathcal{M}, R_F\mathscr{D}_X) = 0$ for $i \neq \dim X$).

Exercise A.59. Assume that $R_F \mathcal{M}$ is strict holonomic. Then there is a unique coherent filtration $F_{\bullet} \mathscr{E}xt_{\mathscr{D}_X}^{\dim X}(\mathcal{M}, \mathscr{D}_X)$ such that

$$R_F \mathscr{E}xt^{\dim X}_{\mathscr{D}_X}(\mathcal{M}, \mathscr{D}_X) = \mathscr{E}xt^{\dim X}_{R_F \mathscr{D}_X}(R_F \mathcal{M}, R_F \mathscr{D}_X).$$

A.10.d. Coherence of the pushforward

Theorem A.10.15 (Coherence of the pushforward). Let $f : X \to X'$ be a holomorphic map between complex manifolds and let \mathscr{M} be a coherent $\widetilde{\mathscr{D}}_X$ -module. Assume that \mathscr{M} admits a coherent filtration $F_{\bullet}\mathscr{M}$. Then, if f is proper, the pushforward complex ${}_{D}f_*\mathscr{M}$ has $\widetilde{\mathscr{D}}_{X'}$ -coherent cohomology.

Proof. Assume first that \mathscr{M} is an induced right $\widehat{\mathscr{D}}_X$ -module $\mathscr{L} \otimes_{\widetilde{\mathscr{O}}_X} \widehat{\mathscr{D}}_X$ where \mathscr{L} is a coherent $\widetilde{\mathscr{O}}_X$ -module. Due to the formula of Exercise A.43(3), the result follows from Grauert's direct image theorem. As a consequence, the same result holds for any bounded complex of such induced right $\widetilde{\mathscr{D}}_X$ -modules.

For \mathscr{M} arbitrary, it is enough by Remark A.8.4(3) to prove the coherence of $\mathscr{H}^{j}{}_{D}f_{*}\mathscr{M}$ for $j \in [-\dim X, 2\dim X]$. Since the $\widetilde{\mathscr{D}}_{X'}$ -coherence is a local property on X', it is enough to prove the coherence property in the neighbourhood of any $x' \in X'$, and therefore it is enough to show the existence, in the neighbourhood of the compact set $f^{-1}(x')$, of a resolution of $\mathscr{M}_{-N-1} \to \cdots \to \mathscr{M}_0 \to \mathscr{M} \to 0$ of sufficiently large length N+2, such that \mathscr{M}_j is a coherent induced $\widetilde{\mathscr{D}}_X$ -module for $j = -N, \ldots, 0$.

Since $f^{-1}(x')$ is compact, there exists p such that $F_p \mathscr{M} \otimes_{\widetilde{\mathscr{O}}_X} \mathscr{D}_X$ is onto in some neighbourhood of $f^{-1}(x')$ (i.e., the coherent $\widetilde{\mathscr{O}}_X$ -module $F_p \mathscr{M}$ generates \mathscr{M} as a $\widetilde{\mathscr{D}}_X$ -module). Set $F_q(F_p \mathscr{M} \otimes_{\widetilde{\mathscr{O}}_X} \widetilde{\mathscr{D}}_X) = F_p \mathscr{M} \otimes_{\widetilde{\mathscr{O}}_X} F_{q-p} \widetilde{\mathscr{D}}_X$. This is a coherent filtration of $F_p \mathscr{M} \otimes_{\widetilde{\mathscr{O}}_X} \widetilde{\mathscr{D}}_X$, which therefore induces a coherent filtration on $\operatorname{Ker}[F_p\mathcal{M} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X \to \mathcal{M}].$ Continuing this way N+2 times, we obtained a resolution of length N+2 of \mathcal{M} by coherent induced right $\widetilde{\mathcal{D}}_X$ -modules on some neighbourhood of $f^{-1}(x')$.

Remark A.10.16 (Pushforward of a holonomic $\widetilde{\mathscr{D}}_X$ -module). Assume that the coherent $\widetilde{\mathscr{D}}_X$ -module \mathscr{M} has a good filtration. For example, assume that \mathscr{M} is strict. Then, the pushforward of a holonomic $\widetilde{\mathscr{D}}_X$ -module by a proper holomorphic map has coherent cohomology. Moreover, a theorem of Kashiwara [Kas76] complements Theorem A.10.15 with an estimate of the characteristic variety of the pushforward cohomology $\widetilde{\mathscr{D}}_{X'}$ -modules in terms of the characteristic variety of the source $\widetilde{\mathscr{D}}_X$ -module. This estimate shows that holonomicity is preserved by proper pushforward. (The theorem of Kashiwara is proved for holonomic \mathscr{D}_X -modules, but it extends in a straightforward way to holonomic $\widetilde{\mathscr{D}}_X$ -modules.)

A.11. Appendix: Differential complexes and the Gauss-Manin connection

In this section we switch to the case of \mathscr{D}_X -modules as in Section A.1 (see Remark A.11.9). Let \mathcal{M} be a *left* \mathscr{D}_X -module and let $f: X \to Y$ be a holomorphic mapping. On the one hand, we have defined the direct images ${}_{\mathsf{D}}f_*\mathcal{M}$ or ${}_{\mathsf{D}}f_!\mathcal{M}$ of \mathcal{M} viewed as \mathscr{D}_X -modules. These are objects in $\mathsf{D}^+(\mathscr{D}_Y)^{\text{left}}$. On the other hand, when f is a smooth holomorphic mapping, a flat connection called the *Gauss-Manin connection* is defined on the relative de Rham cohomology of \mathcal{M} . We will compare both constructions, when f is smooth. Such a comparison has essentially already been done when f is the projection of a product $X = Y \times T \to Y$ (see Example A.8.7 and Exercise A.46(??)).

In this section we also introduce the derived category of differential complexes on a complex manifold X, that is, complexes of \mathcal{O}_X -modules with differential morphisms as differential. This derived category is shown to be equivalent to that of \mathcal{D}_X -modules (Theorem A.11.16). It is sometimes useful to work in this category (see e.g. the proof of Theorem A.11.23).

A.11.a. Differential complexes. Given an \mathscr{O}_X -module \mathscr{L} , there is a natural \mathscr{O}_X -linear morphism (with the right structure on the right-hand term)

$$\mathcal{L} \longrightarrow \mathcal{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X, \quad \ell \longmapsto \ell \otimes 1.$$

There is also a (only) \mathbb{C} -linear morphism

$$(A.11.1) \qquad \qquad \mathcal{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X \longrightarrow \mathcal{L}$$

defined at the level of local sections by $\ell \otimes P \mapsto P(1)\ell$, where P(1) is the result of the action of the differential operator P on 1, which is equal to the degree 0 coefficient of P if P is locally written as $\sum_{\alpha} a_{\alpha}(x)\partial_x^{\alpha}$. In an intrinsic way, consider the natural augmentation morphism $\mathscr{D}_X \to \mathscr{O}_X$, which is left \mathscr{D}_X -linear, hence left \mathscr{O}_X -linear; then apply $\mathcal{L} \otimes_{\mathscr{O}_X} \bullet$ to it. Notice however that (A.11.1) is an \mathscr{O}_X -linear morphism by using the left \mathscr{O}_X -module structure on $\mathcal{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X$.

Exercise A.60. Let \mathcal{L} be an \mathscr{O}_X -module. Show that the terms of the Spencer complex $\operatorname{Sp}(\mathcal{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X)$ are \mathscr{O}_X -modules with the "left" structure, that the differentials are \mathscr{O}_X -linear, that this complex is a resolution of \mathcal{L} as an \mathscr{O}_X -module and that the morphism (A.11.1) is the augmentation morphism $\operatorname{Sp}^0(\mathcal{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X) \to \mathcal{L}$. [*Hint*: use Exercise A.29.]

Let $\mathcal{L}, \mathcal{L}'$ be two \mathscr{O}_X -modules. A (right) \mathscr{D}_X -linear morphism

(A.11.2)
$$v: \mathcal{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X \longrightarrow \mathcal{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X$$

is uniquely determined by the \mathcal{O}_X -linear morphism

$$(A.11.3) w: \mathcal{L} \longrightarrow \mathcal{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X$$

that it induces (where the right \mathscr{O}_X -module structure is chosen on $\mathcal{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X$). In other words, the natural morphism

$$\operatorname{Hom}_{\mathscr{O}_{X}}(\mathcal{L},\mathcal{L}'\otimes_{\mathscr{O}_{X}}\mathscr{D}_{X})\longrightarrow \operatorname{Hom}_{\mathscr{D}_{X}}(\mathcal{L}\otimes_{\mathscr{O}_{X}}\mathscr{D}_{X},\mathcal{L}'\otimes_{\mathscr{O}_{X}}\mathscr{D}_{X})$$

is an isomorphism. We also have, at the sheaf level,

(A.11.4)
$$\mathscr{H}om_{\mathscr{O}_X}(\mathcal{L},\mathcal{L}'\otimes_{\mathscr{O}_X}\mathscr{D}_X) \xrightarrow{\sim} \mathscr{H}om_{\mathscr{D}_X}(\mathcal{L}\otimes_{\mathscr{O}_X}\mathscr{D}_X,\mathcal{L}'\otimes_{\mathscr{O}_X}\mathscr{D}_X).$$

Notice that $\mathscr{H}om_{\mathscr{O}_X}(\mathcal{L}, \mathcal{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X)$ is naturally equipped with an \mathscr{O}_X -module structure by using the left \mathscr{O}_X -module structure on $\mathcal{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X$ (see Remark A.6.1), and similarly $\operatorname{Hom}_{\mathscr{O}_X}(\mathcal{L}, \mathcal{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X)$ is a $\Gamma(X, \mathscr{O}_X)$ -module.

Now, w induces a \mathbb{C} -linear morphism

$$(A.11.5) u: \mathcal{L} \longrightarrow \mathcal{L}',$$

by composition with (A.11.1): $\mathcal{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X \to \mathcal{L}'$. By Exercise A.60, u is nothing but the morphism

$$\mathscr{H}^{0}({}^{\mathrm{p}}\mathrm{DR}(v)): \mathscr{H}^{0}({}^{\mathrm{p}}\mathrm{DR}(\mathcal{L}\otimes_{\mathscr{O}_{X}}\mathscr{D}_{X})) \longrightarrow \mathscr{H}^{0}({}^{\mathrm{p}}\mathrm{DR}(\mathcal{L}'\otimes_{\mathscr{O}_{X}}\mathscr{D}_{X})).$$

Definition A.11.6 (Differential operators between two \mathcal{O}_X -modules)

The \mathbb{C} -vector space $\operatorname{Hom}_{\operatorname{Diff}}(\mathcal{L}, \mathcal{L}')$ of differential operators from \mathcal{L} to \mathcal{L}' is the image of the morphism $\operatorname{Hom}_{\mathscr{D}_X}(\mathcal{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X, \mathcal{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X) \to \operatorname{Hom}_{\mathbb{C}}(\mathcal{L}, \mathcal{L}').$

Similarly we define the sheaf of \mathbb{C} -vector spaces $\mathscr{H}om_{\mathrm{Diff}}(\mathcal{L}, \mathcal{L}')$.

Exercise A.61.

(1) Show that any \mathscr{O}_X -linear morphism $u : \mathcal{L} \to \mathcal{L}'$ is a differential operator from \mathcal{L} to \mathcal{L}' and that a corresponding v is $u \otimes 1$.

(2) Assume that $\mathcal{L}, \mathcal{L}'$ are right \mathscr{D}_X -modules. Let $u : \mathcal{L} \to \mathcal{L}'$ be \mathscr{D}_X -linear. Show that the corresponding v is \mathscr{D}_X -linear for both structures $(\text{right})_{\text{triv}}$ and $(\text{right})_{\text{tens}}$ (see Exercise A.19) on $\mathcal{L}^{(l)} \otimes_{\mathscr{O}_X} \mathscr{D}_X$.

(3) Show that $\mathscr{H}om_{\mathrm{Diff}}(\mathscr{O}_X, \mathscr{O}_X) = \mathscr{D}_X.$

(4) Show that the morphism in Definition A.11.6 is compatible with composition. Conclude that the composition of differential operators is a differential operator and that it is associative.

Exercise A.62 (Integrable connections are differential operators)

Let \mathcal{M} be an \mathscr{O}_X -module and let $\nabla : \mathcal{M} \to \Omega^1_X \otimes_{\mathscr{O}_X} \mathcal{M}$ be an integrable connection on \mathcal{M} .

(1) Show that ∇ is a differential morphism, by considering the right \mathscr{D}_X -linear morphism

$$v(m \otimes P) := \nabla(m) \otimes P + m \otimes \nabla(P),$$

for any local section m of \mathcal{M} and P of \mathscr{D}_X , and where ∇P is defined in Exercise A.5. Extend this result to connections ${}^{(k)}\nabla$.

(2) Let $\mathcal{M}', \mathcal{M}''$ be \mathscr{O}_X -submodules of \mathcal{M} such that ${}^{(k)}\nabla$ induces a \mathbb{C} -linear morphism ${}^{(k)}\nabla': \Omega^k_X \otimes_{\mathscr{O}_X} \mathcal{M}' \to \Omega^{k+1}_X \otimes_{\mathscr{O}_X} \mathcal{M}''$. Show that ${}^{(k)}\nabla'$ is a differential morphism.

Definition A.11.7 (The category $Mod(\mathcal{O}_X, Diff_X)$). We denote by $Mod(\mathcal{O}_X, Diff_X)$ the category whose objects are \mathcal{O}_X -modules and morphisms are differential operators between \mathcal{O}_X -modules (this is justified by Exercise A.61(4)).

In particular, $\mathsf{Mod}(\mathscr{O}_X)$ is a subcategory of $\mathsf{Mod}(\mathscr{O}_X, \mathrm{Diff}_X)$, since any \mathscr{O}_X -linear morphism is a differential operator (of degree zero).

Exercise A.63. Show that $Mod(\mathcal{O}_X, Diff_X)$ is an additive category, i.e.,

- Hom_{Diff} $(\mathcal{L}, \mathcal{L}')$ is a \mathbb{C} -vector space and the composition is \mathbb{C} -bilinear,
- the 0 \mathscr{O}_X -module satisfies $\operatorname{Hom}_{\operatorname{Diff}}(0,0) = 0$,

• $\operatorname{Hom}_{\operatorname{Diff}}(\mathcal{L}_1 \oplus \mathcal{L}_2, \mathcal{L}') = \operatorname{Hom}_{\operatorname{Diff}}(\mathcal{L}_1, \mathcal{L}') \oplus \operatorname{Hom}_{\operatorname{Diff}}(\mathcal{L}_2, \mathcal{L}')$ and similarly with $\mathcal{L}'_1, \mathcal{L}'_2$.

We will now show that the correspondence $\mathcal{L} \mapsto \mathcal{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X$ induces a functor $\mathsf{Mod}(\mathscr{O}_X, \mathrm{Diff}_X) \mapsto \mathsf{Mod}_i(\mathscr{D}_X)$. In order to do so, one first needs to show that to any differential morphism u corresponds at most one v.

Lemma A.11.8. The morphism

$$\operatorname{Hom}_{\mathscr{D}_X}(\mathcal{L}\otimes_{\mathscr{O}_X}\mathscr{D}_X,\mathcal{L}'\otimes_{\mathscr{O}_X}\mathscr{D}_X)\longrightarrow \operatorname{Hom}_{\mathbb{C}}(\mathcal{L},\mathcal{L}')$$
$$v\longmapsto u$$

is injective.

Proof. Recall that, for any multi-index β , we have $\partial_x^{\alpha}(x^{\beta}) = 0$ if $\beta_i < \alpha_i$ for some i, and $\partial_x^{\alpha}(x^{\alpha}) = \alpha$!. Assume that u = 0. Let ℓ be a local section of \mathcal{L} and, using local coordinates (x_1, \ldots, x_n) , write in a unique way $w(\ell) = \sum_{\alpha} w(\ell)_{\alpha} \otimes \partial_x^{\alpha}$, where the sum is taken on multi-indices α and w is as in (A.11.3). If $w(\ell) \neq 0$, let β be minimal (with respect to the usual partial ordering on \mathbb{N}^n) among the multi-indices α such that $w(\ell)_{\alpha} \neq 0$. Then,

$$0 = u(x^{\beta}\ell) = \sum_{\alpha} \partial_x^{\alpha}(x^{\beta})w(\ell)_{\alpha} = \beta!w(\ell)_{\beta},$$

a contradiction.

Remark A.11.9. A similar lemma would not hold in the category of induced graded $R_F \mathscr{D}_X$ -modules because of possible z-torsion: one would only get that $z^k u(x^\beta \ell) = 0$ for some k.

According to Lemma A.11.8, the following definition is meaningful.

Definition A.11.10 (The inverse de Rham functor). The functor

 $^{\operatorname{diff}}\mathrm{DR}^{-1}: \mathsf{Mod}(\mathscr{O}_X, \operatorname{Diff}_X) \longrightarrow \mathsf{Mod}_{\mathrm{i}}(\mathscr{D}_X)$

is defined by ${}^{\operatorname{diff}}\mathrm{DR}^{-1}(\mathcal{L}) = \mathcal{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X$ and ${}^{\operatorname{diff}}\mathrm{DR}^{-1}(u) = v$.

Exercise A.64 (De Rham and inverse de Rham on induced *D*-modules)

(1) Let \mathcal{L} be an \mathscr{O}_X -module. Show that $\mathscr{H}^k({}^{\mathrm{p}}\mathrm{DR}(\mathcal{L}\otimes_{\mathscr{O}_X}\mathscr{D}_X)) = 0$ for $k \neq 0$ and $\mathscr{H}^0({}^{\mathrm{p}}\mathrm{DR}(\mathcal{L}\otimes_{\mathscr{O}_X}\mathscr{D}_X)) = \mathcal{L}$. [*Hint*: use Exercise A.60.]

(2) Show that $\mathscr{H}^{0}({}^{\mathrm{p}}\mathrm{DR})$ defines a functor $\mathsf{Mod}_{i}(\mathscr{D}_{X}) \mapsto \mathsf{Mod}(\mathscr{O}_{X}, \mathrm{Diff}_{X})$, which will be denoted by ${}^{\mathrm{diff}}\mathrm{DR}$.

(3) Show that ${}^{\operatorname{diff}}\mathrm{DR}^{-1}$: $\mathsf{Mod}(\mathscr{O}_X, \operatorname{Diff}_X) \mapsto \mathsf{Mod}_{i}(\mathscr{D}_X)$ is an equivalence of categories, a quasi-inverse functor being ${}^{\operatorname{diff}}\mathrm{DR} : \mathcal{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X \mapsto \mathcal{L}$, ${}^{\operatorname{diff}}\mathrm{DR}(v) = u$.

(4) Show that the composed functor $\mathsf{Mod}(\mathscr{O}_X, \mathrm{Diff}_X) \mapsto \mathsf{Mod}_i(\mathscr{D}_X) \mapsto \mathsf{Mod}(\mathscr{D}_X)$, still denoted by ^{diff}DR⁻¹, is *fully faithful*, i.e., it induces a bijective morphism

$$\operatorname{Hom}_{\operatorname{Diff}}(\mathcal{L},\mathcal{L}') \xrightarrow{\sim} \operatorname{Hom}_{\mathscr{D}_X}(\mathcal{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X, \mathcal{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X).$$

(One may think that we "embed" the additive (nonabelian) category $\mathsf{Mod}(\mathscr{O}_X, \mathrm{Diff}_X)$ in the abelian category $\mathsf{Mod}(\mathscr{D}_X)$; we will use this "embedding" to define below acyclic objects).

Remark A.11.11. By the isomorphism of Exercise A.64, $\operatorname{Hom}_{\operatorname{Diff}}(\mathcal{L}, \mathcal{L}')$ is equipped with the structure of a $\Gamma(X, \mathcal{O}_X)$ -module. Similarly,

 $\mathscr{H}om_{\mathscr{D}_X}(\mathcal{L}\otimes_{\mathscr{O}_X}\mathscr{D}_X,\mathcal{L}'\otimes_{\mathscr{O}_X}\mathscr{D}_X)\longrightarrow \mathscr{H}om_{\mathbb{C}}(\mathcal{L},\mathcal{L}')$

is injective, and and this equips the image sheaf $\mathscr{H}om_{\mathrm{Diff}}(\mathcal{L}, \mathcal{L}')$ with the structure of an \mathscr{O}_X -module.

Remark A.11.12. When considered as taking values in $\mathsf{Mod}(\mathscr{D}_X)$, the functor $^{\text{diff}}\mathsf{DR}^{-1}$ is not, however, an equivalence of categories, i.e., is not essentially surjective. The reason is that, first, not all \mathscr{D}_X -modules are isomorphic to some $\mathcal{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X$ and, next, its natural quasi-inverse would be the de Rham functor ${}^{\mathrm{p}}\mathsf{DR}$ which takes values in a category of complexes. Nevertheless, if one extends suitably these functors to categories of complexes, they become equivalences (see below Theorem A.11.16).

A.11.b. The de Rham complex as a differential complex. According to Exercise A.63, one may consider the category $C^*(\mathscr{O}_X, \operatorname{Diff}_X)$ of *-bounded complexes of objects of $\operatorname{Mod}(\mathscr{O}_X, \operatorname{Diff}_X)$ (with $\star = \varnothing, +, -, \operatorname{b}$), and the category $\mathsf{K}^*(\mathscr{O}_X, \operatorname{Diff}_X)$ of *-bounded complexes up to homotopy (see [**KS90**, Def. 1.3.4]). These are called *-bounded differential complexes.

Exercise A.65 (The de Rham functor $^{\rm diff}DR$).

(1) Show that the de Rham complex of a left \mathscr{D}_X -module \mathcal{M} is a complex in $\mathsf{C}^{\mathrm{b}}(\mathscr{O}_X, \mathrm{Diff}_X)$. [*Hint*: use Exercise A.61(1).]

(2) By using Exercise A.28(1), show that the de Rham complex of a right \mathscr{D}_X -module \mathcal{M} is a complex in $\mathsf{C}^{\mathrm{b}}(\mathscr{O}_X, \mathrm{Diff}_X)$

(3) Show that the de Rham complex of a *-bounded complex of right \mathscr{D}_X -modules has its associated single complex in $C^*(\mathscr{O}_X, \operatorname{Diff}_X)$. [*Hint*: use Exercise A.27.]

(4) Conclude that ^PDR induces a functor ^{diff}DR : $C^{\star}(\mathscr{D}_X) \mapsto C^{\star}(\mathscr{O}_X, \operatorname{Diff}_X)$.

(5) Extend this functor as a functor of triangulated categories $\mathsf{K}^{\star}(\mathscr{D}_X) \to \mathsf{K}^{\star}(\mathscr{O}_X, \mathrm{Diff}_X).$

Exercise A.66. Let \mathcal{M} be a \mathscr{D}_X -module. Show that $\operatorname{God}^{\bullet\operatorname{diff}}\operatorname{DR}\mathcal{M}$ is a differential complex. [*Hint*: Identify this complex with $\operatorname{diff}\operatorname{DR}\operatorname{God}^{\bullet}\mathcal{M}$.]

There is a natural forgetful functor Forget from $\mathsf{Mod}(\mathscr{O}_X, \mathrm{Diff}_X)$ to $\mathsf{Mod}(\mathbb{C}_X)$, and by extension a functor Forget at the level of C^{*} and K^{*}. The previous exercise shows that we can decompose the ^PDR functor as

$$\mathsf{Mod}(\mathscr{D}_X) \xrightarrow[\operatorname{diff} \mathrm{DR}]{} \mathcal{C}^{\mathrm{b}}(\mathscr{O}_X, \mathrm{Diff}_X) \xrightarrow[\operatorname{\mathsf{Forget}}]{} \mathcal{C}^{\mathrm{b}}(\mathbb{C}_X)$$

and

$$\mathsf{K}^{\star}(\mathscr{D}_{X}) \xrightarrow[\operatorname{diff} \mathrm{DR}]{} \mathsf{K}^{\star}(\mathscr{O}_{X}, \mathrm{Diff}_{X}) \xrightarrow[\operatorname{\mathsf{Forget}}]{} \mathsf{K}^{\star}(\mathbb{C}_{X})$$

In order to define the "derived category" of the additive category $\mathsf{Mod}(\mathscr{O}_X, \mathrm{Diff}_X)$, one needs to define the notion of null system in $\mathsf{K}^*(\mathscr{O}_X, \mathrm{Diff}_X)$ and localize the category with respect to the associated multiplicative system. A possible choice would be to say that an object belongs to the null system if it belongs to the null system of $\mathsf{C}^*(\mathbb{C}_X)$ when forgetting the Diff structure, i.e., which has zero cohomology when considered as a complex of sheaves of \mathbb{C} -vector spaces. This is not the choice made below. One says that a differential morphism $u : \mathcal{L} \to \mathcal{L}'$ as in (A.11.5) is a Diffquasi-isomorphism if the corresponding v as in (A.11.2) is a quasi-isomorphism of right \mathscr{D}_X -modules.

The functor $\operatorname{diff} \operatorname{DR}^{-1}$ of Definition A.11.10 extends as a functor $\mathsf{C}^{\star}(\mathscr{O}_X, \operatorname{Diff}_X) \mapsto \mathsf{C}^{\star}_{\mathsf{i}}(\mathscr{D}_X)$ and $\mathsf{K}^{\star}(\mathscr{O}_X, \operatorname{Diff}_X) \mapsto \mathsf{K}^{\star}_{\mathsf{i}}(\mathscr{D}_X)$ in a natural way, and is a functor of triangulated categories on K . Moreover, according to the last part of Exercise A.64, it is an equivalence of triangulated categories.

We wish now to define *acyclic objects* in the triangulated category $\mathsf{K}^*(\mathscr{O}_X, \operatorname{Diff}_X)$, and show that they form a *null system* in the sense of [**KS90**, Def. 1.6.6].

Definition A.11.13. We say that a object \mathcal{L}^{\bullet} of $\mathsf{K}^{\star}(\mathscr{O}_X, \mathrm{Diff}_X)$ is Diff-acyclic if $^{\mathrm{diff}}\mathrm{DR}^{-1}(\mathcal{L}^{\bullet})$ is acyclic in $\mathsf{K}^{\star}_{\mathsf{i}}(\mathscr{D}_X)$ (equivalently, in $\mathsf{K}^{\star}(\mathscr{D}_X)$).

Exercise A.67. Show that the family N of Diff-acyclic objects forms a *null system* in $K^*(\mathscr{O}_X, \operatorname{Diff}_X)$, i.e.,

- the object 0 belongs to N,
- an object \mathcal{L}^{\bullet} belongs to N iff $\mathcal{L}^{\bullet}[1]$ does so,

• if $\mathcal{L}^{\bullet} \to \mathcal{L}'^{\bullet} \to \mathcal{L}''^{\bullet} \to \mathcal{L}^{\bullet}[1]$ is a distinguished triangle of $\mathsf{K}^{\star}(\mathscr{O}_X, \mathrm{Diff}_X)$, and if $\mathcal{L}^{\bullet}, \mathcal{L}'^{\bullet}$ are objects in N , then so is \mathcal{L}''^{\bullet} .

[*Hint*: use that the extension of $^{\text{diff}}$ DR⁻¹ to the categories K^{*} is a functor of triangulated categories.]

Define, as in [**KS90**, (1.6.4)], the family $S(\mathbb{N})$ as the family of morphisms which can be embedded in a distinguished triangle of $\mathsf{K}^*(\mathscr{O}_X, \operatorname{Diff}_X)$, with the third term being an object of N. We call such morphisms Diff -quasi-isomorphisms. Clearly, they correspond exactly via diff DR^{-1} to quasi-isomorphisms in $\mathsf{K}^*(\mathscr{D}_X)$.

We now may localize the category $\mathsf{K}^{\star}(\mathscr{O}_X, \operatorname{Diff}_X)$ with respect to the null system N and get a category denoted by $\mathsf{D}^{\star}(\mathscr{O}_X, \operatorname{Diff}_X)$. By construction, we still get a functor

(A.11.14) $\operatorname{diff} \operatorname{DR}^{-1} : \operatorname{\mathsf{D}}^{\star}(\mathscr{O}_X, \operatorname{Diff}_X) \longrightarrow \operatorname{\mathsf{D}}^{\star}_{i}(\mathscr{D}_X) \longrightarrow \operatorname{\mathsf{D}}^{\star}(\mathscr{D}_X).$

We note that the first component is an equivalence by definition of the null system (since we have an equivalence at the level of the categories K^*). The second component is also an equivalence, according to Corollary A.6.3. We will show below (Theorem A.11.16) that ^{diff}DR is a quasi-inverse functor.

Exercise A.68 (The functor $\mathsf{D}^*(\mathscr{O}_X) \mapsto \mathsf{D}^*(\mathscr{O}_X, \operatorname{Diff}_X)$). Using Exercise A.61(1), define a functor $\mathsf{C}^*(\mathscr{O}_X) \mapsto \mathsf{C}^*(\mathscr{O}_X, \operatorname{Diff}_X)$ and $\mathsf{K}^*(\mathscr{O}_X) \mapsto \mathsf{K}^*(\mathscr{O}_X, \operatorname{Diff}_X)$. By using that \mathscr{D}_X is \mathscr{O}_X -flat, show that if \mathcal{L}^\bullet is acyclic in $\mathsf{K}^*(\mathscr{O}_X)$, then $\mathcal{L}^\bullet \otimes_{\mathscr{O}_X} \mathscr{D}_X$ is acyclic in $\mathsf{K}^*(\mathscr{D}_X)$. Conclude that the previous functor extends as a functor $\mathsf{D}^*(\mathscr{O}_X) \mapsto \mathsf{D}^*(\mathscr{O}_X, \operatorname{Diff}_X)$.

Remark A.11.15. The category $\mathsf{Mod}(\mathscr{O}_X, \mathsf{Diff}_X)$ is also naturally a subcategory of the category $\mathsf{Mod}(\mathbb{C}_X)$ of sheaves of \mathbb{C} -vector spaces because $\operatorname{Hom}_{\operatorname{Diff}}(\mathcal{L}, \mathcal{L}')$ is a subset of $\operatorname{Hom}_{\mathbb{C}}(\mathcal{L}, \mathcal{L}')$. We therefore have a natural functor $\mathsf{Forget} : \mathsf{K}^*(\mathscr{O}_X, \operatorname{Diff}_X) \to \mathsf{K}^*(\mathbb{C}_X)$, forgetting that the differentials of a complex are differential operators, and forgetting also that the homotopies should be differential operators too. As a consequence of Theorem A.11.16, we will see in Exercise A.70 that any object in the null system N defined above is sent to an object in the usual null system of $\mathsf{K}^*(\mathbb{C}_X)$, i.e., objects with zero cohomology. In other words, a Diff-quasi-isomorphism is sent into a usual quasi-isomorphism. But there may exist morphisms in $\mathsf{K}^*(\mathscr{O}_X, \operatorname{Diff}_X)$ which are quasi-isomorphisms when viewed in $\mathsf{K}^*(\mathbb{C}_X)$, but are not Diff-quasi-isomorphisms.

Theorem A.11.16. The functors diff $DR : D^*(\mathscr{D}_X) \to D^*(\mathscr{O}_X, Diff_X)$ and diff $DR^{-1} : D^*(\mathscr{O}_X, Diff_X) \to D^*(\mathscr{D}_X)$ are quasi-inverse functors.

Lemma A.11.17. Let $\mathcal{M}^{\text{right}}$ be a right \mathcal{D}_X -module and let $\mathcal{M}^{\text{left}}$ be the associated left \mathcal{D}_X -module. Then there is a functorial isomorphism of complexes of right \mathcal{D}_X -modules $\left[\stackrel{\text{diff}}{\text{DR}} (\mathcal{M}^{\text{right}} \otimes_{\mathcal{O}_X} \mathcal{D}_X)_{\text{triv}} \right]_{\text{tens}}$ (i.e., the structure $(right)_{\text{triv}}$ is used for $\stackrel{\text{diff}}{\text{DR}}$ and the remaining structure $(right)_{\text{tens}}$ gives the right \mathcal{D}_X -module structure) and $\left[\stackrel{\text{diff}}{\text{DR}} (\mathcal{M}^{\text{left}} \otimes_{\mathcal{O}_X} \mathcal{D}_X)_{\text{tens}} \right]_{\text{triv}} [n]$ (similar meaning), and the latter complex is an object of $C^{\text{b}}_{\text{b}}(\mathcal{D}_X)$.

Moreover, there is an isomorphism of functors diff $DR^{-1} \operatorname{diff} DR \simeq \operatorname{diff} DR^{-1} \operatorname{from} C^{\mathrm{b}}(\mathscr{D}_X)$ to itself.

Proof. We note that $(\mathcal{M}^{\text{left}} \otimes_{\mathscr{O}_X} \mathscr{D}_X)^{\text{right}} \simeq \mathcal{M}^{\text{right}} \otimes_{\mathscr{O}_X} \mathscr{D}_X$, both with the tensor structure, respectively left and right, and this isomorphism is compatible with the right \mathscr{D}_X -structure $(\text{right})_{\text{triv}}$ on both terms. On the other hand, $^{\text{diff}} DR(\mathcal{M}^{\text{left}} \otimes_{\mathscr{O}_X} \mathscr{D}_X)_{\text{tens}}$ is the complex $\Omega^{\bullet}_X \otimes (\mathcal{M}^{\text{left}} \otimes_{\mathscr{O}_X} \mathscr{D}_X)$ with differential $^{\text{diff}} DR^{-1}(\nabla)$ given by a formula like in Exercise A.62, and is clearly a complex in $C^{\text{b}}_{i}(\mathscr{D}_X)$ with respect to the (right)_{triv}-structure.

The isomorphism ${}^{\text{diff}} \text{DR}(\mathcal{M}^{\text{left}} \otimes_{\mathscr{O}_X} \mathscr{D}_X)_{\text{tens}}[n] \simeq {}^{\text{diff}} \text{DR}(\mathcal{M}^{\text{right}} \otimes_{\mathscr{O}_X} \mathscr{D}_X)_{\text{tens}}$ of Exercise A.28(1) is compatible with the (right)_{\text{triv}}-structure, hence

$$\begin{bmatrix} \text{diff} DR(\mathcal{M}^{\text{right}} \otimes_{\mathscr{O}_X} \mathscr{D}_X)_{\text{triv}} \end{bmatrix}_{\text{tens}} \simeq \begin{bmatrix} \text{diff} DR(\mathcal{M}^{\text{right}} \otimes_{\mathscr{O}_X} \mathscr{D}_X)_{\text{tens}} \end{bmatrix}_{\text{triv}} \quad (\text{Exercise A.19}) \\ \simeq \begin{bmatrix} \text{diff} DR(\mathcal{M}^{\text{left}} \otimes_{\mathscr{O}_X} \mathscr{D}_X)_{\text{tens}} \end{bmatrix}_{\text{triv}} [n].$$

For the last assertion, we note that, by definition (see Exercise A.65(1) and (2)),

$${}^{\operatorname{diff}}\mathrm{DR}^{-1}\operatorname{diff}\mathrm{DR}\operatorname{\mathcal{M}}^{\operatorname{right}} = {}^{\operatorname{diff}}\mathrm{DR}^{-1}\operatorname{diff}\mathrm{DR}\operatorname{\mathcal{M}}^{\operatorname{left}}[n].$$

Now,

$$\left[{}^{\rm diff} DR^{^{-1} \, \rm diff} DR \, \mathcal{M}^{\rm left} \right]_{\rm triv} \simeq \left[{}^{\rm diff} DR ({}^{\rm diff} DR^{^{-1}} \, \mathcal{M}^{\rm left})_{\rm tens} \right]_{\rm triv}$$

follows from Exercise A.62: indeed,

$${}^{\operatorname{diff}}\mathrm{DR}^{-1} {}^{\operatorname{diff}}\mathrm{DR} \,\mathcal{M}^{\operatorname{left}} = (\Omega^{\bullet}_X \otimes \mathcal{M}^{\operatorname{left}}) \otimes \mathscr{D}_X$$

with differential $^{\text{diff}}\text{DR}^{-1}(\nabla)$, which is nothing but the complex $\Omega^{\bullet}_X \otimes (\mathcal{M}^{\text{left}} \otimes \mathscr{D}_X)$ where the differential is the connection on the left \mathscr{D}_X -module $(\mathcal{M}^{\text{left}} \otimes \mathscr{D}_X)_{\text{tens}}$. Moreover, this identification is right \mathscr{D}_X -linear with respect to the (right)_{triv} structure on both terms.

Lastly, ${}^{\text{diff}}\text{DR} {}^{\text{diff}}\text{DR} {}^{-1}\mathcal{M}^{\text{left}}[n] = [{}^{\text{diff}}\text{DR}(\mathcal{M}^{\text{left}} \otimes \mathscr{D}_X)_{\text{triv}}]_{\text{tens}}[n]$ is identified with

$$[{}^{\operatorname{diff}}\mathrm{DR}(\mathcal{M}^{\operatorname{right}}\otimes\mathscr{D}_X)_{\operatorname{triv}}]_{\operatorname{tens}} = {}^{\operatorname{diff}}\mathrm{DR}^{\operatorname{diff}}\mathrm{DR}^{-1}\mathcal{M}^{\operatorname{right}}$$

by the previous computation.

Lemma A.11.18. There is an isomorphism of functors ${}^{\text{diff}}\text{DR}^{-1} {}^{\text{diff}}\text{DR} \xrightarrow{\sim} \text{Id}$ from $D^*(\mathscr{D}_X)$ to itself.

Proof. By the previous lemma, we have

$$\begin{split} ^{\text{hiff}} \mathrm{DR}^{\text{-1 diff}} \mathrm{DR}(\bullet) &\simeq ^{\text{diff}} \mathrm{DR}^{\text{-1}}(\bullet) = \left[\begin{smallmatrix} ^{\text{diff}} \mathrm{DR}(\bullet \otimes \mathscr{D}_X)_{\text{triv}} \end{bmatrix}_{\text{tens}} \\ &\simeq \left[\bullet \otimes ^{\text{diff}} \mathrm{DR}(\mathscr{D}_X)_{\text{triv}} \right]_{\text{tens}} \simeq \left[\bullet \otimes \mathscr{O}_X \right]_{\text{tens}} = \bullet. \end{split}$$

Proof of Theorem A.11.16. From the previous lemma, it is now enough to show that, if \mathcal{L}^{\bullet} is a complex in $C^{\star}(\mathscr{O}_X, \operatorname{Diff}_X)$, there exists a a Diff-quasi-isomorphism $\operatorname{diff} \operatorname{DR}^{\operatorname{diff}} \operatorname{DR}^{-1} \mathcal{L}^{\bullet} \to \mathcal{L}^{\bullet}$, and this is equivalent to showing the existence of a quasi-isomorphism $\operatorname{diff} \operatorname{DR}^{\operatorname{-1}\operatorname{diff}} \operatorname{DR}^{\operatorname{-1}\operatorname{diff}} \operatorname{DR}^{-1} \mathcal{L}^{\bullet} \to \operatorname{diff} \operatorname{DR}^{-1} \mathcal{L}^{\bullet}$, that we know from the previous result applied to $\mathcal{M} = \operatorname{diff} \operatorname{DR}^{-1} \mathcal{L}^{\bullet}$.

Remark A.11.19. The functor ${}^{\text{diff}}\mathrm{DR}^{-1} {}^{\text{diff}}\mathrm{DR}$, regarded as a functor $\mathsf{D}^{\star}(\mathscr{D}_X) \to \mathsf{D}^{\star}_{\mathsf{i}}(\mathscr{D}_X)$, is nothing but that of Corollary A.6.3.

Remark A.11.20 (The Godement resolution of a differential complex)

Let \mathcal{L}^{\bullet} be an object of $C^+(\mathscr{O}_X, \operatorname{Diff}_X)$. Then $\operatorname{God}^{\bullet} \mathcal{L}^{\bullet}$ is maybe not a differential complex (see Exercise A.40(2)). However, $\operatorname{God}^{\bullet} \operatorname{diff} \operatorname{DR}^{-1} \mathcal{L}^{\bullet}$ is a differential complex, being equal to $\operatorname{diff} \operatorname{DR} \operatorname{God}^{\bullet} \operatorname{diff} \operatorname{DR}^{-1} \mathcal{L}^{\bullet}$. Therefore, the composite functor $\operatorname{God}^{\bullet} \operatorname{diff} \operatorname{DR}^{-1} \operatorname{plays}$ the role of Godement resolutions in the category of differential complexes.

Exercise A.69. Show that the following diagram commutes:

$$\mathsf{D}^{\star}(\mathscr{D}_X) \xrightarrow[\operatorname{diff} \mathrm{DR}]{} \mathsf{D}^{\star}(\mathscr{O}_X, \mathrm{Diff}_X) \xrightarrow[\operatorname{Forget}]{} \mathsf{Forget} \mathsf{D}^{\star}(\mathbb{C}_X)$$

Exercise A.70. Assume that \mathcal{L}^{\bullet} is Diff-acyclic. Show that Forget \mathcal{L}^{\bullet} is acyclic. [*Hint*: by definition, $^{\text{diff}} DR^{-1}(\mathcal{L}^{\bullet})$ is acyclic; then $^{P}DR^{\text{diff}} DR^{-1}(\mathcal{L}^{\bullet})$ is also acyclic and quasi-isomorphic to Forget \mathcal{L}^{\bullet} .]

Conclude that Forget induces a functor $\mathsf{D}^*(\mathscr{O}_X, \mathrm{Diff}_X) \mapsto \mathsf{D}^*(\mathbb{C}_X)$, and that we have an isomorphism of functors

$${}^{\mathrm{p}}\mathrm{DR}^{\mathrm{diff}}\mathrm{DR}^{-1} \xrightarrow{\sim} \mathsf{Forget} : \mathsf{D}^{\star}(\mathscr{O}_X, \mathrm{Diff}_X) \longmapsto \mathsf{D}^{\star}(\mathbb{C}_X).$$

Compare with Exercise A.60.

Exercise A.71. Let $\mathcal{L}, \mathcal{L}'$ be two \mathcal{O}_X -modules and

 $v: \mathcal{M} = \mathcal{L} \otimes_{\mathscr{O}_X} \mathscr{D}_X \longrightarrow \mathcal{M}' = \mathcal{L}' \otimes_{\mathscr{O}_X} \mathscr{D}_X$

a \mathscr{D}_X -linear morphism. It defines a $f^{-1}\mathscr{D}_Y$ -linear morphism

 $v \otimes \mathbf{1} : \mathfrak{M} \otimes_{\mathscr{D}_X} \mathscr{D}_{X \to Y} \longrightarrow \mathfrak{M}' \otimes_{\mathscr{D}_X} \mathscr{D}_{X \to Y},$

where 1 is the section introduced in Exercise A.44(1). This is therefore a morphism

$$\widetilde{v}: \mathcal{L} \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{D}_Y \longrightarrow \mathcal{L}' \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{D}_Y.$$

Show that ${}^{\operatorname{diff}}\mathrm{DR}_Y(\widetilde{v}) = {}^{\operatorname{diff}}\mathrm{DR}_X(v).$

[*Hint*: since the problem is local, argue with coordinates on X and Y and write $f = (f_1, \ldots, f_m)$. Let ℓ be a local section of \mathcal{L} , and let $\mathbf{1}_X$ be the unit of \mathscr{D}_X . Set $v(\ell \otimes \mathbf{1}_X) = w(\ell) = \sum_{\alpha} w(\ell)_{\alpha} \otimes \partial_x^{\alpha}$ and $\tilde{v}(\ell \otimes \mathbf{1}_X) = v(\ell \otimes \mathbf{1}_X) \otimes \mathbf{1}_{X \to Y}$. Show that, if $\alpha_i \neq 0$,

$$\partial_{x_i}^{\alpha_i} \otimes \mathbf{1}_{X \to Y} = \partial_{x_i}^{\alpha_i - 1} \sum_j \frac{\partial f_j}{\partial x_i} \otimes \partial_{y_j}.$$

Deduce that the image of $\tilde{v}(\ell \otimes \mathbf{1}_X)$ by the map $\mathcal{L} \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{D}_Y \to \mathcal{L}$ is equal to the image of $w(\ell)_0$, which is nothing but $u(\ell)$ by definition of $u := \mathscr{H}^0 \operatorname{DR}_X(v)$.]

A.11.c. The Gauss-Manin connexion. We assume in this section that $f: X \to Y$ is a *smooth* holomorphic map. The cotangent map $T^*f: f^*\Omega_Y^1 \to \Omega_X^1$ is then injective, and we will identify $f^*\Omega_Y^1$ with its image. We set $n = \dim X$, $m = \dim Y$ and d = n - m (we assume that X and Y are pure dimensional, otherwise one works on each connected component of X and Y).

Consider the Leray filtration Ler[•] on the complex (Ω_X^{\bullet}, d) , defined by

$$\operatorname{Ler}^{p}\Omega_{X}^{i} = \operatorname{Im}(f^{*}\Omega_{Y}^{p} \otimes_{\mathscr{O}_{X}} \Omega_{X}^{i-p} \longrightarrow \Omega_{X}^{i}).$$

[With this notation, $\operatorname{Ler}^{p}\Omega_{X}^{i}$ can be nonzero only when $i \in [0, n]$ and $p \in [0, \min(i, m)]$.]

Exercise A.72.

(1) Show that the Leray filtration is a decreasing finite filtration and that it is compatible with the differential.

(2) Show that, locally, being in Ler^p means having at least p factors dy_j in any summand.

Then, as f is smooth, we have (by computing with local coordinates adapted to f),

$$\operatorname{gr}_{\operatorname{Ler}}^p \Omega_X^i = f^* \Omega_Y^p \otimes_{\mathscr{O}_X} \Omega_{X/Y}^{i-p},$$

where $\Omega_{X/Y}^k$ is the sheaf of relative differential forms: $\Omega_{X/Y}^k = \wedge^k \Omega_{X/Y}^1$ and $\Omega_{X/Y}^1 = \Omega_X^1 / f^* \Omega_Y^1$. Notice that $\Omega_{X/Y}^k$ is \mathscr{O}_X -locally free.

Let \mathcal{M} be a left \mathscr{D}_X -module or an object of $\mathsf{D}^+(\mathscr{D}_X)^{\text{left}}$. As f is smooth, the sheaf $\mathscr{D}_{X/Y}$ of relative differential operators is well-defined and by composing the flat connection $\nabla : \mathcal{M} \to \Omega^1_X \otimes_{\mathscr{O}_X} \mathcal{M}$ with the projection $\Omega^1_X \to \Omega^1_{X/Y}$ we get a relative flat connection $\nabla_{X/Y}$ on \mathcal{M} , and thus the structure of a left $\mathscr{D}_{X/Y}$ -module on \mathcal{M} . In particular, the relative de Rham complex is defined as

$$^{p}\mathrm{DR}_{X/Y}\,\mathfrak{M} = (\Omega^{\bullet}_{X/Y} \otimes_{\mathscr{O}_{X}} \mathfrak{M}, \nabla_{X/Y}).$$

We have ${}^{p}DR \mathcal{M} = (\Omega^{\bullet}_{X} \otimes_{\mathscr{O}_{X}} \mathcal{M}, \nabla)$ (see Definition A.5.1) and the Leray filtration $Ler^{p}\Omega^{\bullet}_{X} \otimes_{\mathscr{O}_{X}} \mathcal{M}$ is preserved by the differential ∇ . We can therefore induce the filtration Ler^{\bullet} on the complex ${}^{p}DR \mathcal{M}$. We then have an equality of complexes

$$\operatorname{gr}_{\operatorname{Ler}}^{p} {}^{\operatorname{p}} \operatorname{DR} \mathfrak{M} = f^{*} \Omega_{Y}^{p} \otimes_{\mathscr{O}_{X}} {}^{\operatorname{p}} \operatorname{DR}_{X/Y} \mathfrak{M}[-p].$$

Notice that the differential of these complexes are $f^{-1}\mathcal{O}_Y$ -linear.

The complex $f_* \operatorname{God}^{\bullet p} \operatorname{DR} \mathfrak{M}$ (resp. the complex $f_! \operatorname{God}^{\bullet p} \operatorname{DR} \mathfrak{M}$) is filtered by subcomplexes $f_* \operatorname{God}^{\bullet} \operatorname{Ler}^{p} \operatorname{p} \operatorname{DR} \mathfrak{M}$ (resp. $f_! \operatorname{God}^{\bullet} \operatorname{Ler}^{p} \operatorname{p} \operatorname{DR} \mathfrak{M}$). We therefore get a spectral sequence (the Leray spectral sequence in the category of sheaves of \mathbb{C} -vector spaces, see, e.g. [**God64**]). Using the projection formula for $f_!$ and the fact that Ω_Y^p is \mathscr{O}_Y -locally free, one obtains that the E_1 term for the complex $f_! \operatorname{God}^{\bullet p} \operatorname{DR} \mathfrak{M}$ is given by

(A.11.21)
$$E_{1,!}^{p,q} = \Omega_Y^p \otimes_{\mathscr{O}_Y} R^q f_! {}^{\mathrm{p}}\mathrm{DR}_{X/Y} \mathfrak{M},$$

and the spectral sequence converges to (a suitable graded object associated with) $R^{p+q}f_{!}^{\ p}\mathrm{DR}\mathcal{M}$. If f is proper on $\mathrm{Supp}\mathcal{M}$ or if \mathcal{M} has \mathscr{D}_{X} -coherent cohomology, one can also apply the projection formula to f_{*} (see [**MN93**, §II.5.4]).

By definition of the spectral sequence, the differential $d_1 : E_1^{p,q} \to E_1^{p+1,q}$ is the connecting morphism (see Exercise A.73 below) in the long exact sequence associated to the short exact sequence of complexes

$$0 \longrightarrow \operatorname{gr}_{\operatorname{Ler}}^{p+1} {}^{\operatorname{p}}\mathrm{DR}\, \mathfrak{M} \longrightarrow \operatorname{Ler}^{p} {}^{\operatorname{p}}\mathrm{DR}\, \mathfrak{M} / \operatorname{Ler}^{p+2} {}^{\operatorname{p}}\mathrm{DR}\, \mathfrak{M} \longrightarrow \operatorname{gr}_{\operatorname{Ler}}^{p} {}^{\operatorname{p}}\mathrm{DR}\, \mathfrak{M} \longrightarrow 0$$

after applying $f_! \operatorname{God}^{\bullet}$ (or $f_* \operatorname{God}^{\bullet}$ if one of the previous properties is satisfied).

Exercise A.73 (The connecting morphism). Let $0 \to C_1^{\bullet} \to C_2^{\bullet} \to C_3^{\bullet} \to 0$ be an exact sequence of complexes. Let $[\mu] \in H^k C_3^{\bullet}$ and choose a representative in C_3^k with $d\mu = 0$. Lift μ as $\tilde{\mu} \in C_2^k$.

(1) Show that $d\tilde{\mu} \in C_1^{k+1}$ and that its differential is zero, so that the class $[d\tilde{\mu}] \in H^{k+1}C_1^{\bullet}$ is well-defined.

(2) Show that $\delta : [\mu] \mapsto [d\widetilde{\mu}]$ is a well-defined morphism $H^k C_3^{\bullet} \to H^{k+1} C_1^{\bullet}$.

(3) Deduce the existence of the cohomology long exact sequence, having δ as its connecting morphism.

Lemma A.11.22 (The Gauss-Manin connection). The morphism

$$\nabla^{\mathrm{GM}} := d_1 : R^q f_! \,^{\mathrm{p}} \mathrm{DR}_{X/Y} \, \mathfrak{M} = E_1^{0,q} \longrightarrow E_1^{1,q} = \Omega^1_Y \otimes_{\mathscr{O}_Y} R^q f_! \,^{\mathrm{p}} \mathrm{DR}_{X/Y} \, \mathfrak{M}$$

is a flat connection on $R^q f_! {}^{\mathrm{p}}\mathrm{DR}_{X/Y} \mathfrak{M}$, called the Gauss-Manin connection and the complex $(E_1^{\bullet,q}, d_1)$ is equal to the de Rham complex diff $\mathrm{DR}_Y(R^q f_! {}^{\mathrm{p}}\mathrm{DR}_{X/Y} \mathfrak{M}, \nabla^{\mathrm{GM}})$.

Sketch of proof of Lemma A.11.22. Instead of using the Godement resolution, one can use the C^{∞} de Rham complex $\mathscr{E}_X^{\bullet} \otimes_{\mathscr{O}_X} \mathcal{M}$, with the differential D defined by

$$D(\eta \otimes m) = \mathrm{d}\eta \otimes m + (-1)^k \eta \wedge \nabla m,$$

if η is C^{∞} differential k-form, that is, a local section of \mathscr{E}_X^k ($k \leq 0$). By a standard argument (Dolbeault resolution) analogous to that of Exercise A.46(??), this C^{∞} de Rham complex is quasi-isomorphic to the holomorphic one, and is equipped with the Leray filtration. The quasi-isomorphism is strict with respect to Ler[•]. One can therefore compute with the C^{∞} de Rham complex. Moreover, the assertion is local with respect to Y.

Assume first that, in the neighbourhood of $f^{-1}(y)$, X is diffeomorphic to a product $X \simeq Z \times Y$. This occurs for example if f is proper (Ehresmann's theorem). Then we identify \mathscr{E}_X^{p+q} with $\mathscr{E}_Y^p \otimes \mathscr{E}_{X/Y}^q$ and the differential D decomposes accordingly as $D_Y + D_{X/Y}$. The flatness of D implies the flatness of $D_{X/Y}$ and D_Y . Given a section μ of $f_!(\mathscr{E}_Y^p \otimes (\mathscr{E}_{X/Y}^q \otimes \mathcal{M}))$ which is closed with respect to $D_{X/Y}$, we can identify it with its lift $\tilde{\mu}$ (see Exercise A.73), and $d_1\mu$ is thus the class of $D_Y\mu$, so the C^{∞} Gauss-Manin connection D^{GM} in degree zero induces d_1 in any degree.

In general, choose a partition of unity (χ_{α}) such that for every α , when restricted to some open neighbourhood of $\operatorname{Supp} \chi_{\alpha}$, f is locally the projection from a product to one of its factors. We set $D = \sum_{\alpha} \chi_{\alpha} D = \sum_{\alpha} D^{(\alpha)}$ and we apply the previous argument to each $D^{(\alpha)}$.

Theorem A.11.23. Let $f : X \to Y$ be a smooth holomorphic map and let \mathfrak{M} be left \mathscr{D}_X -module—or more generally an object of $\mathsf{D}^+(\mathscr{D}_X)^{\text{left}}$. Then there is a functorial isomorphism of left \mathscr{D}_Y -modules

$$R^k f_! \, {}^{\mathrm{p}}\mathrm{DR}_{X/Y} \, \mathfrak{M} \longrightarrow \mathscr{H}^k_{\,\,\mathrm{D}} f_! \mathfrak{M}$$

when one endows the left-hand term with the Gauss-Manin connection ∇^{GM} . The same result holds for ${}_{\mathrm{D}}f_*$ instead of ${}_{\mathrm{D}}f_!$ if f is proper on Supp \mathcal{M} or \mathcal{M} is \mathscr{D}_X -coherent (or has coherent cohomology).

Proof. Recall that, for a left \mathscr{D}_X -module \mathcal{M} , we have

$$\mathcal{A}^{\mathrm{right}} \otimes_{\mathscr{D}_X} \mathrm{Sp}^{\bullet}_{X \to Y}(\mathscr{D}_X) \simeq \Omega^{\bullet}_X(\mathfrak{M} \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{D}_Y)[n],$$

so that the direct image of \mathcal{M} , regarded as a right \mathcal{D}_Y -module, is

(A.11.24)
$$({}_{\mathrm{D}}f_!\mathfrak{M})^{\mathrm{right}} = \boldsymbol{R}f_! {}^{\mathrm{p}}\mathrm{DR}_X(\mathfrak{M} \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{D}_Y)[m],$$

by Exercise A.46(1). We conclude that

٦

$${}^{\operatorname{diff}}\mathrm{DR}_{Y \operatorname{D}} f_! \mathfrak{M} \simeq {}^{\operatorname{diff}}\mathrm{DR}_Y \big(\boldsymbol{R} f_! {}^{\operatorname{p}}\mathrm{DR}_X (\mathfrak{M} \otimes_{f^{-1} \mathscr{O}_Y} f^{-1} \mathscr{D}_Y) \big).$$

There is a Leray filtration Ler[•] ${}^{p}DR_{X}(\mathcal{M} \otimes_{f^{-1}\mathscr{O}_{Y}} f^{-1}\mathscr{D}_{Y})$. Notice that the graded complex $\operatorname{gr}_{\operatorname{Ler}}^{p} {}^{p}DR_{X}(\mathcal{M} \otimes_{f^{-1}\mathscr{O}_{Y}} f^{-1}\mathscr{D}_{Y})$ is equal to the complex

$$f^{-1}\Omega^p_Y \otimes_{f^{-1}\mathscr{O}_Y} {}^{\mathrm{p}}\mathrm{DR}_{X/Y} \mathfrak{M} \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{D}_Y[-p]$$

with differential induced by $\nabla_{X/Y}$ on \mathcal{M} (remark that the part of the differential involving T^*f is killed by taking $\operatorname{gr}_{\operatorname{Ler}}^p$). The differential is now $f^{-1}\mathcal{O}_Y$ -linear.

The filtered complex $\mathbf{R}f_! \operatorname{Ler}^{\bullet p} \operatorname{DR}_X(\mathcal{M} \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{D}_Y)$ gives rise to a spectral sequence in the category of right \mathscr{D}_Y -modules. By the previous computation, the $E_1^{p,q}$ term of this spectral sequence is the right \mathscr{D}_Y -module

$$R^{p+q}f_!(f^{-1}\Omega^p_Y \otimes_{f^{-1}\mathscr{O}_Y} {}^{\mathrm{p}}\mathrm{DR}_{X/Y} \mathfrak{M} \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{D}_Y[-p]) \simeq \Omega^p_Y \otimes_{\mathscr{O}_Y} R^q f_! {}^{\mathrm{p}}\mathrm{DR}_{X/Y} \mathfrak{M} \otimes_{\mathscr{O}_Y} \mathscr{D}_Y,$$

which is an induced \mathscr{D}_Y -module, whose ${}^{\text{diff}}\mathrm{DR}_Y$ is equal to the corresponding Gauss-Manin term (A.11.21). We claim, as will show below, that the differential d_1 becomes

the Gauss-Manin d_1 after applying ^{diff}DR_Y. This will prove that the Gauss-Manin E_1 complex is equal to ^{diff}DR_Y of the E_1 complex of right \mathscr{D}_Y -modules.

Notice now that Lemma A.11.22 shows in particular that the E_1 complex considered there is a complex in $C^+(\mathscr{O}_Y, \operatorname{Diff}_Y)$, and

$${}^{\operatorname{diff}}\mathrm{DR}_Y^{-1}(E_1^{\bullet,q},d_1) \simeq (R^q f_! {}^{\mathrm{p}}\mathrm{DR}_{X/Y}\,\mathfrak{M},\nabla^{\mathrm{GM}})^{\operatorname{right}}[-m],$$

since, for a left \mathscr{D}_Y -module \mathcal{N} , we have, according to Theorem A.11.16,

$${}^{\mathrm{diff}}\mathrm{DR}_{Y}^{-1}{}^{\mathrm{diff}}\mathrm{DR}_{Y}(\mathcal{N}) = {}^{\mathrm{diff}}\mathrm{DR}_{Y}^{-1}{}^{\mathrm{diff}}\mathrm{DR}_{Y}(\mathcal{N}^{\mathrm{right}})[-m] \simeq \mathcal{N}^{\mathrm{right}}[-m]$$

The claim above, together with Lemma A.11.18, implies that the E_1 complex of the \mathscr{D}_Y -Leray spectral sequence has cohomology in degree m only, hence this spectral sequence degenerates at E_2 , this cohomology being isomorphic to $(R^q f_! {}^p \mathrm{DR}_{X/Y} \mathfrak{M}, \nabla^{\mathrm{GM}})^{\mathrm{right}}[-m]$. But the spectral sequences converges (the Leray filtration is finite) and its limit is $\bigoplus_p \mathrm{gr}^p \mathscr{H}^{q-m}({}_{\mathrm{D}}f_!\mathfrak{M})^{\mathrm{right}}$ for the induced filtration on $\mathscr{H}^{q-m}({}_{\mathrm{D}}f_!\mathfrak{M})^{\mathrm{right}}$, according to (A.11.24). We conclude that this implicit filtration is trivial and that $\mathscr{H}^q({}_{\mathrm{D}}f_!\mathfrak{M})^{\mathrm{right}} = (R^q f_! {}^p \mathrm{DR}_{X/Y} \mathfrak{M}, \nabla^{\mathrm{GM}})^{\mathrm{right}}$, as wanted, after side changing.

Let us now compare the d_1 of both spectral sequences. As the construction is clearly functorial with respect to \mathcal{M} , we can replace \mathcal{M} by the flabby sheaf $\operatorname{God}^{\ell} \mathcal{M}$ for every ℓ . We then have

$$\begin{split} \boldsymbol{R} f_! \big(\Omega_X^{\bullet} \otimes_{\mathscr{O}_X} \operatorname{God}^{\ell} \mathfrak{M} \otimes_{f^{-1}\mathscr{O}_Y} f^{-1} \mathscr{D}_Y \big) \\ &= \boldsymbol{R} f_! \big(\operatorname{God}^{\ell} (\Omega_X^{\bullet} \otimes_{\mathscr{O}_X} \mathfrak{M}) \otimes_{f^{-1}\mathscr{O}_Y} f^{-1} \mathscr{D}_Y \big) \quad (\text{Exercise A.40}) \\ &= \boldsymbol{R} f_! \big(\operatorname{God}^{\ell} (\Omega_X^{\bullet} \otimes_{\mathscr{O}_X} \mathfrak{M}) \big) \otimes_{\mathscr{O}_Y} \mathscr{D}_Y \quad (\text{projection formula}) \\ &= f_! \big(\operatorname{God}^{\ell} (\Omega_X^{\bullet} \otimes_{\mathscr{O}_X} \mathfrak{M}) \big) \otimes_{\mathscr{O}_Y} \mathscr{D}_Y \quad (\text{flabbiness of God}^{\ell}) \\ &= f_! \big(\Omega_X^{\bullet} \otimes_{\mathscr{O}_X} \operatorname{God}^{\ell} \mathfrak{M} \big) \otimes_{\mathscr{O}_Y} \mathscr{D}_Y \quad (\text{Exercise A.40}) \\ &= f_! \big(\Omega_X^{\bullet} \otimes_{\mathscr{O}_X} \operatorname{God}^{\ell} \mathfrak{M} \otimes_{f^{-1}\mathscr{O}_Y} f^{-1} \mathscr{D}_Y \big) \quad (\text{projection formula}). \end{split}$$

It is also enough to make the computation locally on Y, so that we can write $f = (f_1, \ldots, f_m)$, using local coordinates (y_1, \ldots, y_m) . If μ is a section of $\Omega_X^k \otimes \mathcal{M}$ and $\mathbf{1}_Y$ is the unit of \mathscr{D}_Y , then (A.7.1) can be written as

$$\nabla^X(\mu\otimes\mathbf{1}_Y)=(\nabla\mu)\otimes\mathbf{1}_Y+\sum_{j=1}^m\mu\wedge df_j\otimes\partial_{y_j}.$$

Using the definition of d_1 given by Exercise A.73 and an argument similar to that of Exercise A.71, one gets the desired assertion.

A.12. Comments

Here come the references to the existing work which has been the source of inspiration for this chapter.