CONVENTIONS AND NOTATION

0.1. Numbers.

- $\mathbb{N}$: non-negative integers.
- $\mathbb{Z}$: integers.
- $\mathbb{Q}$: rational numbers.
- $\mathbb{R}$: real numbers.
- $\mathbb{C} = \mathbb{R} + \mathsf{i}\,\mathbb{R}$: complex numbers.
- $\mathbb{S}^1 \subset \mathbb{C}$: complex numbers with absolute value equal to 1.

0.2. Signs. For every $k \in \mathbb{Z}$, we define the orientation number by the equality $\varepsilon(k) = (-1)^{k(k-1)/2}$. We have

$$\begin{aligned}\varepsilon(k+1) = -\,\varepsilon(-k) = (-1)^k\,\varepsilon(k), \qquad \varepsilon(k+2) = -\,\varepsilon(k),\\ \varepsilon(k+\ell) = (-1)^{k\ell}\,\varepsilon(k)\,\varepsilon(\ell), \qquad \varepsilon(n-k) = (-1)^k\,\varepsilon(n+k).\end{aligned}$$

If we take complex coordinates $z_1, \dots, z_n$ on $\mathbb{C}^n$ and set $z_j = x_j + \mathsf{i}\,y_j$, then

$$(-1)^n \frac{\varepsilon(n)}{(2\pi\mathsf{i})^n}\,(\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n) \wedge (\mathrm{d}\overline{z}_1 \wedge \cdots \wedge \mathrm{d}\overline{z}_n) = (1/\pi)^n \mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}x_n \wedge \mathrm{d}y_n.$$

The following coefficients will often occur later and deserve a specific notation:

$$\begin{aligned}\operatorname{Sgn}(n,k) &= (-1)^n \frac{\varepsilon(n+k)}{(2\pi\mathsf{i})^n} \quad (n,k \in \mathbb{Z}),\\ \operatorname{Sgn}(n) &= \operatorname{Sgn}(n,0) = (-1)^n \frac{\varepsilon(n)}{(2\pi\mathsf{i})^n} = \frac{\varepsilon(n+1)}{(2\pi\mathsf{i})^n}.\end{aligned} \tag{0.2 $*$}$$

Let us notice the following relations:

$$\begin{aligned}\operatorname{Sgn}(n) &= \operatorname{Sgn}(n-1) \cdot \frac{(-1)^{n-1}}{2\pi\mathsf{i}},\\ \operatorname{Sgn}(n,k) &= (-1)^{m(p+k)} \operatorname{Sgn}(m) \operatorname{Sgn}(p,k), \quad n = m+p,\ m,p \geqslant 0,\\ \operatorname{Sgn}(n,-k) &= (-1)^k \operatorname{Sgn}(n,k).\end{aligned} \tag{0.2 $**$}$$

0.3. Categories. Given a category A, we say that a subcategory A' is *full* if, for every pair of objects A, B of A', we have $\mathrm{Hom}_{\mathsf{A}'}(A,B) = \mathrm{Hom}_{\mathsf{A}}(A,B)$. A stronger notion is that of a *strictly full* subcategory, which has the supplementary condition that every object in A which is isomorphic (in A) to an object in A' is already in A'. All along this text, we will use the latter notion, but simply call it a full subcategory, in order to avoid too many different meanings for the word "strict". This should not cause any trouble, since all full subcategories we define consist of all objects satisfying some properties which are obviously stable by isomorphism.

For a sheaf of rings $\mathcal{A}_X$ on a topological space X, we denote by $\mathsf{Mod}(\mathcal{A}_X)$, or simply $\mathsf{Mod}(\mathcal{A}_X)$ the category of $\mathcal{A}_X$-modules. It is an abelian category. The category of complexes on $\mathsf{Mod}(\mathcal{A}_X)$ is denoted by $\mathsf{C}^\star(\mathcal{A}_X)$, with

- $\star$ empty: no condition,
- $\star = +$: complexes bounded from below,
- $\star = -$: complexes bounded from above,
- $\star = \mathrm{b}$: bounded complexes.

By considering the morphisms up to homotopy we obtain the category $\mathsf{K}^\star(\mathcal{A}_X)$ of complexes up to homotopy. Finally, $\mathsf{D}^\star(\mathcal{A}_X)$ denotes the corresponding derived category. We refer for example to [**KS90**, Chap. 1] for the fundamental properties of the derived category of a triangulated category.

0.4. Filtrations.

- Filtrations denoted by F are indexed by $\mathbb{Z}$.
- Increasing filtrations are indicated by a lower index, and decreasing filtrations by an upper index. The usual rule for passing from one kind to the other one is to set $F^\bullet = F_{-\bullet}$.
- However, this is not the rule used for V-filtrations, which are indexed by $\mathbb{R}$, and where we set $V^\bullet = V_{-\bullet-1}$.
- The shift $[k]$ of a filtration by an integer k is defined by

$$F[k]_\bullet := F_{\bullet-k}, \quad \text{equivalently,} \quad F[k]^\bullet := F^{\bullet+k}.$$

- Given a filtered sheaf of rings $(\mathcal{A}_X, F_\bullet\mathcal{A}_X)$ on a topological space, the Rees sheaf of rings $R_F\mathcal{A}_X := \bigoplus_p F_p\mathcal{A}_X \cdot z^p$ is denoted by a calligraphic letter $\mathcal{A}_X$.
- The Rees ring attached to the field $\mathbb{C}$ of complex numbers equipped with the filtration $F_0\mathbb{C} = \mathbb{C}$ and $F_{-1}\mathbb{C} = 0$ is $\widetilde{\mathbb{C}} = \mathbb{C}[z]$.
- In general, for a filtered object $(\mathcal{M}, F_\bullet\mathcal{M})$, the associated Rees object is denoted by the calligraphic letter $\mathcal{M}$. The Rees construction for a filtered morphism φ is indicated by the decoration $\widetilde{\varphi}$.

0.5. Vector spaces and sesquilinear pairings

- The *conjugate* $\overline{\mathcal{H}}$ of a $\mathbb{C}$-vector space $\mathcal{H}$ is $\mathcal{H}_\mathbb{R}$, i.e., $\mathcal{H}$ considered as an $\mathbb{R}$-vector space, together with the $\mathbb{C}$ action defined by $\lambda \cdot x = \overline{\lambda} x$ (for all $\lambda \in \mathbb{C}$ and $x \in \mathcal{H}_\mathbb{R}$) (see Exercise 2.1). In order to make clear the structure, we denote by $\overline{x}$ the element x of $\mathcal{H}_\mathbb{R}$ when considered as being in $\overline{\mathcal{H}}$. Conjugation is a covariant functor on the category of $\mathbb{C}$-vector spaces. For a morphism $\varphi : \mathcal{H}_1 \to \mathcal{H}_2$, we denote by $\overline{\varphi} : \overline{\mathcal{H}}_1 \to \overline{\mathcal{H}}_2$ the corresponding morphism.

• We denote by $\langle \bullet, \bullet \rangle : \mathcal{H} \otimes_{\mathbb{C}} \mathcal{H}^\vee \to \mathbb{C}$ the tautological duality pairing. We set $\mathcal{H}^* = \overline{\mathcal{H}}^\vee = (\overline{\mathcal{H}})^\vee$, that we call the *Hermitian dual vector space*. Duality and Hermitian duality are contravariant functors on the category of $\mathbb{C}$-vector spaces. For a morphism $\varphi : \mathcal{H}_1 \to \mathcal{H}_2$, we denote by $\varphi^* : \mathcal{H}_2^* \to \mathcal{H}_1^*$ the corresponding morphism.

• A *sesquilinear pairing* between $\mathbb{C}$-vector spaces $\mathcal{H}', \mathcal{H}''$ is a $\mathbb{C}$-linear morphism

$$\mathfrak{s} : \mathcal{H}' \otimes_{\mathbb{C}} \overline{\mathcal{H}''} \longrightarrow \mathbb{C}.$$

We identify such a sesquilinear pairing $\mathfrak{s}$ with a linear morphism

$$\mathfrak{s} : \mathcal{H}'' \longrightarrow \mathcal{H}'^*$$

by setting, for $x \in \mathcal{H}'$, $y \in \mathcal{H}''$,

$$\mathfrak{s}(x, \overline{y}) := \langle x, \overline{\mathfrak{s}(y)} \rangle_{\mathcal{H}'} \quad (\overline{\mathfrak{s}(y)} \in \mathcal{H}'^\vee).$$

If $\varphi'' : \mathcal{H}_1'' \to \mathcal{H}''$ is a linear morphism, then $\mathfrak{s} \circ \varphi'' : \mathcal{H}_1'' \to \mathcal{H}'^*$ corresponds to the pairing $\mathcal{H}' \otimes \overline{\mathcal{H}_1''} \to \mathbb{C}$ given by $\mathfrak{s}(x, \overline{\varphi''(y_1)})$, and if $\varphi' : \mathcal{H}_1' \to \mathcal{H}'$ is a linear morphism, then $\varphi'^* \circ \mathfrak{s} : \mathcal{H}'' \to \mathcal{H}_1'^*$ corresponds to the pairing $\mathfrak{s}(\varphi'(x_1), \overline{y})$.

The Hermitian adjoint $\mathfrak{s}^* : \mathcal{H}' \to \mathcal{H}''^*$ of a linear morphism $\mathfrak{s} : \mathcal{H}'' \to \mathcal{H}'^*$ corresponds to the sesquilinear pairing

$$\mathfrak{s}^*(y, \overline{x}) = \overline{\mathfrak{s}(x, \overline{y})}.$$

Assume now that $\mathcal{H}' = \mathcal{H}'' = \mathcal{H}$. We say that $\mathfrak{s}$ is $\pm$-Hermitian if $\mathfrak{s}^* = \pm\mathfrak{s}$. For a linear morphism $\varphi : \mathcal{H} \to \mathcal{H}$, we say that φ is self- resp. skew-adjoint (with respect to $\mathfrak{s}$) if

$$\mathfrak{s}(\varphi(x), \overline{y}) = \mathfrak{s}(x, \overline{\varphi(y)}) \quad \text{resp.} \ -\mathfrak{s}(x, \overline{\varphi(y)}).$$

Considering $\mathfrak{s}$ as a linear morphism as above and denoting by φ^* the Hermitian adjoint of φ, this is translated as

$$\varphi^* \circ \mathfrak{s} = \pm\mathfrak{s} \circ \varphi.$$

Then φ is self- resp. skew-adjoint (with respect to $\mathfrak{s}$) if and only if it is so with respect to $\mathfrak{s}^*$.

If $\mathfrak{s}$ is *non-degenerate*, that is, if $\mathfrak{s} : \mathcal{H} \to \mathcal{H}^*$ is an isomorphism, then one can define the $\mathfrak{s}$-adjoint $\varphi^\star$ (not to be confused with φ^*) of any linear morphism $\varphi : \mathcal{H} \to \mathcal{H}$ by the formula

$$\varphi^\star = \mathfrak{s}^{(-1)} \circ \varphi^* \circ \mathfrak{s} : \mathcal{H} \longrightarrow \mathcal{H}.$$

In such a case, φ is self- (resp. skew-) adjoint with respect to $\mathfrak{s}$ if and only if $\varphi^\star = \pm\varphi$.

0.6. Complex manifolds and their basic sheaves of rings

• We consider complex manifolds, usually denoted by X, Y of complex dimension d_X, d_Y, and holomorphic maps $f : X \to Y$ between them, of relative dimension $d_{X/Y} = d_X - d_Y$. We will often use the following shortcuts:

$$\boxed{n = d_X = \dim X, \quad m = d_Y = \dim Y, \quad n - m = d_X - d_Y = d_{X/Y}.}$$

• A smooth hypersurface of X (i.e., a closed complex submanifold everywhere of codimension 1 in X) will usually be denoted by H.

• A divisor D is a reduced complex analytic subspace of X everywhere of codimension 1. A local reduced defining equation for D or H is usually denoted by g.

• At many places, the divisor D is assume to have only normal crossings as singularities. It is however in general not necessary to assume that its irreducible components are smooth.

• The structure sheaf of holomorphic functions on X is denoted by $\mathcal{O}_X$. The sheaf of holomorphic differential forms is Ω^1_X and the sheaf of holomorphic vector fields is Θ_X. Their k-th wedge product is Ω^k_X resp. $\Theta_{X,k} = \Theta_{X,k}$. The dualizing sheaf $\wedge^n\Omega^1_X$ is denoted by ω_X. The sheaf of holomorphic differential operators is $\mathcal{D}_X$.

• The sheaf of C^∞ function on the underlying C^∞ manifold is denoted by $\mathcal{C}^\infty_X$. Correspondingly, the sheaf of C^∞ differential forms of degree k is $\mathcal{E}^k_X = \mathcal{C}^\infty_X \otimes_{\mathcal{O}_X} \Omega^k_X$ and is decomposed with respect to the holomorphic and anti-holomorphic degrees with components $\mathcal{E}^{(p,q)}_X$. The sheaf of C^∞ poly-vector fields of degree k is denoted by $\mathcal{T}_{X,k} = \mathcal{C}^\infty_X \otimes_{\mathcal{O}_X} \Theta_{X,k}$ and is similarly decomposed with components $\mathcal{T}_{X,(p,q)}$.

• The Rees construction for $\mathcal{O}_X$, equipped with the filtration $F_0\mathcal{O}_X = \mathcal{O}_X$ and $F_{-1}\mathcal{O}_X = 0$ leads to $\widetilde{\mathcal{O}}_X = \mathcal{O}_X[z]$.

• Correspondingly, we set $\widetilde{\Omega}^1_X = z^{-1}\Omega^1_X[z]$ and $\widetilde{\Theta}_X = z\Theta_X[z]$. We have $\widetilde{\Omega}^k_X = \wedge^k\widetilde{\Omega}^1_X$ and $\widetilde{\Theta}_{X,k} = \wedge^k\widetilde{\Theta}_X$.

• The Rees construction for $\omega_X = \Omega^n_X$ equipped with the filtration $F_{-n}\omega_X = \omega_X$ and $F_{-n-1}\omega_X = 0$ gives rise to $\widetilde{\omega}_X = \widetilde{\Omega}^n_X$.

• The Rees construction for $\mathcal{D}_X$ with its filtration by the order of differential operators $F_\bullet\mathcal{D}_X$ gives rise to $\widetilde{\mathcal{D}}_X$.

• The C^∞ analogues are $\widetilde{\mathcal{C}}^\infty_X = \mathcal{C}^\infty_X[z]$, $\widetilde{\mathcal{E}}^k_X = \widetilde{\mathcal{C}}^\infty_X \otimes_{\mathcal{O}_X} \widetilde{\Omega}^k_X$, and $\mathcal{T}^k_X = \widetilde{\mathcal{C}}^\infty_X \otimes_{\mathcal{O}_X} \widetilde{\Theta}_X$.

0.7. Sheaves of rings and modules

• $\mathbb{C}$-vector spaces are denoted by $\mathcal{H}, \mathcal{H}', \mathcal{H}''$.

• C^∞ vector bundles on X are denoted by $\mathcal{H}$, and a C^∞ connection is denoted by $D : \mathcal{H} \to \mathcal{E}^1_X \otimes \mathcal{H}$. It is decomposed into its $(1,0)$ and $(0,1)$ components $D' : \mathcal{H} \to \mathcal{E}^{1,0}_X \otimes \mathcal{H}$ and $D'' : \mathcal{H} \to \mathcal{E}^{0,1}_X \otimes \mathcal{H}$.

• A holomorphic vector bundle is denoted by $\mathcal{H}', \mathcal{H}''$ or $\mathcal{V}$. For a C^∞ vector bundle $\mathcal{H}$ with integrable connection D, we regard $\mathcal{H}' = \operatorname{Ker} D''$ as a holomorphic vector bundle with a integrable holomorphic connection $\nabla : \mathcal{H}' \to \Omega^1_X \otimes \mathcal{H}'$.

• Modules over a sheaf of rings $\mathcal{A}_X = \mathcal{O}_X, \mathcal{D}_X$ are denoted by $\mathcal{M}, \mathcal{N}$.

• The Rees objects associated to filtered holomorphic vector bundles, $\mathcal{O}_X$-modules or $\mathcal{D}_X$-modules are denoted by the corresponding calligraphic letter $\mathcal{H}', \mathcal{H}'', \mathcal{M}, \mathcal{N}$.

0.8. Basic operators of Hodge theory. Some operators will have an invariable notation, whatever the category they belong to.

• Sesquilinear pairings used for categories of triples are denoted by $\mathfrak{s}$.

• Nilpotent endomorphisms are denoted by N, and their monodromy filtration is denoted by $\mathrm{M}_\bullet(\mathrm{N})$ or simply $\mathrm{M}_\bullet$.

• Polarizations are denoted by S, but their components may be denoted by $\mathcal{S}$ or $\widetilde{\mathcal{S}}$, depending on the context.

Cohomology functors.

• Whatever derived category it acts on, the k-th cohomology functor is denoted by H^k.

• Pushforward or pullback functors are mostly defined as functors on derived categories, but not defined as right or left derived functors. In order to simplify the notation, the cohomology functors like $\mathcal{H}^k \boldsymbol{R} f_*$ or ${}_{\mathrm{D}}f_*$ are denoted by $f_*^{(k)}$ or ${}_{\mathrm{D}}f_*^{(k)}$.