

CONTENTS

Preface	1
Part 0. Introductory part	3
Conventions and notation	5
1. Overview	11
2. Hodge theory: review of classical results	13
2.1. Introduction.....	13
2.2. Hodge-Tate structure and highest dimensional cohomology.....	14
2.3. Complex Hodge theory on compact Riemann surfaces.....	15
2.4. Complex Hodge theory of smooth projective varieties.....	16
2.5. Polarizable Hodge structures.....	20
2.6. Mixed Hodge structures.....	27
2.7. Exercises.....	30
2.8. Comments.....	33
3. Hodge-Lefschetz structures	35
3.1. \mathfrak{sl}_2 -representations and quivers.....	35
3.2. Polarized \mathfrak{sl}_2 -Hodge structures.....	40
3.3. A-Lefschetz structures.....	51
3.4. Polarizable Hodge-Lefschetz structures.....	54
3.5. Exercises.....	61
3.6. Comments.....	65
4. Variations of Hodge structure on a complex manifold	67
4.1. Variations of Hodge structure.....	67
4.2. The Hodge theorem.....	72
4.3. Semi-simplicity.....	83
4.4. Exercises.....	90
4.5. Comments.....	93
5. The Rees construction for Hodge structures	95
5.1. Filtered objects and the Rees construction.....	95
5.2. The category of $\tilde{\mathbb{C}}$ -triples.....	100
5.3. Hodge-Lefschetz triples.....	105
5.4. Variations of Hodge triple.....	110
5.5. Exercises.....	112
5.6. Comments.....	114

Part I. Polarizable Hodge modules on curves	115
6. Variations of Hodge structure on curves	117
Part 1: Metric properties near punctures	
6.1. Introduction.....	117
6.2. Variations of Hodge structure on a punctured disc.....	118
6.3. Metric properties near a puncture.....	122
6.4. Semi-simplicity.....	132
6.5. Exercises.....	136
6.6. Comments.....	140
Part 2: Limiting Hodge properties	
6.7. The holomorphic Hodge filtration.....	141
6.8. The limiting Hodge-Lefschetz structure.....	142
6.9. Exercises.....	147
6.10. Comments.....	148
Part 3: The Hodge-Zucker theorem	
6.11. Introduction.....	149
6.12. The holomorphic de Rham complexes.....	150
6.13. The L^2 de Rham complex and the L^2 Poincaré lemma.....	154
6.14. The Hodge filtration.....	163
6.15. Exercises.....	172
6.16. Comments.....	173
7. Polarizable Hodge modules on curves	175
7.1. Introduction.....	175
7.2. Basics on holonomic \mathcal{D} -modules in one variable.....	177
7.3. Sesquilinear pairings between \mathcal{D} -modules on a Riemann surface.....	189
7.4. Hodge \mathcal{D} -modules on a Riemann surface and the Hodge-Saito theorem...	198
7.5. Semi-simplicity.....	206
7.6. Numerical invariants of variations of \mathbb{C} -Hodge structure.....	207
7.7. Exercises.....	212
7.8. Comments.....	220
Part II. Tools from the theory of \mathcal{D}-modules	221
8. Training on \mathcal{D}-modules	223
8.1. The sheaf of holomorphic differential operators.....	223
8.2. Left and right.....	226
8.3. Examples of $\tilde{\mathcal{D}}$ -modules.....	228
8.4. The de Rham functor.....	230
8.5. Induced $\tilde{\mathcal{D}}$ -modules.....	236
8.6. Pullback and external product of $\tilde{\mathcal{D}}$ -modules.....	236
8.7. Pushforward of $\tilde{\mathcal{D}}$ -modules.....	242
8.8. Coherent $\tilde{\mathcal{D}}_X$ -modules and coherent filtrations.....	255
8.9. Appendix: Differential complexes and the Gauss-Manin connection.....	266
8.10. Exercises.....	275
8.11. Comments.....	303

9. Nearby and vanishing cycles of $\tilde{\mathcal{D}}$-modules	305
9.1. Introduction.....	306
9.2. The filtration $V_{\bullet}\tilde{\mathcal{D}}_X$ relative to a smooth hypersurface.....	307
9.3. Specialization of coherent $\tilde{\mathcal{D}}_X$ -modules.....	313
9.4. Nearby and vanishing cycle functors.....	324
9.5. Strictly non-characteristic restrictions in codimension one.....	327
9.6. Strict Kashiwara's equivalence.....	328
9.7. Support-decomposable $\tilde{\mathcal{D}}$ -modules.....	332
9.8. Direct image of strictly \mathbb{R} -specializable coherent $\tilde{\mathcal{D}}_X$ -modules.....	340
9.9. Examples of computations of nearby and vanishing cycles.....	348
9.10. Exercises.....	359
9.11. Comments.....	370
10. Specialization of filtered \mathcal{D}-modules	373
10.1. Introduction.....	373
10.2. Strict and bi-strict complexes.....	374
10.3. Bi-filtered \mathcal{D}_X -modules.....	376
10.4. The direct image of bi-filtered \mathcal{D}_X -modules.....	382
10.5. Specializability of filtered \mathcal{D}_X -modules.....	385
10.6. A strictness criterion for complexes of filtered \mathcal{D} -modules.....	387
10.7. Comparison with the results of Chapter 9.....	392
10.8. Exercises.....	401
10.9. Comments.....	402
11. Localization, dual localization and maximal extension	403
11.1. Introduction.....	403
11.2. Localization and dual localization in the strictly non-characteristic case.....	405
11.3. Localization of $\tilde{\mathcal{D}}_X$ -modules.....	417
11.4. Dual localization.....	423
11.5. D -localizable $\tilde{\mathcal{D}}_X$ -modules and middle extension.....	428
11.6. Beilinson's maximal extension and applications.....	429
11.7. Localizability, maximalizability and pushforward.....	436
11.8. The Thom-Sebastiani formula for the vanishing cycles.....	437
11.9. The Kodaira-Saito vanishing property.....	442
11.10. Exercises.....	451
11.11. Comments.....	454
12. The category of triples of $\tilde{\mathcal{D}}_X$-modules	455
12.1. Introduction.....	455
12.2. Distributions and currents on a complex manifold.....	456
12.3. Sesquilinear pairings between \mathcal{D}_X -modules.....	460
12.4. Pushforward of sesquilinear pairings.....	464
12.5. Pullback, specialization and localization of sesquilinear pairings.....	473
12.6. Compatibility between functors on sesquilinear pairings.....	494
12.7. The category $\tilde{\mathcal{D}}$ -Triples and its functors.....	495
12.8. Exercises.....	506
12.9. Comments.....	510

13. Duality, real structures and perverse sheaves: an overview	511
13.1. Introduction.....	511
13.2. Perverse sheaves and the de Rham functor.....	512
13.3. The Hermitian duality functor for holonomic \mathcal{D}_X -modules.....	515
13.4. Duality for nondegenerate regular holonomic triples.....	519
13.5. Comments.....	523
 Part III. Polarizable Hodge modules	 525
14. Polarizable Hodge modules and their direct images	527
14.1. Introduction.....	527
14.2. Definition and first properties of polarizable Hodge modules.....	528
14.3. Introduction to the direct image theorem.....	540
14.4. Behaviour of the Hodge module properties by projective pushforward...	542
14.5. End of the proof of the Hodge-Saito theorem.....	546
14.6. Variations of Hodge structure are Hodge modules.....	554
14.7. Some properties of the category of W -filtered Hodge modules.....	556
14.8. Exercises.....	559
14.9. Comments.....	559
15. $\tilde{\mathcal{D}}$-modules of normal crossing type	561
Part 1: Distributive filtrations and strictness	
15.1. Distributive filtrations.....	561
15.2. Reformulation in terms of flatness.....	564
15.3. Strictness of morphisms.....	570
15.4. Appendix. Compatible filtrations.....	572
15.5. Exercises.....	575
Part 2: Fundamental properties	
15.6. Introduction.....	579
15.7. Normal crossing type.....	581
15.8. Filtered normal crossing type.....	587
15.9. Sesquilinear pairings of normal crossing type.....	596
15.10. Exercises.....	599
Part 3: Nearby cycles along a monomial	
15.11. Introduction.....	601
15.12. Nearby cycles for coherently F -filtered \mathcal{D}_X -modules of normal crossing type.....	615
15.13. Comments.....	624
 16. The structure theorem for polarizable Hodge modules	 625
16.1. Introduction.....	625
16.2. The structure theorem.....	625
16.3. Applications of the structure theorem.....	627
16.4. Polarized Hodge modules in the normal crossing case.....	629
16.5. Comments.....	630
 Appendix. Sign conventions for Hodge modules	 631
A.1. General principles.....	631

A.2. Hodge structures and polarizations.....	632
A.3. Cohomology of compact Kähler manifolds.....	634
A.4. \mathfrak{sl}_2 -Hodge structures and polarizations.....	638
A.5. Pairings on \mathcal{D} -modules.....	641
A.6. Direct images.....	647
A.7. Variations of Hodge structure and polarizations.....	649
A.8. Degenerating variations of Hodge structure.....	651
A.9. Hodge modules on the unit disk.....	657
A.10. Nearby and vanishing cycles.....	659
Bibliography.....	665

