APPENDIX.
SIGN CONVENTIONS FOR HODGE MODULES

A.1. General principles

In this appendix, we explain how one can arrive at the correct sign conventions for polarized Hodge modules. This is a bit of a detective story, fortunately with a happy ending. Finding the correct signs looks difficult at the beginning, because there are many places in the theory where one might have to choose a sign factor, and it is not clear that all those choices can be made consistently. For example, should there be a sign in the conversion between left and right $\mathcal{D}$-modules? What are the correct signs to use for direct images? For nearby and vanishing cycles? For the duality functor?

Before going into any details, we think it may be helpful to list a few general principles that have turned out to be useful in the solution:

(1) Make all definitions in such a way that they do not depend on the choice of the imaginary unit $\mathsf{i} = \sqrt{-1}$.

(2) Make all constructions compatible with closed embeddings, and therefore independent of the choice of ambient complex manifold.

(3) In particular, work consistently with right $\mathcal{D}$-modules and currents (instead of with left $\mathcal{D}$-modules and distributions).

(4) When defining a current, choose the sign in such a way that the resulting current is positive, if possible.

(5) Use Deligne's Koszul sign rule for graded objects. Under this rule, switching two quantities x and y produces a sign factor of $(-1)^{\deg x \deg y}$.

A.1.1. Example. The integral over a complex manifold X depends on the orientation; the orientation is induced by the standard orientation on $\mathbb{C}$, in which $1, \mathsf{i}$ is a positively oriented basis over $\mathbb{R}$. To make the integral independent of the choice of i, it is better to work with the expression

$$\frac{1}{(2\pi\mathsf{i})^{\dim X}} \int_X$$

instead. Similarly, the Lefschetz operator $L_\omega \alpha = \omega \wedge \alpha$ on the cohomology of a compact Kähler manifold depends on the choice of i, because the Kähler form ω is minus the imaginary part of the Kähler metric. It is therefore better to work with the operator $(2\pi\mathsf{i})L_\omega$ instead.

A.1.2. Example. The fundamental group of the punctured disk

$$\Delta^* = \{t \in \mathbb{C} \mid 0 < |t| < 1\}$$

is naturally the group $\mathbb{Z}(1) = (2\pi\mathsf{i})\mathbb{Z}$. Indeed, independently of the choice of i, the universal covering space of the punctured disk is $\exp\colon \mathbb{H} \to \Delta^*$, where

$$\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Re} z < 0\}$$

is the *left* halfplane. The group $\mathbb{Z}(1)$ acts on this space by translations.

A.1.3. Example. Polarizations are defined as Hermitian pairings with values in the sheaf of currents. The following collection of basic currents on the unit disk Δ plays an important role in the theory. Define $\mathrm{L}(t) = -\log|t|^2$, with a minus sign to make the function positive on Δ^*. For $\alpha < 0$ and $p \in \mathbb{N}$, the formula

$$\big\langle C_{\alpha,p}, \varphi\big\rangle = \frac{\varepsilon(2)}{2\pi\mathsf{i}} \int_\Delta \frac{\mathrm{L}(t)^p}{p!} |t|^{-2(1+\alpha)} \varphi \, \mathrm{d}t \wedge \mathrm{d}\bar{t} \tag{A.1.4}$$

defines a current on Δ. The factor $2\pi\mathsf{i}$ makes the current independent of the choice of $\mathsf{i} = \sqrt{-1}$, and the sign factor $\varepsilon(2) = -1$ makes it positive, as suggested by the general principles above. For different values of $p \in \mathbb{N}$, the basic currents are related by the identity

$$C_{\alpha,p}(t\partial_t - \alpha) = C_{\alpha,p}(\bar{t}\partial_{\bar{t}} - \alpha) = C_{\alpha,p-1},$$

which can be proved using integration by parts. The delta function

$$\big\langle \delta_0, \varphi\big\rangle = \varphi(0)$$

can be expressed in terms of the basic currents as

$$\delta_0 = -C_{-1,1}\partial_t\partial_{\bar{t}}; \tag{A.1.5}$$

the proof is again by integration by parts.

A.2. Hodge structures and polarizations

The first place where a sign factor appears is in the definition of complex Hodge structures. Let H be a finite-dimensional complex vector space. Recall that a *Hodge structure* of weight k on H is a decomposition

$$H = \bigoplus_{p+q=k} H^{p,q}. \tag{A.2.1}$$

A *polarization* of H is a Hermitian form

$$\mathrm{S}\colon H \otimes_{\mathbb{C}} \overline{H} \longrightarrow \mathbb{C},$$

with the following two properties:

(a) The decomposition in (A.2.1) is orthogonal with respect to S.
(b) The Hermitian form $c_k(-1)^q\mathrm{S}$ is positive definite on the subspace $H^{p,q}$.

In this definition, c_k is a sign factor depending on the weight of the Hodge structure. We will see below that there are only two choices: either $c_k = (-1)^k$, which is the convention used in classical Hodge theory; or $c_k = 1$, which is the convention used in Saito's work. We will find that $c_k = 1$ is indeed the correct choice for the theory of Hodge modules, but we shall give all formulas with c_k for the time being, so as not to prejudge the issue.

A.2.2. Example. On $\mathbb{C} = \mathbb{C}^{0,0}$, the natural Hermitian form is $\mathrm{S}(a, b) = a\bar{b}$. If we want this to be a polarization, we have to use $c_0 = 1$.

A.2.3. Example. If H is a Hodge structure of weight k, then the conjugate vector space $\overline{H}$ inherits a Hodge structure of weight k, with Hodge decomposition

$$\overline{H}^{p,q} = \overline{H^{q,p}}.$$

The *Tate twist* $H(n)$ is the Hodge structure of weight $k - 2n$ with

$$H(n)^{p,q} = H^{p+n,q+n}.$$

The first condition in the definition of a polarization is equivalent to saying that

$$\mathrm{S}\colon H \otimes_{\mathbb{C}} \overline{H} \longrightarrow \mathbb{C}(-k)$$

is a morphism of Hodge structures of weight $2k$.

A.2.4. Example. Let $H_{\mathbb{R}}$ be a finite-dimensional real vector space. Cattani, Kaplan, and Schmid define a *real Hodge structure* of weight k to be a decomposition

$$H = \mathbb{C} \otimes_{\mathbb{R}} H_{\mathbb{R}} = \bigoplus_{p+q=k} H^{p,q}$$

with the property that $\overline{H^{p,q}} = H^{q,p}$. They say that a bilinear pairing

$$\mathrm{Q}_{\mathbb{R}}\colon H_{\mathbb{R}} \otimes_{\mathbb{R}} H_{\mathbb{R}} \longrightarrow \mathbb{R}$$

is a *polarization* if the following conditions are satisfied: $\mathrm{Q}_{\mathbb{R}}$ is $(-1)^k$-symmetric; the Hodge decomposition is orthogonal with respect to $\mathrm{Q}_{\mathbb{R}}$; and $\mathrm{Q}_{\mathbb{R}}(i^{p-q}v, \overline{v}) > 0$ for every nonzero $v \in H^{p,q}$. In that case, the Hermitian pairing

$$\mathrm{S}\colon H \otimes_{\mathbb{C}} \overline{H} \longrightarrow \mathbb{C}, \quad \mathrm{S}(v, w) = c_k(-1)^k \cdot (2\pi \mathrm{i})^{-k} \mathrm{Q}_{\mathbb{R}}(v, \overline{w})$$

is a polarization in our sense. Indeed, for nonzero $v \in H^{p,q}$, one gets

$$c_k(-1)^q \cdot \mathrm{S}(v, v) = c_k(-1)^k i^k i^{p-q} \cdot \mathrm{S}(v, v) = (2\pi)^{-k} \mathrm{Q}_{\mathbb{R}}(i^{p-q}v, \overline{v}) > 0.$$

One can interpret the factor $(2\pi \mathrm{i})^{-k}$ as saying that $\mathrm{Q}_{\mathbb{R}}\colon H_{\mathbb{R}} \otimes_{\mathbb{R}} H_{\mathbb{R}} \to \mathbb{R}(-k)$ is a morphism of real Hodge structures of weight $2k$; in classical Hodge theory, it is therefore more natural to take $c_k = (-1)^k$.

A.3. Cohomology of compact Kähler manifolds

We can pin down some of the signs by working out what happens for the cohomology of compact Kähler manifolds. Let X be a compact Kähler manifold of dimension n. For each $k \in \{0, 1, \dots, 2n\}$, the k-th cohomology has a Hodge structure of weight k, with Hodge decomposition

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X).$$

A choice of Kähler metric h determines a polarization of the Hodge structure; it also determines the Lefschetz operator, which makes the direct sum of all cohomology groups into a representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. Our goal will be to describe all this information as concisely as possible.

Note. The advantage of this example – and the reason for putting it at the beginning of our analysis – is that there are no choices involved in constructing a positive definite pairing. Indeed, the Kähler metric induces a positive definite Hermitian inner product on the space of harmonic k-forms, hence on $H^k(X, \mathbb{C})$. All we have to do is figure out what signs appear when we compare this inner product to the pairing given by wedge product and integration over X.

Fix a choice of $\mathsf{i} = \sqrt{-1}$. The Kähler form $\omega = -\operatorname{Im} h \in A^2(X, \mathbb{R})$ and its cohomology class $[\omega] \in H^2(X, \mathbb{R})$ depend on the choice of i, because the imaginary part $\operatorname{Im}\colon \mathbb{C} \to \mathbb{R}$ does. The choice of i endows the two-dimensional real vector space $\mathbb{C}$ with an orientation, by declaring that $1, \mathsf{i}$ is a positively-oriented basis; the induced orientation on X has the property that

$$\int_X \frac{\omega^n}{n!} = \operatorname{vol}(X) > 0.$$

We can remove the dependence on the choice of i by defining $\mathbb{R}(1) = 2\pi\mathsf{i} \cdot \mathbb{R} \subseteq \mathbb{C}$, and working with the closed two-form $2\pi\mathsf{i}\, \omega \in A^2(X, \mathbb{R}(1))$; its cohomology class is $[2\pi\mathsf{i}\, \omega] \in H^2(X, \mathbb{R}(1))$. Instead of the usual integral, we use

$$\frac{1}{(2\pi\mathsf{i})^n} \int_X : A^{2n}(X, \mathbb{C}) \longrightarrow \mathbb{C}.$$

Now all terms in the identity

$$\frac{1}{(2\pi\mathsf{i})^n} \int_X \frac{(2\pi\mathsf{i}\, \omega)^n}{n!} = \operatorname{vol}(X)$$

are independent of the choice of i.

A.3.1. Example. On $\mathbb{P}^1$, with the Fubini-Study metric, one has

$$2\pi\mathsf{i}\, \omega_{\mathrm{FS}} = c_1\bigl(\mathcal{O}_{\mathbb{P}^1}(1)\bigr) \in H^2\bigl(\mathbb{P}^1, \mathbb{Z}(1)\bigr),$$

and the volume comes out to

$$\operatorname{vol}(\mathbb{P}^1) = \frac{1}{2\pi\mathsf{i}} \int_{\mathbb{P}^1} 2\pi\mathsf{i}\, \omega_{\mathrm{FS}} = 1.$$

This is the reason for including the factor 2π into the definition. Some of the formulas below would look nicer without the 2π, but we shall keep it for the sake of tradition.

Let $A^k(X) = A^k(X, \mathbb{C})$ be the space of smooth complex-valued k-forms. The Kähler metric h induces on $A^k(X)$ a Hermitian inner product

$$\langle \alpha, \beta \rangle = \int_X \alpha \wedge *\overline{\beta},$$

where $*\colon A^k(X) \to A^{2n-k}(X)$ is the Hodge $*$-operator. Like the integral, the Hodge $*$-operator depends on the orientation, whereas the inner product only depends on the Kähler metric h. We define the *Lefschetz operator*

$$L_\omega \colon A^\bullet(X) \longrightarrow A^{\bullet+2}(X)$$

by the formula $L_\omega \alpha = \omega \wedge \alpha$, and its adjoint

$$\Lambda_\omega \colon A^\bullet(X) \longrightarrow A^{\bullet-2}(X)$$

by the formula $\langle L_\omega \alpha, \beta \rangle = \langle \alpha, \Lambda_\omega \beta \rangle$. The main tool for describing the polarization is the following result, known as *Weil's identity.*

A.3.2. Proposition. *If $\alpha \in A^{p,q}(X)$ is primitive, in the sense that $\Lambda_\omega \alpha = 0$, then*

$$*\alpha = \mathsf{i}^{q-p}\, \varepsilon(k)\, \frac{L_\omega^{n-k}}{(n-k)!}\, \alpha, \tag{A.3.3}$$

where $\varepsilon(k) = (-1)^{k(k-1)/2}$ and $k = p + q$.

We can use Weil's identity to express the Hodge $*$-operator in terms of representation theory. The complex Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ has the standard basis

$$\mathrm{H} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathrm{X} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathrm{Y} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

In the complex Lie group $\mathrm{SL}_2(\mathbb{C})$, consider the *Weil element*

$$\mathrm{w} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = e^{\mathrm{X}} e^{-\mathrm{Y}} e^{\mathrm{X}}.$$

It has the property that $\mathrm{w}^{-1} = -\mathrm{w}$, and under the adjoint action of $\mathrm{SL}_2(\mathbb{C})$ on its Lie algebra, one has the identities

$$\mathrm{wHw}^{-1} = -\mathrm{H}, \quad \mathrm{wXw}^{-1} = -\mathrm{Y}, \quad \mathrm{wYw}^{-1} = -\mathrm{X}.$$

From this, one deduces that $e^{\mathrm{X}} = \mathrm{w} e^{-\mathrm{X}} e^{\mathrm{Y}} = e^{\mathrm{Y}} \mathrm{w} e^{\mathrm{Y}}$, which gives another way to remember the formula for w.

The (infinite-dimensional) vector space

$$A^\bullet(X) = \bigoplus_{k=0}^{2n} A^k(X)$$

becomes a representation of $\mathfrak{sl}_2(\mathbb{C})$ if we set

$$\mathrm{X} = 2\pi \mathsf{i}\, L_\omega \quad \text{and} \quad \mathrm{Y} = (2\pi \mathsf{i})^{-1} \Lambda_\omega,$$

and let H act as multiplication by $k-n$ on the subspace $A^k(X)$. The reason for this (non-standard) definition is that it makes the representation not depend on the choice of i. It is easy to see how the Weil element w acts on primitive forms. Suppose that $\alpha \in A^{n-k}(X)$ satisfies $\mathrm{Y}\alpha = 0$. Then $\mathrm{w}\alpha \in A^{n+k}(X)$, and if we expand both sides of the identity

$$e^{\mathrm{X}}\alpha = e^{\mathrm{Y}}\mathrm{w}e^{\mathrm{Y}}\alpha = e^{\mathrm{Y}}\mathrm{w}\alpha$$

into power series, and compare terms in degree $n+k$, we get

$$\mathrm{w}\alpha = \frac{\mathrm{X}^k}{k!}\alpha.$$

This formula is the reason for using w (instead of the otherwise equivalent w^{-1}): there is no sign on the right-hand side.

Note. One should be careful: the element

$$\mathrm{w}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$$

acts on $A^k(X)$ as $(-1)^{k-n}$, and *not* just as -1.

We deduce the following generalization of Weil's identity, which shows again how the Hodge $*$-operator depends on the choice of i.

A.3.4. Proposition. *For every $\alpha \in A^{p,q}(X)$, one has*

$$*\,\alpha = \frac{1}{(2\pi\mathsf{i})^n}\cdot(-1)^q\,\varepsilon(p+q)(2\pi)^{p+q}\cdot\mathrm{w}\alpha.$$

Proof. Suppose first that $\mathrm{Y}\alpha = 0$. Setting $k = p+q$, we have

$$\mathrm{w}\alpha = \frac{\mathrm{X}^{n-k}}{(n-k)!}\,\alpha.$$

On the other hand, Weil's identity (A.3.3) becomes

$$*\alpha = i^{q-p}\,\varepsilon(k)\cdot(2\pi\mathsf{i})^{k-n}\frac{\mathrm{X}^{n-k}}{(n-k)!}\,\alpha = (2\pi\mathsf{i})^{-n}\cdot(-1)^q\,\varepsilon(k)(2\pi)^k\cdot\mathrm{w}\alpha,$$

as claimed. The general case follows by using the relations

$$*\,\mathrm{X} = -(2\pi)^2\,\mathrm{Y}\,* \quad \text{and} \quad \mathrm{wX} = -\mathrm{Yw},$$

the Lefschetz decomposition for α, and the identity $\varepsilon(k+2) = -\,\varepsilon(k)$. ∎

Now we can easily derive the Hodge-Riemann bilinear relations. Suppose that $\alpha, \beta \in A^{p,q}(X)$, and set $k = p+q$. Then

$$\alpha \wedge *\overline{\beta} = \alpha \wedge \overline{(*\beta)} = \frac{1}{(2\pi\mathsf{i})^n}\cdot(-1)^{n+q}\,\varepsilon(k)(2\pi)^k\cdot\alpha\wedge\overline{(\mathrm{w}\beta)}.$$

If we put this into the formula for the inner product, we get

$$\langle\alpha,\beta\rangle = \int_X \alpha\wedge *\overline{\beta} = (-1)^{n+q}\,\varepsilon(k)(2\pi)^k\cdot\frac{1}{(2\pi\mathsf{i})^n}\int_X \alpha\wedge\overline{(\mathrm{w}\beta)}. \tag{A.3.5}$$

According to our definition, this means that the Hermitian pairing

$$(\alpha, \beta) \longmapsto (-1)^n c_k \, \varepsilon(k) \cdot \frac{1}{(2\pi \mathsf{i})^n} \int_X \alpha \wedge \overline{(\mathrm{w}\beta)},$$

polarizes the Hodge structure on $H^k(X, \mathbb{C})$.

It turns out that there is a much more concise way of describing the polarization. Let us set $H_k = H^{n+k}(X, \mathbb{C})$; this has a Hodge structure of weight $n+k$, and its weight with respect to the action by H is equal to k. Also set

$$H = \bigoplus_{k \in \mathbb{Z}} H_k,$$

with the induced action by the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ and the Lie group $\mathrm{SL}_2(\mathbb{C})$. For each $k \in \{-n, \dots, n\}$, we have a sesquilinear pairing

$$\mathrm{S}_k \colon H_k \otimes_{\mathbb{C}} \overline{H_{-k}} \longrightarrow \mathbb{C}, \quad \mathrm{S}_k(\alpha, \beta) = b_k \cdot \frac{1}{(2\pi \mathsf{i})^n} \int_X \alpha \wedge \overline{\beta}. \tag{A.3.6}$$

Here b_k is a sign factor; our goal will be to choose b_k in such a way that all the formulas become as simple as possible. We can put all of the S_k together into one big sesquilinear pairing

$$\mathrm{S} \colon H \otimes_{\mathbb{C}} \overline{H} \longrightarrow \mathbb{C}, \quad \mathrm{S}_{|H_k \otimes_{\mathbb{C}} \overline{H_\ell}} = \begin{cases} \mathrm{S}_k & \text{if } \ell = -k, \\ 0 & \text{otherwise.} \end{cases}$$

The following two identities can be checked with a brief calculation:

$$\overline{\mathrm{S}_k(\alpha, \beta)} = (-1)^k b_{-k} b_k \cdot \mathrm{S}_{-k}(\beta, \alpha) \quad \text{and} \quad \mathrm{S}_k(\mathrm{H}\alpha, \beta) = -\mathrm{S}_k(\alpha, \mathrm{H}\beta)$$

for $\alpha \in H_k$ and $\beta \in H_{-k}$. Since $\mathrm{X} = 2\pi \mathsf{i}\, L_\omega$, it is also not hard to show that

$$\mathrm{S}_{k+2}(\mathrm{X}\alpha, \beta) = -b_k b_{k+2} \cdot \mathrm{S}_k(\alpha, \mathrm{X}\beta)$$

for $\alpha \in H_k$ and $\beta \in H_{-(k+2)}$. A slightly longer calculation, based on Proposition A.3.4, is required to prove the identity

$$\mathrm{S}_k(\alpha, \mathrm{w}\beta) = (-1)^k b_{-k} b_k \cdot \mathrm{S}_{-k}(\mathrm{w}\alpha, \beta)$$

for every $\alpha, \beta \in H_k$. Now the fact that $\mathrm{Y} = -\mathrm{w}\mathrm{X}\mathrm{w}^{-1}$ can be used to deduce the following more surprising identity:

$$\mathrm{S}_{k-2}(\mathrm{Y}\alpha, \beta) = -b_k b_{k-2} \cdot \mathrm{S}_k(\alpha, \mathrm{Y}\beta)$$

for every $\alpha \in H_k$ and every $\beta \in H_{-(k-2)}$. We write "surprising" because it is not at all clear, at first glance, that one can move the adjoint Λ_ω of the Lefschetz operator from one factor of the integral to the other.

Clearly, we should require $b_{k+2} = -b_k$ for every $k \in \mathbb{Z}$, in order to eliminate all the sign factors from the above formulas. Let us restate the resulting identities in terms of the sesquilinear pairing $\mathrm{S} \colon H \otimes_{\mathbb{C}} \overline{H} \to \mathbb{C}$: first, S is Hermitian symmetric; second, one has the four identities

$$\begin{aligned}
\mathrm{S}\circ(\mathrm{H}\otimes\mathrm{Id}) &= -\mathrm{S}\circ(\mathrm{Id}\otimes\mathrm{H}),\\
\mathrm{S}\circ(\mathrm{X}\otimes\mathrm{Id}) &= \mathrm{S}\circ(\mathrm{Id}\otimes\mathrm{X}),\\
\mathrm{S}\circ(\mathrm{Y}\otimes\mathrm{Id}) &= \mathrm{S}\circ(\mathrm{Id}\otimes\mathrm{Y}),\\
\mathrm{S}\circ(\mathrm{w}\otimes\mathrm{Id}) &= \mathrm{S}\circ(\mathrm{Id}\otimes\mathrm{w}).
\end{aligned}\tag{A.3.7}$$

Now suppose that $\alpha, \beta \in A^{p,q}(X)$ are harmonic forms. It will be convenient to define $k = (p+q) - n$, so that $[\alpha], [\beta] \in H_k$. Then

$$\mathrm{S}_k(\alpha, \mathrm{w}\beta) = b_k \cdot \frac{1}{(2\pi\mathrm{i})^n} \int_X \alpha \wedge \overline{(\mathrm{w}\beta)}.$$

Going back to (A.3.5), we can rewrite this in the form

$$\mathrm{S}_k(\alpha, \mathrm{w}\beta) = (-1)^q \cdot b_k (-1)^n \, \varepsilon(n+k) \cdot \frac{\langle \alpha, \beta \rangle}{(2\pi)^{n+k}}.$$

The conclusion is H_k has a Hodge structure of weight $n+k$, which is polarized by the Hermitian form $\mathrm{S}_k \circ (\mathrm{Id} \otimes \mathrm{w})$, provided that

$$b_k = (-1)^n \, \varepsilon(n+k) c_{n+k}. \tag{A.3.8}$$

Note. Recall that $b_{k+2} = -b_k$. Since $\varepsilon(k+2) = -\varepsilon(k)$, it follows that $c_{k+2} = c_k$ for every $k \in \mathbb{Z}$; together with the normalization $c_0 = 1$, this leaves the two values $c_k = 1$ and $c_k = (-1)^k$ as the only possibilities. We will see below that $c_k = 1$ is the better choice for the theory of Hodge modules.

Let us summarize our findings. Setting $H_k = H^{n+k}(X, \mathbb{C})$, the vector space

$$H = \bigoplus_{k \in \mathbb{Z}} H_k$$

is a representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. Each weight space H_k has a Hodge structure of weight $n+k$, and the two operators

$$\mathrm{X}\colon H_k \longrightarrow H_{k+2}(1) \quad \text{and} \quad \mathrm{Y}\colon H_k \longrightarrow H_{k-2}(-1)$$

are morphisms of Hodge structure. All of these Hodge structures are simultaneously polarized by the Hermitian form $\mathrm{S} \circ (\mathrm{Id} \otimes \mathrm{w})$, where $\mathrm{S}\colon H \otimes_{\mathbb{C}} \overline{H} \to \mathbb{C}$ is assembled from the individual sesquilinear pairings

$$\mathrm{S}_k\colon H_k \otimes_{\mathbb{C}} \overline{H_{-k}} \longrightarrow \mathbb{C}, \quad \mathrm{S}_k(\alpha, \beta) = (-1)^n c_{n+k} \, \varepsilon(n+k) \cdot \frac{1}{(2\pi\mathrm{i})^n} \int_X \alpha \wedge \overline{\beta},$$

and satisfies the identities in (A.3.7).

A.4. $\mathfrak{sl}_2$-Hodge structures and polarizations

The cohomology of a compact Kähler manifold is both a representation of $\mathfrak{sl}_2(\mathbb{C})$ and a direct sum of polarized Hodge structures, in a compatible way. Since the same kind of structure also appears in the analysis of polarized variations of Hodge structure on the punctured disk, it will be useful to give it a name.

A.4.1. Definition. An $\mathfrak{sl}_2$*-Hodge structure* on a finite-dimensional complex vector space H is a representation of $\mathfrak{sl}_2(\mathbb{C})$ on H with the following properties:

(a) Each weight space $H_k = E_k(\mathrm{H})$ has a Hodge structure of weight $n+k$; the integer n is called the *(central) weight* of the $\mathfrak{sl}_2$-Hodge structure.

(b) The two operators

$$\mathrm{X}\colon H_k \longrightarrow H_{k+2}(1) \quad \text{and} \quad \mathrm{Y}\colon H_k \longrightarrow H_{k-2}(-1)$$

are morphisms of Hodge structure.

Equivalently, an $\mathfrak{sl}_2$-Hodge structure of weight n is a bigraded vector space

$$H = \bigoplus_{p,q\in\mathbb{Z}} H^{p,q}$$

that is simultaneously a representation of $\mathfrak{sl}_2(\mathbb{C})$, in a way that is compatible with the bigrading. This means that

$$\mathrm{X}\colon H^{p,q} \longrightarrow H^{p+1,q+1} \quad \text{and} \quad \mathrm{Y}\colon H^{p,q} \longrightarrow H^{p-1,q-1},$$

and that H acts on the subspace $H^{p,q}$ as multiplication by the integer $(p+q)-n$. This makes each of the weight spaces

$$H_k = \bigoplus_{p+q=n+k} H^{p,q}$$

into a Hodge structure of weight $n+k$. In this abstract setting, we can again define the *Weil element*

$$\mathrm{w} = e^{\mathrm{X}} e^{-\mathrm{Y}} e^{\mathrm{X}} \in \mathrm{GL}(H).$$

The Weil element induces isomorphisms $\mathrm{w}\colon H_k \to H_{-k}$ among opposite weight spaces, due to the fact that $\mathrm{wHw}^{-1} = -\mathrm{H}$.

A.4.2. Lemma. *If H is an $\mathfrak{sl}_2$-Hodge structure, then* $\mathrm{w}\colon H_k \to H_{-k}(-k)$ *is an isomorphism of Hodge structures (of weight $n+k$).*

Proof. We first prove an auxiliary formula. Suppose that $b \in H_{-\ell}$ is primitive, in the sense that $\mathrm{Y}b = 0$ (and therefore $\ell \geqslant 0$). From $\mathrm{w}e^{-\mathrm{X}} = e^{\mathrm{X}}e^{-\mathrm{Y}}$, we get $\mathrm{w}e^{-\mathrm{X}}b = e^{\mathrm{X}}b$, and after expanding and comparing terms in degree $\ell - 2j$, also

$$\mathrm{w}\frac{\mathrm{X}^j}{j!}b = (-1)^j \frac{\mathrm{X}^{\ell-j}}{(\ell-j)!}b. \tag{A.4.3}$$

Now any $a \in H_k$ has a unique Lefschetz decomposition

$$a = \sum_{j\geqslant \max(k,0)} \frac{\mathrm{X}^j}{j!}a_j,$$

where $a_j \in H_{k-2j}$ satisfies $Ya_j = 0$. Here we only need to consider $j \geqslant k$ in the sum because $\mathrm{X}^{2j-k+1}a_j = 0$, which implies that $\mathrm{X}^j a_j = 0$ for $j < k$. Suppose further that $a \in H^{p,q}$, where $p+q = n+k$. Then $\mathrm{X}^i a_j \in H^{p+i,q+i}$, and by descending induction on $j \geqslant \max(k,0)$, we deduce that

$$a_j \in H^{p-j,q-j}.$$

In other words, the Lefschetz decomposition holds in the category of Hodge structures. We can now check what happens when we apply w. Using (A.4.3),

$$\mathrm{w}a = \sum_{j \geqslant \max(k,0)} \mathrm{w}\frac{\mathrm{X}^j}{j!}a_j = \sum_{j \geqslant \max(k,0)} (-1)^j \frac{\mathrm{X}^{j-k}}{(j-k)!}a_j \in H^{p-k,q-k},$$

and so w is a morphism of Hodge structures. Since w is bijective, it must be an isomorphism of Hodge structures, as claimed. □

We define polarizations of $\mathfrak{sl}_2$-Hodge structures by analogy with the case of compact Kähler manifolds.

A.4.4. Definition. A *polarization* of an $\mathfrak{sl}_2$-Hodge structure H is a Hermitian form

$$\mathrm{S}\colon H \otimes_{\mathbb{C}} \overline{H} \longrightarrow \mathbb{C}$$

that satisfies the four identities

$$\begin{aligned} \mathrm{S} \circ (\mathrm{H} \otimes \mathrm{Id}) &= -\mathrm{S} \circ (\mathrm{Id} \otimes \mathrm{H}), \\ \mathrm{S} \circ (\mathrm{X} \otimes \mathrm{Id}) &= \mathrm{S} \circ (\mathrm{Id} \otimes \mathrm{X}), \\ \mathrm{S} \circ (\mathrm{Y} \otimes \mathrm{Id}) &= \mathrm{S} \circ (\mathrm{Id} \otimes \mathrm{Y}), \\ \mathrm{S} \circ (\mathrm{w} \otimes \mathrm{Id}) &= \mathrm{S} \circ (\mathrm{Id} \otimes \mathrm{w}), \end{aligned}$$

such that $\mathrm{S} \circ (\mathrm{Id} \otimes \mathrm{w})$ polarizes the Hodge structure of weight $n+k$ on each H_k.

The relation $\mathrm{S} \circ (\mathrm{H} \otimes \mathrm{Id}) = -\mathrm{S} \circ (\mathrm{Id} \otimes \mathrm{H})$ implies that

$$\mathrm{S}_{|H_k \otimes_{\mathbb{C}} \overline{H_\ell}} = \begin{cases} \mathrm{S}_k & \text{if } \ell = -k, \\ 0 & \text{otherwise.} \end{cases}$$

and so S is actually given by a collection of sesquilinear pairings

$$\mathrm{S}_k\colon H_k \otimes_{\mathbb{C}} \overline{H_{-k}} \longrightarrow \mathbb{C},$$

exactly as in the previous section.

A.4.5. Example. With the exception of positivity, all the conditions in the definition have a nice functorial interpretation. The conjugate complex vector space $\overline{H}$ is again an $\mathfrak{sl}_2$-Hodge structure of weight n: the action of H is unchanged, but X and Y act with an extra minus sign. This sign change is dictated by the geometric case, where $\mathrm{X} = 2\pi\mathrm{i}\, L_\omega$ and $\mathrm{Y} = (2\pi i)^{-1}\Lambda_\omega$. Likewise, if H' and H'' are $\mathfrak{sl}_2$-Hodge structures of weights n' and n'', then the tensor product $H' \otimes_{\mathbb{C}} H''$ is naturally an $\mathfrak{sl}_2$-Hodge structure of weight $n'+n''$: to be precise,

$$\big(H' \otimes_{\mathbb{C}} H''\big)_k = \bigoplus_{i+j=k} H'_i \otimes_{\mathbb{C}} H''_j,$$

and the $\mathfrak{sl}_2(\mathbb{C})$-action is given by the usual formulas

$$\begin{aligned} \mathrm{X}(v' \otimes v'') &= \mathrm{X}v' \otimes v'' + v' \otimes \mathrm{X}v'', \\ \mathrm{Y}(v' \otimes v'') &= \mathrm{Y}v' \otimes v'' + v' \otimes \mathrm{Y}v'', \\ \mathrm{H}(v' \otimes v'') &= \mathrm{H}v' \otimes v'' + v' \otimes \mathrm{H}v''. \end{aligned}$$

Lastly, we can turn $\mathbb{C}(-n)$ into an $\mathfrak{sl}_2$-Hodge structure of weight $2n$ by letting $\mathfrak{sl}_2(\mathbb{C})$ act trivially. Then all the identities in Definition A.4.4 can be summarized in one line by saying that the Hermitian form

$$\mathrm{S}\colon H \otimes_{\mathbb{C}} \overline{H} \longrightarrow \mathbb{C}(-n)$$

is a morphism of $\mathfrak{sl}_2$-Hodge structures of central weight $2n$. This shows that the choice of the sign factor b_k in (A.3.8) is the only natural one.

A.5. Pairings on $\mathcal{D}$-modules

Let us return to the cohomology of compact Kähler manifolds, in particular, to the formula for the sesquilinear pairing

$$\mathrm{S}_k\colon H_k \otimes_{\mathbb{C}} \overline{H_{-k}} \longrightarrow \mathbb{C}, \quad \mathrm{S}_k(\alpha,\beta) = (-1)^n c_{n+k}\,\varepsilon(n+k)\cdot \frac{1}{(2\pi\mathsf{i})^n}\int_X \alpha\wedge\overline{\beta}.$$

The sign factor $(-1)^n c_{n+k}\,\varepsilon(n+k)$ in this formula represents an interesting puzzle, whose solution is another important step in finding the correct sign conventions for Hodge modules, especially for direct images.

Recall that $H_k = H^{n+k}(X,\mathbb{C})$ is isomorphic to the $(n+k)$-th hypercohomology of the holomorphic de Rham complex $\mathrm{DR}(\mathcal{O}_X)$; this is the complex

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\mathrm{d}} \Omega_X^1 \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega_X^n \longrightarrow 0,$$

which naturally lives in degrees $0,\dots,n$. Equivalently, H_k is the k-th hypercohomology of the shifted de Rham complex $\mathrm{DR}(\mathcal{O}_X)[n]$; under the Koszul sign rule, the differential in the complex $\mathrm{DR}(\mathcal{O}_X)[n]$ has to be $(-1)^n d$.

Now the left $\mathcal{D}_X$-module $\mathcal{O}_X$ comes with a natural Hermitian pairing, given by taking two local sections $f,g\in\mathcal{O}_X$ to the product $f\overline{g}$. What should the corresponding pairing on the right $\mathcal{D}_X$-module ω_X be? The correct answer to this question turns out to be

$$\text{(A.5.1)}\qquad \mathrm{S}_X\colon \omega_X \otimes_{\mathbb{C}} \overline{\omega_X} \longrightarrow \mathfrak{C}_X, \quad \big\langle \mathrm{S}_X(\omega',\omega''),\varphi\big\rangle = \frac{\varepsilon(n+1)}{(2\pi\mathsf{i})^n}\int_X \varphi\cdot\omega'\wedge\overline{\omega''},$$

where $\mathfrak{C}_X$ is the sheaf of currents of maximal degree, and $\varepsilon(k) = (-1)^{k(k-1)/2}$. Note that S_X is a morphism of right $\mathcal{D}_X \otimes_{\mathbb{C}} \mathcal{D}_{\overline{X}}$-modules. It is also Hermitian symmetric and, with the sign factor $\varepsilon(n+1)$ in front, positive definite: if the test function φ is real-valued and nonnegative, then

$$\big\langle \mathrm{S}_X(\omega,\omega),\varphi\big\rangle \geqslant 0,$$

and equality for every φ implies that $\omega = 0$. Following the general principle that currents should be defined to be positive where possible, this is clearly the most natural choice for the pairing on ω_X.

Note. With this definition of S_X, the induced pairing on the space $H^{n,0}(X) = H^0(X,\omega_X)$ is already positive definite. If we do not want to add any additional sign factors, then we need to use $c_k = 1$ and not $c_k = (-1)^k$; in other words, in a polarized Hodge structure, the sign of the polarization on the subspace $H^{p,q}$ should be $(-1)^q$. We will see below that this choice works well in all cases.

Back to the puzzle of the sign factor $(-1)^n\,\varepsilon(n+k)$. We have

$$(-1)^n\,\varepsilon(n+k) = (-1)^n\,\varepsilon(n)\,\varepsilon(k)(-1)^{nk} = \varepsilon(n+1)\,\varepsilon(k)(-1)^{nk},$$

which means that we can write the sesquilinear pairing from above as

$$\mathrm{S}_k(\alpha,\beta) = \varepsilon(k)\cdot(-1)^{nk}\cdot\frac{\varepsilon(n+1)}{(2\pi\mathrm{i})^n}\int_X \alpha\wedge\overline{\beta}. \tag{A.5.2}$$

The third factor is consistent with the pairing S_X on the right $\mathcal{D}_X$-module ω_X, and the first factor $\varepsilon(k)$ only depends on the degree of the cohomology (which is what we need to get a pairing that is embedding-independent); the question is where the extra factor of $(-1)^{nk}$ comes from. Deligne gave a technical answer in a letter to Saito (in terms of tensor products and shifts of complexes), but a more natural answer in our setting is that it is caused by the conversion between right and left $\mathcal{D}$-modules. Namely, in order to convert the natural pairing on the *right* $\mathcal{D}_X$-module ω_X into a pairing on the de Rham complex of the *left* $\mathcal{D}_X$-module $\mathcal{O}_X$, some sign changes are needed, and these sign changes nicely account for the factor $(-1)^{nk}$ in the above formula.

Since this is an important issue, we shall spend the remainder of this section going through the details. To begin with, we describe a naive way for getting a pairing on cohomology, in the setting of right $\mathcal{D}$-modules. Let $\mathcal{M}$ be a right $\mathcal{D}_X$-module, and suppose that we have a flat Hermitian pairing

$$\mathrm{S}\colon \mathcal{M}\otimes_{\mathbb{C}}\overline{\mathcal{M}} \longrightarrow \mathfrak{C}_X\,.$$

We use the notation $\mathrm{Sp}_X(\mathcal{M})$ for the Spencer complex

$$0\longrightarrow \mathcal{M}\otimes_{\mathcal{O}_X}\wedge^n\Theta_X \xrightarrow{\ \delta\ }\cdots\xrightarrow{\ \delta\ }\mathcal{M}\otimes_{\mathcal{O}_X}\Theta_X\xrightarrow{\ \delta\ }\mathcal{M}\longrightarrow 0,$$

which naturally lives in degrees $-n,\dots,0$. The formula for the differential is

$$\delta(m\otimes\partial_J) = \sum_{i=1}^{p}(-1)^{i-1}(m\partial_{j_i})\otimes\partial_{J\setminus\{j_i\}},$$

where, given an ordered index set $J=\{j_1,\dots,j_p\}$ with $j_1<\cdots<j_p$, we set

$$\partial_J = \partial_{j_1}\wedge\cdots\wedge\partial_{j_p}\quad\text{and}\quad \mathrm{d}x_J = \mathrm{d}x_{j_1}\wedge\cdots\wedge\mathrm{d}x_{j_p}$$

Here and elsewhere, we always stick to the Koszul sign rule: on the i-th term in the sum, we need to commute ∂_{j_i} past $(i-1)$ other vector fields, hence the sign of $(-1)^{i-1}$. We are going to write out all the formulas involving signs in what follows, to be sure that everything works out correctly.

Now for the definition of the naive pairing. The tensor product

$$\mathrm{Sp}_X(\mathcal{M})\otimes_{\mathbb{C}}\overline{\mathrm{Sp}_X(\mathcal{M})}$$

is naturally a double complex, with term in bidegree $(-p, -q)$ given by

$$\Big(\mathcal{M} \otimes_{\mathcal{O}_X} \wedge^p \Theta_X\Big) \otimes_{\mathbb{C}} \Big(\overline{\mathcal{M}} \otimes_{\mathcal{O}_{\overline{X}}} \wedge^q \Theta_{\overline{X}}\Big) \cong \Big(\mathcal{M} \otimes_{\mathbb{C}} \overline{\mathcal{M}}\Big) \otimes_{\mathcal{O}_{X,\overline{X}}} \Big(\wedge^p \Theta_X \otimes_{\mathbb{C}} \wedge^q \Theta_{\overline{X}}\Big).$$

Here $\mathcal{O}_{X,\overline{X}}$ is a convenient shorthand for the sheaf of algebras $\mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{O}_{\overline{X}}$. The associated simple complex, with Deligne's sign rule for the differential, lives in degrees $-2n, \dots, 0$, and its term in degree $-k$ is

$$\big(\mathcal{M} \otimes_{\mathbb{C}} \overline{\mathcal{M}}\big) \otimes_{\mathcal{O}_{X,\overline{X}}} \wedge^k \Theta_{X,\overline{X}}.$$

To simplify the notation, we have introduced the additional sheaf

$$\Theta_{X,\overline{X}} = \big(\Theta_X \otimes_{\mathbb{C}} \mathcal{O}_{\overline{X}}\big) \oplus \big(\mathcal{O}_X \otimes_{\mathbb{C}} \Theta_{\overline{X}}\big),$$

which is locally free of rank $2n$ over $\mathcal{O}_{X,\overline{X}}$; in the formula above, the wedge product is over $\mathcal{O}_{X,\overline{X}}$. We denote the associated simple complex by the symbol

$$\mathrm{Sp}_{X,\overline{X}}\big(\mathcal{M} \otimes_{\mathbb{C}} \overline{\mathcal{M}}\big),$$

because the formula for the differential is exactly the same as in the usual Spencer complex, but where $\mathcal{M} \otimes_{\mathbb{C}} \overline{\mathcal{M}}$ is now considered as a right module over $\mathcal{D}_{X,\overline{X}} = \mathcal{D}_X \otimes_{\mathbb{C}} \mathcal{D}_{\overline{X}}$, and where Θ_X is replaced by $\Theta_{X,\overline{X}}$.

A.5.3. Example. Indeed, say we have a local section

$$m' \otimes m'' \otimes \partial_J \otimes \overline{\partial}_K,$$

with $|J| = p$ and $|K| = q$; it lives in bidegree $(-p, -q)$ in the double complex, and in degree $-(p+q)$ in the associated simple complex. Under Deligne's sign conventions, the differential of the simple complex takes this element to

$$\sum_{i=1}^{p} (-1)^{i-1} (m'\partial_{j_i}) \otimes m'' \otimes \partial_{J\setminus\{j_i\}} \otimes \overline{\partial}_K \\ + (-1)^p \sum_{i=1}^{q} (-1)^{i-1} m' \otimes (m''\overline{\partial}_{k_i}) \otimes \partial_J \otimes \overline{\partial}_{K\setminus\{k_i\}}.$$

But this is exactly the image of $m' \otimes m'' \otimes \partial_J \wedge \overline{\partial}_K$ under the differential of the Spencer complex, and so the notation $\mathrm{Sp}_{X,\overline{X}}(\mathcal{M} \otimes_{\mathbb{C}} \overline{\mathcal{M}})$ is justified.

Since our Hermitian pairing

$$\mathrm{S} \colon \mathcal{M} \otimes_{\mathbb{C}} \overline{\mathcal{M}} \longrightarrow \mathfrak{C}_X$$

is a morphism of right $\mathcal{D}_{X,\overline{X}}$-modules, it induces a morphism of complexes

$$\mathrm{Sp}_{X,\overline{X}}\big(\mathcal{M} \otimes_{\mathbb{C}} \overline{\mathcal{M}}\big) \longrightarrow \mathrm{Sp}_{X,\overline{X}}(\mathfrak{C}_X),$$

simply by applying S termwise. The net result is that we have a morphism

$$\mathrm{Sp}_X(\mathcal{M}) \otimes_{\mathbb{C}} \overline{\mathrm{Sp}_X(\mathcal{M})} \longrightarrow \mathrm{Sp}_{X,\overline{X}}(\mathfrak{C}_X). \tag{A.5.4}$$

The Poincaré lemma for distributions implies that the complex $\mathrm{Sp}_{X,\overline{X}}(\mathfrak{C}_X)$ is a fine resolution of the constant sheaf $\mathbb{C}[2n]$. So the morphism in (A.5.4) induces, without any further work, sesquilinear pairings

$$H^k\big(X, \mathrm{Sp}_X(\mathcal{M})\big) \otimes_{\mathbb{C}} \overline{H^{-k}\big(X, \mathrm{Sp}_X(\mathcal{M})\big)} \longrightarrow H^{2n}(X, \mathbb{C}) \cong \mathbb{C}$$

on the level of cohomology. In fact, one can be more precise about the identification between $H^{2n}(X, \mathbb{C})$ and $\mathbb{C}$: the isomorphism

$$H^0\big(X, \mathrm{Sp}_{X,\overline{X}}(\mathfrak{C}_X)\big) \xrightarrow{\cong} \mathbb{C}$$

is given by evaluating currents on the constant test function 1.

Now we can formulate the answer to the puzzle in a more precise way: the Hermitian pairing S on the right $\mathcal{D}_X$-module ω_X induces naive pairings between the cohomology spaces

$$H_k \cong H^k\big(X, \mathrm{Sp}_X(\omega_X)\big),$$

and the claim is that this procedure explains the mysterious factor $(-1)^{nk}$ in (A.5.2). To understand why, we need to work through the conversion between the Spencer complex $\mathrm{Sp}_X(\omega_X)$ and the (shifted) de Rham complex $\mathrm{DR}(\mathcal{O}_X)[n]$. That is to say, we need to a formula for the pairing on the de Rham complex, induced by the naive pairing

$$\mathrm{Sp}_X(\omega_X) \otimes_{\mathbb{C}} \overline{\mathrm{Sp}_X(\omega_X)} \longrightarrow \mathrm{Sp}_{X,\overline{X}}(\mathfrak{C}_X)$$

under the isomorphism between the de Rham complex and the Spencer complex. As before, it is important to use the Koszul sign rule consistently.

To keep the notation simple, let us suppose more generally that $\mathcal{N}$ is any left $\mathcal{D}_X$-module. Its de Rham complex is the complex

$$0 \longrightarrow \mathcal{N} \xrightarrow{\nabla} \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{N} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Omega^n_X \otimes_{\mathcal{O}_X} \mathcal{N} \longrightarrow 0,$$

which naturally lives in degrees $0, \ldots, n$. We shall insist on using the notation $\mathrm{DR}_X(\mathcal{N})[n]$ for the shifted de Rham complex, as a reminder that the differential in this complex is $(-1)^n\nabla$. Concretely, the formula for the differential is

$$\mathrm{d}x_J \otimes m \longmapsto (-1)^n(-1)^{|J|}\sum_{j=1}^n \mathrm{d}x_J \wedge \mathrm{d}x_j \otimes (\partial_j m),$$

where the $(-1)^{|J|}$ comes from the fact that we had to move the differential in the complex (which has degree 1) past the form $\mathrm{d}x_J$.

A.5.5. Lemma. *The shifted de Rham complex of $\mathcal{N}$ is isomorphic to the Spencer complex of the associated right $\mathcal{D}_X$-module $\omega_X \otimes_{\mathcal{O}_X} \mathcal{N}$.*

Proof. Since it matters in what follows, let us carefully write down the exact formulas for the morphism of complexes

$$\mathrm{Sp}_X(\omega_X \otimes_{\mathcal{O}_X} \mathcal{N}) \longrightarrow \mathrm{DR}_X(\mathcal{N})[n].$$

They are determined by the condition that, in degree zero, we want the morphism $\omega_X \otimes_{\mathcal{O}_X} \mathcal{N} \to \Omega_X^n \otimes_{\mathcal{O}_X} \mathcal{N}$ to be the identity. This forces us to define

$$(\omega_X \otimes_{\mathcal{O}_X} \mathcal{N}) \otimes_{\mathcal{O}_X} \wedge^p \Theta_X \longrightarrow \Omega_X^{n-p} \otimes_{\mathcal{O}_X} \mathcal{N}$$

by the following rule:

$$\omega \otimes m \otimes \partial_J \longmapsto (-1)^{(n-j_1)+\cdots+(n-j_p)} \mathrm{d}x_{J^c} \otimes m \tag{A.5.6}$$

Here $J = \{j_1, \ldots, j_p\}$ is an ordered index set, $J^c = \{1, \ldots, n\} \setminus J$ is the complement, with the natural ordering, and $\omega = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$. The sign factor is explained by the fact that we have to move ∂_{j_1} past $\mathrm{d}x_{j_1+1}, \ldots, \mathrm{d}x_n$, before we can contract it against $\mathrm{d}x_{j_1}$, causing a factor of $(-1)^{n-j_1}$ to appear, and so on.

Let us verify that (A.5.6) really defines a morphism of complexes: each square

$$\begin{tikzcd}
(\omega_X \otimes_{\mathcal{O}_X} \mathcal{N}) \otimes_{\mathcal{O}_X} \wedge^p \Theta_X \arrow[r] \arrow[d, "\delta"'] & \Omega_X^{n-p} \otimes_{\mathcal{O}_X} \mathcal{N} \arrow[d, "(-1)^n \nabla"] \\
(\omega_X \otimes_{\mathcal{O}_X} \mathcal{N}) \otimes_{\mathcal{O}_X} \wedge^{p-1} \Theta_X \arrow[r] & \Omega_X^{n-p+1} \otimes_{\mathcal{O}_X} \mathcal{N}
\end{tikzcd}$$

commutes. Starting from $\omega \otimes m \otimes \partial_J$ with $|J| = p$, and going along the arrows on the top and right, we obtain

$$\begin{aligned}
&(-1)^{(n-j_1)+\cdots+(n-j_p)}(-1)^n(-1)^{|J^c|}\sum_{j=1}^n \mathrm{d}x_{J^c} \wedge \mathrm{d}x_j \otimes (\partial_j m) \\
&\qquad = (-1)^{(n+1)p}(-1)^{j_1+\cdots+j_p}\sum_{i=1}^p \mathrm{d}x_{J^c} \wedge \mathrm{d}x_{j_i} \otimes (\partial_{j_i} m).
\end{aligned}\tag{A.5.7}$$

Going along the arrow on the left, we obtain

$$\sum_{i=1}^p (-1)^{i-1}(\omega \otimes m)\partial_{j_i} \otimes \partial_{J\setminus\{j_i\}} = \sum_{i=1}^p (-1)^i \omega \otimes (\partial_{j_i} m) \otimes \partial_{J\setminus\{j_i\}},$$

and the arrow on the bottom turns this into

$$\begin{aligned}
&(-1)^{(n-j_1)+\cdots+(n-j_p)}\sum_{i=1}^p (-1)^i(-1)^{n-j_i}\mathrm{d}x_{(J\setminus\{j_i\})^c} \otimes (\partial_{j_i} m) \\
&= (-1)^{np}(-1)^{j_1+\cdots+j_p}\sum_{i=1}^p (-1)^p \mathrm{d}x_{J^c} \wedge \mathrm{d}x_{j_i} \otimes (\partial_{j_i} m).
\end{aligned}\tag{A.5.8}$$

The point is that $\mathrm{d}x_{(J\setminus\{j_i\})^c} = (-1)^{(p-i)+(n-j_i)}\mathrm{d}x_{J^c} \wedge \mathrm{d}x_{j_i}$, because putting the expression $\mathrm{d}x_{J^c} \wedge \mathrm{d}x_{j_i}$ into the correct order requires moving $\mathrm{d}x_{j_i}$ past a form of degree $(n - j_i) - (p - i)$. In any case, the two expressions in (A.5.7) and (A.5.8) are equal, and so we do have a morphism of complexes. $\square$

For the same reason, we have an isomorphism of complexes

$$\mathrm{Sp}_{X,\overline{X}}(\mathfrak{C}_X) \longrightarrow \mathrm{DR}_{X,\overline{X}}(\mathfrak{Db}_X)[2n],$$

where $\mathfrak{Db}_X$ is the sheaf of distributions on X, considered as a left module over $\mathcal{D}_{X,\overline{X}}$, and where the (shifted) de Rham complex is defined in the same way as for $\mathcal{D}_X$-modules, but using the wedge powers of the locally free $\mathcal{O}_{X,\overline{X}}$-module

$$\Omega^1_{X,\overline{X}} = \big(\Omega^1_X \otimes_{\mathbb{C}} \mathcal{O}_{\overline{X}}\big) \oplus \big(\mathcal{O}_X \otimes_{\mathbb{C}} \Omega^1_{\overline{X}}\big),$$

and the differential $(-1)^{2n}\nabla$. Concretely, the morphism of complexes is defined on the terms in degree $-k$, which are

$$\mathfrak{C}_X \otimes_{\mathcal{O}_{X,\overline{X}}} \wedge^k \Theta_{X,\overline{X}} \longrightarrow \Omega^{2n-k}_{X,\overline{X}} \otimes_{\mathcal{O}_{X,\overline{X}}} \mathfrak{Db}_X,$$

by the following formula (dictated by the Koszul sign rule): write a given current locally as $D\omega \wedge \overline{\omega}$, for a unique distribution D; then

$$\begin{aligned}(\text{A.5.9}) \qquad \big(D\omega \wedge \overline{\omega}\big) \otimes \partial_J \wedge \overline{\partial}_K \longmapsto \\ (-1)^{(j_1+\cdots+j_p)+(k_1+\cdots+k_q)}(-1)^{nq} \mathrm{d}x_{J^c} \wedge \mathrm{d}\overline{x}_{K^c} \otimes D\end{aligned}$$

where $|J| = p$ and $|K| = q$, and $p+q = k$. The sign factor is again explained by the number of swaps that are needed to move everything into the right place, which is $(2n-j_1)+\cdots+(2n-j_p)+(n-k_1)+\cdots+(n-k_q)$.

We can now derive a formula for the induced pairing

$$(\text{A.5.10}) \qquad \mathrm{DR}_X(\mathcal{O}_X)[n] \otimes_{\mathbb{C}} \overline{\mathrm{DR}_X(\mathcal{O}_X)[n]} \longrightarrow \mathrm{DR}_{X,\overline{X}}(\mathfrak{Db}_X).$$

Take two local sections $\alpha = \mathrm{d}x_{J^c}$ and $\beta = \mathrm{d}x_{K^c}$, where $|J| = p$ and $|K| = q$. Under the isomorphism $\mathrm{DR}_X(\mathcal{O}_X)[n] \cong \mathrm{Sp}_X(\omega_X)$ in Lemma A.5.5, the holomorphic $(n-p)$-form α goes to

$$(-1)^{np}(-1)^{j_1+\cdots+j_p} \cdot \omega \otimes \partial_J,$$

and the holomorphic $(n-q)$-form β goes to

$$(-1)^{nq}(-1)^{k_1+\cdots+k_q} \cdot \omega \otimes \partial_K.$$

The naive pairing on $\mathrm{Sp}_X(\omega_X)$ takes those two sections to

$$(-1)^{n(p+q)}(-1)^{(j_1+\cdots+j_p)+(k_1+\cdots+k_q)} S(\omega,\omega) \otimes \partial_J \wedge \overline{\partial}_K,$$

where S is defined in (A.5.1). Now $\mathrm{S}(\omega,\omega) = D\omega \wedge \overline{\omega}$, where D is the distribution

$$D = \frac{\varepsilon(n+1)}{(2\pi \mathsf{i})^n} \int_X \in H^0(X, \mathfrak{Db}_X).$$

Under the isomorphism in (A.5.9), the section from above therefore goes to

$$(-1)^{np} \mathrm{d}x_{J^c} \wedge \mathrm{d}\overline{x}_{K^c} \otimes D = (-1)^{n(\deg \alpha - n)} \alpha \wedge \overline{\beta} \otimes D.$$

The formula we have just derived also works for smooth forms. In other words, the same formula can be used to extend (A.5.10) to a pairing on the de Rham complex of smooth forms (which is the usual Dolbeault resolution used to compute cohomology). The resulting pairings on cohomology

$$H^{n+k}(X,\mathbb{C}) \otimes_{\mathbb{C}} \overline{H^{n-k}(X,\mathbb{C})} \longrightarrow \mathbb{C}$$

are of course given by the same formula

$$(\alpha, \beta) \longmapsto (-1)^{n(\deg \alpha - n)} \frac{\varepsilon(n+1)}{(2\pi \mathrm{i})^n} \int_X \alpha \wedge \overline{\beta}.$$

Since $\deg \alpha = n+k$, we have succeeded in explaining the mysterious sign factor $(-1)^{nk}$ in (A.5.2), in a very natural way!

Let us summarize the result of this rather lengthy computation. If we define the Hermitian pairing S_X on the right $\mathcal{D}_X$-module ω_X as in (A.5.1), and if we use the naive pairing on the Spencer complex, we obtain a collection of pairings

$$\mathrm{S}_k \colon H^k\big(X, \mathrm{Sp}_X(\omega_X)\big) \otimes_{\mathbb{C}} \overline{H^{-k}\big(X, \mathrm{Sp}_X(\omega_X)\big)} \longrightarrow \mathbb{C},$$

with all signs dictated by the Koszul sign rule alone. The conclusion is then that these pairings polarize the $\mathfrak{sl}_2$-Hodge structure of weight $n = \dim X$ on the graded vector space

$$\bigoplus_{k \in \mathbb{Z}} H^k\big(X, \mathrm{Sp}_X(\omega_X)\big),$$

provided that we multiply the k-th pairing S_k by the factor $\varepsilon(k)$. This is good news, because it describes the $\mathfrak{sl}_2$-Hodge structure and its polarization in a way that does not mention the dimension of the compact Kähler manifold X, a crucial point if we want a theory that is independent of the choice of ambient complex manifold.

A.6. Direct images

It is now an easy matter to figure out the sign conventions for direct images. Since every morphism between complex manifolds factors into a closed embedding followed by a projection, we only need to consider those two cases.

The first case is that of a closed embedding $i \colon X \hookrightarrow Y$. Suppose that $\mathcal{M}$ is a coherent right $\mathcal{D}_X$-module, and $\mathrm{S} \colon \mathcal{M} \otimes_{\mathbb{C}} \overline{\mathcal{M}} \to \mathfrak{C}_X$ a Hermitian pairing. Let

$$\mathcal{D}_{X \to Y} = \mathcal{O}_X \otimes_{i^{-1}\mathscr{O}_Y} i^{-1}\mathcal{D}_Y$$

be the transfer module, which is a $(\mathcal{D}_X, i^{-1}\mathcal{D}_Y)$-bimodule. The direct image

$$i_+\mathcal{M} = i_*\big(\mathcal{M} \otimes_{\mathcal{D}_X} \mathcal{D}_{X \to Y}\big)$$

is a coherent right $\mathcal{D}_Y$-module. There is an induced Hermitian pairing

$$i_+\mathrm{S} \colon i_+\mathcal{M} \otimes_{\mathbb{C}} \overline{i_+\mathcal{M}} \longrightarrow \mathfrak{C}_Y,$$

that can be described in a coordinate-free way as follows. Since the tensor product over $\mathbb{C}$ is exact, we have a natural isomorphism

$$\big(\mathcal{M} \otimes_{\mathcal{D}_X} \mathcal{D}_{X \to Y}\big) \otimes_{\mathbb{C}} \overline{\big(\mathcal{M} \otimes_{\mathcal{D}_X} \mathcal{D}_{X \to Y}\big)} \cong \big(\mathcal{M} \otimes_{\mathbb{C}} \overline{\mathcal{M}}\big) \otimes_{\mathcal{D}_{X,\overline{X}}} \mathcal{D}_{X \to Y, \overline{X} \to \overline{Y}},$$

where $\mathcal{D}_{X,\overline{X}} = \mathcal{D}_X \otimes_{\mathbb{C}} \mathcal{D}_{\overline{X}}$ and $\mathcal{D}_{X \to Y, \overline{X} \to \overline{Y}} = \mathcal{D}_{X \to Y} \otimes_{\mathbb{C}} \mathcal{D}_{\overline{X} \to \overline{Y}}$. Applying the sheaf theoretic direct image i_*, and composing with $\mathrm{S} \colon \mathcal{M} \otimes_{\mathbb{C}} \overline{\mathcal{M}} \to \mathfrak{C}_X$, we get

$$i_+\mathcal{M} \otimes_{\mathbb{C}} \overline{i_+\mathcal{M}} \longrightarrow i_*\big(\mathfrak{C}_X \otimes_{\mathcal{D}_{X,\overline{X}}} \mathcal{D}_{X \to Y, \overline{X} \to \overline{Y}}\big).$$

Pushforward of currents defines a morphism $i_* \mathfrak{C}_X \to \mathfrak{C}_Y$, according to the rule

$$\langle i_* C, \varphi \rangle = \langle C, i^* \varphi \rangle.$$

From this, we obtain another natural morphism

$$i_*\big(\mathfrak{C}_X \otimes_{\mathcal{D}_{X,\overline{X}}} \mathcal{D}_{X\to Y,\overline{X}\to\overline{Y}}\big) \longrightarrow \mathfrak{C}_Y, \quad C \otimes (f \otimes P) \otimes (\bar{g} \otimes \bar{Q}) \longmapsto i_*(Cf\bar{g}) \cdot P\bar{Q}.$$

After composing the two morphisms, we arrive at the desired Hermitian pairing

$$i_+\mathrm{S}\colon i_+\mathcal{M} \otimes_{\mathbb{C}} \overline{i_+\mathcal{M}} \longrightarrow \mathfrak{C}_Y\,.$$

All of the currents in the image are supported on X; for two sections in the subsheaf $i_*\mathcal{M}$, the current is just obtained by pushforward from X to Y, but in general, the construction involves some derivatives in directions normal to Y.

The second case is that of a projection $f\colon X \to Y$, say with $X = F \times Y$ and $f = p_2$. Let $\mathcal{M}$ be a coherent right $\mathcal{D}_X$-module, and $\mathrm{S}\colon \mathcal{M} \otimes_{\mathbb{C}} \overline{\mathcal{M}} \to \mathfrak{C}_X$ a Hermitian pairing. The direct image

$$f_+\mathcal{M} = \mathbf{R}f_*\big(\mathcal{M} \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}\big)$$

is computed using the relative Spencer complex $\mathrm{Sp}_f(\mathcal{M})$. This is the complex

$$0 \longrightarrow \mathcal{M} \otimes_{p_1^{-1}\mathcal{O}_F} \wedge^r p_1^{-1}\Theta_F \xrightarrow{\ \delta\ } \cdots \xrightarrow{\ \delta\ } \mathcal{M} \otimes_{p_1^{-1}\mathcal{O}_F} p_1^{-1}\Theta_F \xrightarrow{\ \delta\ } \mathcal{M} \longrightarrow 0,$$

which naturally lives in degrees $-r, \dots, 0$, where $r = \dim F$; the formula for the differential is the same as in the absolute case. By a similar construction as in the previous section, we obtain a naive pairing on the complex $\mathrm{Sp}_f(\mathcal{M})$, which we may write by analogy with the absolute case as

$$\mathrm{Sp}_f(\mathcal{M}) \otimes_{\mathbb{C}} \overline{\mathrm{Sp}_f(\mathcal{M})} \longrightarrow \mathrm{Sp}_{f,\overline{f}}(\mathfrak{C}_X).$$

Here the complex on the right-hand side lives in degrees $-2r, \dots 0$, and with similar notation as in the previous section, the term in degree $-k$ looks like

$$\mathfrak{C}_X \otimes_{p_1^{-1}\mathcal{O}_{F,\overline{F}}} \wedge^k p_1^{-1}\Theta_{F,\overline{F}}.$$

By a relative version of the Poincaré lemma for distributions, this complex is a fine resolution of $f^{-1}\mathfrak{C}_Y[2r]$, and and so we obtain induced sesquilinear pairings

$$\mathrm{S}_k\colon R^k f_* \mathrm{Sp}_f(\mathcal{M}) \otimes_{\mathbb{C}} \overline{R^{-k} f_* \mathrm{Sp}_f(\mathcal{M})} \longrightarrow \mathfrak{C}_Y\,.$$

The isomorphism $R^{2r} f^{-1}\mathfrak{C}_Y \cong \mathfrak{C}_Y$ is given, in terms of the explicit fine resolution from above, simply by pushforward of currents.

Now for the general case. Suppose that $f\colon X \to Y$ is a holomorphic mapping between two complex manifolds. Let $\mathcal{M}$ be a coherent right $\mathcal{D}_X$-module, and $\mathrm{S}\colon \mathcal{M} \otimes_{\mathbb{C}} \overline{\mathcal{M}} \to \mathfrak{C}_X$ a Hermitian pairing. Suppose that f is proper, or at least proper on the support of $\mathcal{M}$. By factoring f as

$$X \xrightarrow{\ i\ } X \times Y \xrightarrow{\ p_2\ } Y$$

and applying the two constructions from above, we obtain a collection of induced sesquilinear pairings

$$\mathrm{S}_k\colon \mathcal{H}^k f_+\mathcal{M} \otimes_{\mathbb{C}} \overline{\mathcal{H}^{-k} f_+\mathcal{M}} \longrightarrow \mathfrak{C}_Y$$

The sign conventions are then easy to state: for each $k \in \mathbb{Z}$, we should multiply the naive pairing S_k by the factor $\varepsilon(k) = (-1)^{k(k-1)/2}$. In the special case of a closed embedding, this means that we simply use the pairing $i_+\mathrm{S}$ induced by pushforward of currents (because $\varepsilon(0) = 1$). This convention is suggested by the analysis in the previous section. The direct image theorem for polarized Hodge modules then takes the following form:

A.6.1. Theorem. *Let $f\colon X \to Y$ be a projective morphism between two complex manifolds. Let $M \in \mathsf{HM}(X, w)$ be a polarized Hodge module of weight w. Then*

$$\bigoplus_{k\in\mathbb{Z}} H^k f_* M$$

is a polarized $\mathfrak{sl}_2$-Hodge module of weight w; here $\mathrm{X} \in \mathfrak{sl}_2(\mathbb{C})$ acts as $2\pi\mathsf{i}\, L_\omega$, and the polarization is given by the sesquilinear pairings $\varepsilon(k)\mathrm{S}_k$ from above.

A.7. Variations of Hodge structure and polarizations

Recall that a *variation of Hodge structure* of weight n on a complex manifold X is a smooth vector bundle E with a decomposition into smooth subbundles

$$E = \bigoplus_{p+q=n} E^{p,q},$$

and a flat connection $d\colon A^0(X, E) \to A^1(X, E)$ that maps the space of sections $A^0(X, E^{p,q})$ of the subbundle $E^{p,q}$ into the direct sum

$$A^{1,0}(X, E^{p,q}) \oplus A^{1,0}(X, E^{p-1,q+1}) \oplus A^{0,1}(X, E^{p,q}) \oplus A^{0,1}(X, E^{p+1,q-1}).$$

Note that we are describing the connection in terms of its action on the space of smooth sections of E; equivalently, we could consider d as a morphism from the sheaf of smooth sections of E to the sheaf of smooth 1-forms with coefficients in E. Lastly, a *polarization* of E is a Hermitian form

$$\mathrm{S}_E\colon A^0(X, E) \otimes_{A^0(X)} \overline{A^0(X, E)} \longrightarrow A^0(X)$$

that is flat with respect to d, and whose restriction to each fiber E_x polarizes the Hodge structure of weight n on the vector space

$$E_x = \bigoplus_{p+q=n} E_x^{p,q}.$$

From the polarization, we obtain a smooth Hermitian metric h_E on the bundle E, called the *Hodge metric*, by setting

$$h_E(v, w) = c_n \sum_{p+q=n} (-1)^q \mathrm{S}_E(v^{p,q}, w^{p,q}).$$

Most of the results about variations of Hodge structure are, directly or indirectly, statements about the Hodge metric.

If we decompose the connection by type as $d = d' + d''$, then the $(0,1)$-part d'' gives E the structure of a holomorphic vector bundle that we denote by the symbol $\mathcal{E}$, and the $(1,0)$-part d' defines a flat holomorphic connection

$$\nabla \colon \mathcal{E} \longrightarrow \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{E}.$$

The condition above says that the Hodge bundles

$$F^p E = E^{p,q} \oplus E^{p+1,q-1} \oplus E^{p+2,q-2} \oplus \cdots$$

have the structure of holomorphic subbundles $F^p\mathcal{E} \subseteq \mathcal{E}$, and that the holomorphic connection ∇ satisfies the Griffiths transversality condition

$$\nabla(F^p\mathcal{E}) \subseteq \Omega_X^1 \otimes_{\mathcal{O}_X} F^{p-1}\mathcal{E}. \tag{A.7.1}$$

The process for converting a polarized variation of Hodge structure of weight n into a polarized Hodge module of weight $n + \dim X$ is as follows. First, consider the associated right $\mathcal{D}_X$-module

$$\mathcal{M} = \omega_X \otimes_{\mathcal{O}_X} \mathcal{E},$$

with the action by vector fields defined in terms of the connection as

$$(\omega \otimes s) \cdot \xi = (\omega \cdot \xi) \otimes s - \omega \otimes \nabla_\xi s.$$

The Hodge filtration on $\mathcal{E}$ defines an increasing filtration

$$F_\bullet\mathcal{M} = \omega_X \otimes_{\mathcal{O}_X} F^{-\bullet-\dim X}\mathcal{E},$$

which is compatible with the $\mathcal{D}_X$-module structure because of (A.7.1).

Note. The shift by $\dim X$ is necessary in order to make the isomorphism in Lemma A.5.5 between the Spencer complex $\mathrm{Sp}_X(\mathcal{M})$ and the shifted de Rham complex $\mathrm{DR}_X(\mathcal{E})$ into a *filtered* isomorphism.

Finally, we should define the Hermitian pairing

$$\mathrm{S}_M \colon \mathcal{M} \otimes_{\mathbb{C}} \overline{\mathcal{M}} \longrightarrow \mathfrak{C}_X$$

by the following formula, suggested by (A.5.1):

$$\big\langle \mathrm{S}_M(\omega' \otimes s', \omega'' \otimes s''), \varphi \big\rangle = \frac{\varepsilon(n+1)}{(2\pi \mathrm{i})^n} \int_X \varphi \cdot \mathrm{S}_E(s', s'')\, \omega' \wedge \overline{\omega''} \tag{A.7.2}$$

Assuming that X is compact, and that p is the largest integer such that $F^p\mathcal{E} \neq 0$, the induced pairing on the space $H^0(X, \omega_X \otimes F^p\mathcal{E})$ is then $c_n(-1)^{n-p}$-positive definite with this definition.

A.8. Degenerating variations of Hodge structure

In this section, we are going to check our sign conventions against another real-world example: polarized variations of Hodge structure on the punctured disk. This is another instance where $\mathfrak{sl}_2$-Hodge structures appear, and we will see that the sign conventions we have developed so far also work nicely in this case.

Using the notation from the previous section, let us consider a polarized variation of Hodge structure E of weight n on the punctured unit disk

$$\Delta^* = \{t \in \mathbb{C} \mid 0 < |t| < 1\}.$$

In order to have a fixed reference frame, we introduce the complex vector space V of all multivalued flat section of (E, d); equivalently, these are the flat sections of the pullback $\exp^* E$ to the universal covering space $\exp\colon \mathbb{H} \to \Delta^*$. Note that the universal covering space of Δ^* is naturally the *left* half plane

$$\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Re} z < 0\},$$

and the group of deck transformations is

$$\mathbb{Z}(1) = \{z \in \mathbb{C} \mid e^z = 1\} = (2\pi \mathsf{i})\mathbb{Z} \subseteq \mathbb{C}.$$

Translation by elements of $\mathbb{Z}(1)$ defines a group homomorphism

$$\rho\colon \mathbb{Z}(1) \longrightarrow \mathrm{GL}(V);$$

to be specific, Schmid's convention is that $\rho(\zeta)$ takes a flat section $v(z)$ of the bundle $\exp^* E$ to the flat section $v(z - \zeta)$. In particular, we have the (positively oriented) monodromy transformation

$$T = \rho(2\pi \mathsf{i}) \in \mathrm{GL}(V),$$

which depends on the choice of $i = \sqrt{-1}$. If we write its Jordan decomposition in the form

$$T = T_s \cdot e^{2\pi \mathsf{i}\, N},$$

with $T_s \in \mathrm{GL}(V)$ semisimple, $N \in \mathrm{End}(V)$ nilpotent, and $[T_s, N] = 0$, then N is independent of the choice of i. The polarization induces a Hermitian pairing

$$\mathrm{S}\colon V \otimes_{\mathbb{C}} \overline{V} \longrightarrow \mathbb{C},$$

and since T preserves the pairing, one easily checks that

$$\mathrm{S} \circ (T_s \otimes T_s) = \mathrm{S} \quad \text{and} \quad \mathrm{S} \circ (N \otimes \mathrm{Id}) = \mathrm{S} \circ (\mathrm{Id} \otimes N).$$

According to the *monodromy theorem*, all eigenvalues of the monodromy transformation T have absolute value 1. After fixing an interval $[\alpha, \alpha+1) \subseteq \mathbb{R}$, we can therefore write the semisimple operator T_s uniquely as

$$T_s = e^{2\pi \mathsf{i}\, S_\alpha},$$

where $S_\alpha \in \mathrm{End}(V)$ is semisimple with real eigenvalues contained in $[\alpha, \alpha + 1)$. We then have $T = e^{2\pi \mathsf{i}(S_\alpha + N)}$, and the operator $S_\alpha + N$ in the exponent does not depend on the choice of i.

A.8.1. Example. The definition of the monodromy operator appears unmotivated – why not use $v(z+\zeta)$ instead? – but the operator $S_\alpha+N$ does have a natural interpretation in terms of the connection. Let $\widetilde{\mathcal{E}}^\alpha$ be the canonical extension of $(\mathcal{E},\nabla)$, characterized by the property that ∇ extends to a logarithmic connection

$$\nabla\colon \widetilde{\mathcal{E}}^\alpha \longrightarrow \Omega^1_\Delta(\log 0)\otimes_{\mathcal{O}} \widetilde{\mathcal{E}}^\alpha$$

whose residue at the origin

$$R_\alpha = \mathrm{Res}_{t=0}(\nabla)\in \mathrm{End}\big(\widetilde{\mathcal{E}}^\alpha_{|0}\big)$$

has eigenvalues in the interval $[\alpha,\alpha+1)$. There is a distinguished trivialization

$$\mathcal{O}_\Delta\otimes_{\mathbb{C}} \widetilde{\mathcal{E}}^\alpha_{|0}\cong \widetilde{\mathcal{E}}^\alpha,$$

depending only on the choice of coordinate t on the disk, with the property that

$$\nabla(1\otimes v) = \frac{\mathrm{d}t}{t}\otimes R_\alpha v, \quad \text{for } v\in \widetilde{\mathcal{E}}^\alpha_{|0}.$$

After pulling everything back to the universal covering space $\mathbb{H}$, we obtain

$$\nabla(1\otimes v) = dz\otimes R_\alpha v,$$

where ∇ denotes the induced flat holomorphic connection on the pullback of $\mathcal{E}$. A brief computation shows that the expression

$$\sigma_v(z) = e^{-zR_\alpha}(1\otimes v) = \sum_{j=0}^{\infty}\frac{(-1)^j}{j!}z^j\otimes R_\alpha^j v$$

defines a global section of $\exp^*\mathcal{E}$ that is annihilated by ∇. This sets up an isomorphism between $\widetilde{\mathcal{E}}^\alpha_{|0}$ and the space of multivalued flat sections of $(\mathcal{E},\nabla)$, and so we can describe the canonical extension as

$$\widetilde{\mathcal{E}}^\alpha\cong \mathcal{O}_\Delta\otimes_{\mathbb{C}} V.$$

With this identification, the monodromy transformation is $T=e^{2\pi \mathsf{i}\, R_\alpha}$, because

$$\sigma_v(z-2\pi\mathsf{i}) = e^{2\pi\mathsf{i}\, R_\alpha}\sigma_v(z).$$

It follows that the operator $S_\alpha+N=R_\alpha$ is exactly the residue of the logarithmic connection on $\widetilde{\mathcal{E}}^\alpha$.

The main result is that the vector space V has an $\mathfrak{sl}_2$-Hodge structure of weight n, polarized by the Hermitian pairing S. This is not entirely canonical, though, because the representation of $\mathfrak{sl}_2(\mathbb{C})$ depends on the choice of a splitting for the weight filtration. First, recall that the nilpotent operator $N\in\mathrm{End}(V)$ determines the *monodromy weight filtration* $W_\bullet$, which is the unique increasing filtration with $NW_\ell\subseteq W_{\ell-2}$ for all $\ell\in\mathbb{Z}$, such that

$$N^\ell\colon \mathrm{gr}^W_\ell \longrightarrow \mathrm{gr}^W_{-\ell}$$

is an isomorphism for every $\ell\geqslant 0$. The weight filtration governs the asymptotic behavior of the Hodge metric, in the sense that

$$v\in W_\ell\setminus W_{\ell-1} \quad\Longleftrightarrow\quad h(v,v)\sim |\mathrm{Re}\, z|^\ell,$$

at least as long as $|\operatorname{Im} z|$ stays bounded. These asymptotic formulas for the norm of multivalued flat sections are known as the "Hodge norm estimates". Looking at these formulas, a natural idea is to rescale the Hodge metric, in order even out the different powers of $|\operatorname{Re} z|$. For that purpose, we have to choose a *splitting* for the weight filtration. By this, we mean a semisimple operator $H \in \operatorname{End}(V)$ with integer eigenvalues, such that

$$W_\ell = E_\ell(H) \oplus W_{\ell-1} \quad \text{and} \quad [H, N] = -2N.$$

In addition, we can easily arrange that $\mathrm{S} \circ (H \otimes \mathrm{Id}) + \mathrm{S} \circ (\mathrm{Id} \otimes H) = 0$ and that $[H, T_s] = 0$; note that, even with these extra conditions, the splitting H is far from unique in general. For $v \in E_\ell(H)$, we now have

$$e^{-\frac{1}{2} \log|\operatorname{Re} z| \, H} v = |\operatorname{Re} z|^{-\frac{\ell}{2}} v.$$

It turns out that rescaling by the operator $e^{-\frac{1}{2} \log|\operatorname{Re} z| \, H}$ not only removes the singular behavior of the Hodge metric, but it also makes the family of polarized Hodge structures of weight n converge to a limit.

To describe the convergence, we need to introduce two additional pieces of notation. The first is the *period domain* D. The points of D parametrize all possible Hodge structures

$$V = \bigoplus_{p+q=n} V^{p,q}$$

of weight n on the vector space V that are polarized by the Hermitian form S and have the appropriate set of Hodge numbers $\dim V^{p,q}$. The polarization being fixed, a Hodge structure is uniquely determined by its Hodge filtration

$$F^p V = V^{p,q} \oplus V^{p+1,q-1} \oplus V^{p+2,q-2} \oplus \cdots,$$

and this makes D a subset of the "compact dual" $\check{D}$, the space of all decreasing filtrations on V by subspaces of the appropriate dimensions $\dim F^p V$. The compact dual $\check{D}$ is a compact complex manifold, and a homegeneous space for the complex Lie group $\mathrm{GL}(V)$; the period domain D is an open subset, and a homogeneous space for the real Lie group

$$G = \{g \in \mathrm{GL}(V) \mid \mathrm{S} \circ (g \otimes g) = \mathrm{S}\}.$$

The polarized variation of Hodge structure E determines a *period mapping*

$$\Phi \colon \mathbb{H} \longrightarrow D,$$

where $\Phi(z)$ is the Hodge structure on V induced by the isomorphism $V \cong E_{e^z}$.

Note. For clarity, we are going to use the notation

$$V = \bigoplus_{p+q=n} V^{p,q}_{\Phi(z)}$$

for the Hodge decomposition in the Hodge structure $\Phi(z)$, and $\Phi^p(z)$ for the subspaces in the Hodge filtration. We also write

$$\langle v, w \rangle_{\Phi(z)} = c_n \sum_{p+q=n} (-1)^q \mathrm{S}(v^{p,q}, w^{p,q}) = h_E(v, w)(z)$$

for the resulting Hermitian inner product on V. The action by the real group G works in such a way that $\langle gv, gw\rangle_{g\Phi(z)} = \langle v, w\rangle_{\Phi(z)}$.

Since $T \in G$, the definition of the monodromy operator implies that

$$\Phi(z + 2\pi \mathsf{i}) = T \cdot \Phi(z),$$

This means that the expression $e^{-z(S_\alpha+N)}\Phi(z)$ is invariant under translation by $2\pi\mathsf{i}$, and so it descends to a holomorphic mapping

$$\Psi_\alpha \colon \Delta^* \longrightarrow \check{D}, \quad \Psi_\alpha(e^z) = e^{-z(S_\alpha+N)}\Phi(z).$$

The following result is known as the "nilpotent orbit theorem".

A.8.2. Theorem. *The holomorphic mapping Ψ_α extends over the origin, and the limiting value $\Psi_\alpha(0) \in \check{D}$ satisfies $N\Psi_\alpha^p(0) \subseteq \Psi_\alpha^{p-1}(0)$ for all $p \in \mathbb{Z}$.*

Note. An equivalent formulation is that the Hodge bundles $F^p\mathcal{E}$ extend to holomorphic subbundles $F^p\widetilde{\mathcal{E}}^\alpha$ of the canonical extension. Under the isomorphism

$$\widetilde{\mathcal{E}}^\alpha_{|0} \cong V$$

with the space of multivalued flat sections, the filtration $\Psi_\alpha(0)$ is then simply the filtration induced by these subbundles,

$$F^p\widetilde{\mathcal{E}}^\alpha_{|0} \cong \Psi_\alpha^p(0),$$

and the second half of the nilpotent orbit theorem is asserting that the residue R_α maps the subspace $F^p\widetilde{\mathcal{E}}^\alpha_{|0}$ into the subspace $F^{p-1}\widetilde{\mathcal{E}}^\alpha_{|0}$.

Now we are ready to discuss the convergence properties of the period mapping. As suggested above, we consider the *rescaled period mapping*

$$\hat{\Phi}_H \colon \mathbb{H} \longrightarrow D, \quad \hat{\Phi}_H(z) = e^{\frac{1}{2}\log|\operatorname{Re} z|\, H} e^{-\frac{1}{2}(z-\bar{z})(S_\alpha+N)}\Phi(z).$$

Since both exponential factors belong to the real group G, the rescaled period mapping still takes values in the period domain D. It is also invariant under translation by $2\pi\mathsf{i}$, and for any multivalued flat section $v \in V$, the expression

$$\big\|v\big\|^2_{\hat{\Phi}_H(z)} = \big\|e^{\frac{1}{2}(z-\bar{z})(S_\alpha+N)}e^{-\frac{1}{2}\log|\operatorname{Re} z|\, H} v\big\|_{\Phi(z)}$$

remains bounded as $\operatorname{Re} z \to -\infty$ (due to the Hodge norm estimates). The nice thing is that this rescaling also makes the polarized Hodge structures converge.

A.8.3. Theorem. *The rescaled period mapping $\hat{\Phi}_H$ converges to a limit*

$$e^{-N}F_H = \lim_{\operatorname{Re} z \to -\infty} \hat{\Phi}_H(z) \in D.$$

Moreover, the filtration $F_H \in \check{D}$ has the property that, for all $p \in \mathbb{Z}$,

$$NF_H^p \subseteq F_H^{p-1}, \quad HF_H^p \subseteq F_H^p, \quad \textit{and} \quad T_sF_H^p \subseteq F_H^p.$$

The filtration F_H in the statement of the theorem is obtained from the filtration $\Psi_\alpha(0)$ in the nilpotent orbit theorem in two steps. One can check that

$$\hat{\Phi}_H(z) = e^{-N} \cdot e^{\frac{1}{2}\log|\operatorname{Re} z|\, H} e^{-|\operatorname{Re} z| S_\alpha} \Psi_\alpha(e^z),$$

and since $\Psi_\alpha(e^z)$ converges to its limit $\Psi_\alpha(0)$ at a rate $|e^z| = e^{|\operatorname{Re} z|}$, this gives

$$F_H = \lim_{|\operatorname{Re} z| \to \infty} e^{\frac{1}{2}\log|\operatorname{Re} z|\, H} e^{-|\operatorname{Re} z| S_\alpha} \Psi_\alpha(0). \tag{A.8.4}$$

Let us briefly digress on the effect of the two exponential factors, since this may be helpful for understanding where the filtration F_H comes from. Suppose for a moment that $S \in \operatorname{End}(V)$ is an arbitrary semisimple endomorphism with real eigenvalues $\alpha_1 < \alpha_2 < \cdots < \alpha_r$. Then for any filtration $F \in \check{D}$, the limit

$$F_S = \lim_{x \to \infty} e^{xS} F \in \check{D}$$

exists and is compatible with S, in the sense that $SF_S^p \subseteq F_S^p$ for all $p \in \mathbb{Z}$. The effect of the limit can be understood concretely as follows. Consider the filtration by increasing eigenvalues of S, with terms

$$G_j = E_{\alpha_1}(S) \oplus \cdots \oplus E_{\alpha_j}(S).$$

The filtration F induces a filtration on each subquotient G_j/G_{j-1}, and under the obvious isomorphism $E_{\alpha_j}(S) \cong G_j/G_{j-1}$, we have

$$F_S^p \cap E_{\alpha_j}(S) \cong (F^p \cap G_j + G_{j-1})/G_{j-1}.$$

In the specific case in (A.8.4) that we care about, this means:

(1) The effect of the exponential factor $e^{-|\operatorname{Re} z| S_\alpha}$ is to produce a filtration

$$F_{\lim} = \lim_{x \to \infty} e^{-xS_\alpha} F \in \check{D} \tag{A.8.5}$$

that is compatible with the eigenspace decomposition of the semisimple operator $T_s = e^{2\pi \mathrm{i}\, S_\alpha}$. Because of the minus sign in the exponent, the relevant filtration is by *decreasing* eigenvalues of S_α.

(2) The effect of the exponential factor $e^{\frac{1}{2}\log|\operatorname{Re} z|\, H}$ is to produce a filtration

$$F_H = \lim_{x \to \infty} e^{\frac{1}{2}\log x\, H} F_{\lim} \in \check{D}$$

that is also compatible with the eigenspace decomposition of the semisimple operator H. The relevant filtration is the monodromy weight filtration $W_\bullet$, which is exactly the filtration by increasing eigenvalues of H.

The fact that $e^{-N} F_H \in D$ is a polarized Hodge structure of weight n implies, after some linear algebra, that the filtration F_H is the Hodge filtration of a polarized $\mathfrak{sl}_2$-Hodge structure of weight n. We now describe the relevant objects. Because of the relation $[H, N] = -2N$, the two operators $H, N \in \operatorname{End}(V)$ are part of a representation of $\mathfrak{sl}_2(\mathbb{C})$. With respect to the standard basis

$$\mathrm{H} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathrm{X} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathrm{Y} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

we let $\mathrm{H} \in \mathfrak{sl}_2(\mathbb{C})$ act as the semisimple operator $H \in \operatorname{End}(V)$, and we let $\mathrm{Y} \in \mathfrak{sl}_2(\mathbb{C})$ act as the nilpotent operator $-N$. By construction, the semisimple part T_s of the monodromy transformation commutes with the action by $\mathfrak{sl}_2(\mathbb{C})$.

Note. The minus sign in $\mathrm{Y} = -N$ is important; we shall justify in a minute why it has to be there and why it is the natural choice.

To match our earlier notation, let us write

$$V_\ell = E_\ell(\mathrm{H}) \cong \operatorname{gr}_\ell^W V$$

for the weight spaces of the semisimple operator H. Recall that the Hermitian form $\mathrm{S}\colon V \otimes_{\mathbb{C}} \overline{V} \to \mathbb{C}$ has the property that

$$\mathrm{S} \circ (\mathrm{H} \otimes \mathrm{Id}) = -\mathrm{S} \circ (\mathrm{Id} \otimes \mathrm{H}) \quad \text{and} \quad \mathrm{S} \circ (\mathrm{Y} \otimes \mathrm{Id}) = \mathrm{S} \circ (\mathrm{Id} \otimes \mathrm{Y}),$$

as required by (A.3.7). The main result is then the following.

A.8.6. Theorem. *With notation as above, the space of multivalued flat sections*

$$V = \bigoplus_{\ell \in \mathbb{Z}} V_\ell$$

becomes an $\mathfrak{sl}_2$-Hodge structure of weight n, polarized by the Hermitian form S. *Its Hodge filtration is the filtration F_H, in the sense that*

$$F_H^p \cap V_\ell = V_\ell^{p,\ell-p} \oplus V_\ell^{p+1,\ell-(p+1)} \oplus V_\ell^{p+2,\ell-(p+2)} \oplus \cdots$$

for all integers $p, \ell \in \mathbb{Z}$. Moreover, the operator $T_s \in \operatorname{End}(V)$ is an endomorphism of the polarized $\mathfrak{sl}_2$-Hodge structure.

A.8.7. Example. Here is a simple example that shows the sign conventions at work. Consider the standard representation of $\mathfrak{sl}_2(\mathbb{C})$ on the vector space $V = \mathbb{C}^2$, with the standard Hermitian form

$$\mathsf{S} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

If we set $F^1 = \mathbb{C}(1,0)$ and $F^0 = \mathbb{C}^2$, then $e^{\mathrm{Y}} F$ is the Hodge filtration of a polarized Hodge structure of weight 1: the Hodge decomposition is

$$V = V^{1,0} \oplus V^{0,1} = \mathbb{C}(1,1) \oplus \mathbb{C}(1,-1),$$

and S is clearly positive on the first subspace and negative on the second one (in agreement with our convention that $c_n = 1$). On the other hand, the Weil element $\mathrm{w} \in \mathrm{SL}_2(\mathbb{C})$ satisfies

$$\mathrm{w}(0,1) = (1,0) \quad \text{and} \quad \mathrm{w}(1,0) = -(0,1),$$

and so we do get a polarized $\mathfrak{sl}_2$-Hodge structure of weight 1, with

$$V_1 = V_1^{1,1} = \mathbb{C}(1,0) \quad \text{and} \quad V_{-1} = V_{-1}^{0,0} = \mathbb{C}(0,1),$$

since for example $\mathrm{S}\big((0,1), \mathrm{w}(0,1)\big) = 2$. Note that the signs do not work out properly if we use the Hodge filtration $e^{-\mathrm{Y}} F$ instead; this is one reason why it is necessary to define the $\mathfrak{sl}_2(\mathbb{C})$-representation using $\mathrm{Y} = -N$.

A.9. Hodge modules on the unit disk

Before we turn to the sign conventions for nearby and vanishing cycles, it may be useful to summarize the results of the previous section in the language of Hodge modules. The polarized variation of Hodge structure E of weight n on Δ^* determines a polarized Hodge module $M \in \mathsf{HM}(\Delta, n+1)$, with pure support Δ. Let us denote by $(\mathcal{M}, F_\bullet\mathcal{M})$ its underlying filtered $\mathcal{D}_\Delta$-module, and by $\mathrm{S}_M\colon \mathcal{M} \otimes_{\mathbb{C}} \overline{\mathcal{M}} \to \mathfrak{C}_\Delta$ the Hermitian pairing giving the polarization.

We briefly review the construction of M. The various canonical extensions $\widetilde{\mathcal{E}}^\alpha$ and $\widetilde{\mathcal{E}}^{>\alpha}$ embed into Deligne's meromorphic extension $\widetilde{\mathcal{E}}$, which is naturally a left $\mathcal{D}_\Delta$-module, with ∂_t acting through the logarithmic connection. The subsheaves $\widetilde{\mathcal{E}}^\alpha$ define a decreasing filtration on $\widetilde{\mathcal{E}}$, and

$$\widetilde{\mathcal{E}}^\alpha/\widetilde{\mathcal{E}}^{\alpha+1} = \widetilde{\mathcal{E}}^\alpha/t\widetilde{\mathcal{E}}^\alpha \widetilde{\mathcal{E}}^\alpha_{|0} \cong V.$$

Under this isomorphism, the operator $t\partial_t$ goes to the residue $R_\alpha = S_\alpha + N$ of the logarithmic connection. The corresponding right $\mathcal{D}_\Delta$-module $\omega_\Delta \otimes_{\mathcal{O}_\Delta} \widetilde{\mathcal{E}}$ has a unique maximal submodule with pure support Δ, namely

$$\mathcal{M} = \bigl(\omega_\Delta \otimes_{\mathcal{O}_\Delta} \widetilde{\mathcal{E}}^{>-1}\bigr) \cdot \mathcal{D}_\Delta \subseteq \omega_\Delta \otimes_{\mathcal{O}_\Delta} \widetilde{\mathcal{E}}.$$

For $\alpha < 0$, the V-filtration with respect to $t = 0$ is given by the formula

$$V_\alpha\mathcal{M} = \omega_\Delta \otimes_{\mathcal{O}_\Delta} \widetilde{\mathcal{E}}^{-(\alpha+1)}.$$

In particular, this leads to a canonical isomorphism

$$V_\alpha\mathcal{M}/V_{\alpha-1}\mathcal{M} = V_\alpha\mathcal{M}/V_\alpha\mathcal{M}\cdot t \cong \widetilde{\mathcal{E}}^{-(\alpha+1)}_{|0} \cong V,$$

under which right multiplication by $t\partial_t$ becomes left multiplication by $-\partial_t t = -(t\partial_t + 1)$, hence goes to the operator $-(R_{-(\alpha+1)} + \mathrm{Id})$. Moreover, the induced filtration $V_\bullet\mathcal{M}/V_{\alpha-1}\mathcal{M}$ becomes, on the vector space V, the filtration by *decreasing* eigenvalues of $S_{-(\alpha+1)}$. For $\alpha < 0$, this gives

$$\mathrm{gr}^V_\alpha\mathcal{M} \cong E_{e^{-2\pi i\alpha}}(T_s), \tag{A.9.1}$$

and under this isomorphism, the nilpotent operator $t\partial_t - \alpha$ on the left-hand side corresponds to the nilpotent operator $\mathrm{Y} = -N$ on the right-hand side (which is therefore the natural choice for the $\mathfrak{sl}_2(\mathbb{C})$-representation).

Note. This is another instance of the general principle that one can arrive at the correct signs simply by working consistently with right $\mathcal{D}$-modules.

The filtration $F_\bullet\mathcal{M}$ is constructed in such a way that

$$F_pV_\alpha\mathcal{M} = F_p\mathcal{M} \cap V_\alpha\mathcal{M} = \omega_\Delta \otimes_{\mathcal{O}_\Delta} F^{-p-1}\widetilde{\mathcal{E}}^{-\alpha-1}$$

for $\alpha < 0$. It induces a filtration on $\mathrm{gr}^V_\alpha\mathcal{M}$, with terms

$$F_p\mathrm{gr}^V_\alpha\mathcal{M} = \bigl(F_pV_\alpha\mathcal{M} + V_{<\alpha}\mathcal{M}\bigr)/V_{<\alpha}\mathcal{M}.$$

Since the V-filtration corresponds, on the vector space V, to the filtration by decreasing eigenvalues of $S_{-(\alpha+1)}$, this matches up nicely with our earlier discussion: under the isomorphism in (A.9.1), the filtration $F_\bullet \mathrm{gr}_\alpha^V \mathcal{M}$ becomes the limiting Hodge filtration $F_{\lim}^{-\bullet-1}$, defined in (A.8.5). Consequently, after choosing a splitting $H \in \mathrm{End}(V)$ for the weight filtration, the induced filtration on

$$\mathrm{gr}_\ell^W \mathrm{gr}_\alpha^V \mathcal{M} \cong E_\ell(H) \cap E_{e^{-2\pi \mathrm{i}\,\alpha}}(T_s)$$

is precisely the filtration $F_H^{-\bullet-1}$. We can therefore restate Theorem A.8.6 by saying that, for each $\alpha \in [-1,0)$, the graded vector space

$$\mathrm{gr}^W \mathrm{gr}_\alpha^V \mathcal{M} = \bigoplus_{\ell \in \mathbb{Z}} \mathrm{gr}_\ell^W \mathrm{gr}_\alpha^V \mathcal{M} \tag{A.9.2}$$

has an $\mathfrak{sl}_2$-Hodge structure of central weight n; here the representation by $\mathfrak{sl}_2(\mathbb{C})$ is defined by letting Y act as $t\partial_t - \alpha$, and the Hodge filtration is the filtration

$$F_{-\bullet-1} \mathrm{gr}^W \mathrm{gr}_\alpha^V \mathcal{M}$$

induced by the filtration $F_\bullet \mathcal{M}$.

Note. This formulation of Theorem A.8.6 does not require choosing a splitting for the weight filtration (because it is a result about the associated graded object).

Since it is instructive, let us also review how to recover the polarization on the $\mathfrak{sl}_2$-Hodge structure from the Hermitian pairing S_M on the $\mathcal{D}$-module $\mathcal{M}$. Recall from above that we have a preferred trivialization

$$\widetilde{\mathcal{E}}^{>-1} \cong \mathcal{O}_\Delta \otimes_\mathbb{C} V$$

for the canonical extension. In this frame, the polarization S_E on the variation of Hodge structure takes the form

$$\mathrm{S}_E(1 \otimes v', 1 \otimes v'') = \sum_{\beta \in (-1,0]} \sum_{j=0}^{\infty} |t|^{2\beta} \mathrm{L}(t)^j \cdot \frac{(-1)^j}{j!} \mathrm{S}\big(v'_\beta, N^j v''_\beta\big).$$

Note that the expression on the right-hand side is locally integrable precisely for $\beta > -1$. The Hermitian pairing on $\mathcal{M}$ is defined in such a way that, on the subsheaf $V_{<0}\mathcal{M} \cong \omega_\Delta \otimes_\mathbb{C} V$, one has

$$\big\langle \mathrm{S}_M(\mathrm{d}t \otimes v', \mathrm{d}t \otimes v''), \varphi \big\rangle = \frac{\varepsilon(2)}{2\pi \mathrm{i}} \int_\Delta \varphi \cdot \mathrm{S}_E(1 \otimes v', 1 \otimes v'') \mathrm{d}t \wedge \mathrm{d}\bar{t}.$$

The constants in this formula are of course dictated by (A.7.2). In terms of the basic currents $C_{\alpha,p}$ from (A.1.4), the definition of the pairing reads

$$\mathrm{S}_M(\mathrm{d}t \otimes v', \mathrm{d}t \otimes v'') = \sum_{\beta \in (-1,0]} \sum_{j=0}^{\infty} (-1)^j \mathrm{S}\big(v'_\beta, N^j v''_\beta\big) \cdot C_{-(\beta+1),j}, \tag{A.9.3}$$

From this asymptotic expansion, we can recover the restriction of the Hermitian pairing $\mathrm{S}\colon V \otimes_\mathbb{C} \overline{V} \to \mathbb{C}$ to the subspace

$$\mathrm{gr}_\alpha^V \mathcal{M} \cong E_{e^{-2\pi \mathrm{i}\,\alpha}}(T_s)$$

by taking the coefficient of the basic current $C_{\alpha,0}$; here $\alpha = -(\beta+1) \in [-1,0)$.

Note. There are no additional signs in this description; this is due to our principle of defining currents to be positive where possible.

So the conclusion is that $\mathfrak{sl}_2$-Hodge structure on (A.9.2) is polarized by the Hermitian pairing that we get by taking the coefficient of the basic current $C_{\alpha,0}$ in the asymptotic expansion of the pairing S_M. One can extract this coefficient, without writing down the asymptotic expansion, by using the Mellin transform.

A.9.4. Example. Another useful example is the direct image of a polarized Hodge structure H under the closed embedding $i\colon \{0\} \hookrightarrow \Delta$. If the weight of H is equal to n, then $i_*H \in \mathsf{HM}(\Delta, n)$. Using the notation from above, let us write

$$\mathrm{S}_0\colon i_+H \otimes_{\mathbb{C}} \overline{i_+H} \longrightarrow \mathfrak{C}_\Delta$$

for the induced Hermitian pairing. For two vectors $h', h'' \in H$, we have

$$\mathrm{S}_0(h', h'') = \mathrm{S}(h', h'') \cdot \delta_0,$$

where δ_0 is the delta function. So in this case, we can recover the polarization on H from the Hermitian pairing on i_+H as the coefficient in front of δ_0.

A.10. Nearby and vanishing cycles

In this section, we discuss the sign conventions for nearby and vanishing cycles, taking the example in the previous section as a model. Let us begin with a brief review of the general construction and its properties. Fix a complex manifold X. On the product $X \times \mathbb{C}$, we have the holomorphic function $t\colon X \times \mathbb{C} \to \mathbb{C}$, and the corresponding holomorphic vector field ∂_t. Suppose that $M \in \mathsf{HM}(X \times \mathbb{C}, w)$ is a polarized Hodge module of weight w on the product $X \times \mathbb{C}$. As usual, we denote by $(\mathcal{M}, F_\bullet\mathcal{M})$ the underlying filtered right $\mathcal{D}_{X\times\mathbb{C}}$-module, and by

$$\mathrm{S}_M\colon \mathcal{M} \otimes_{\mathbb{C}} \overline{\mathcal{M}} \longrightarrow \mathfrak{C}_{X\times\mathbb{C}}\,.$$

the Hermitian pairing giving the polarization. Lastly, we use the notation $V_\bullet\mathcal{M}$ for the V-filtration on $\mathcal{M}$ relative to $t = 0$.

(1) For every $\alpha \in [-1, 0)$, one has the *nearby cycles* $\psi_{t,\lambda}M$ for the eigenvalue $\lambda = e^{-2\pi\mathrm{i}\,\alpha}$. This is an object on X. The underlying filtered $\mathcal{D}_X$-module

$$\bigl(\mathrm{gr}_\alpha^V\mathcal{M}, F_{\bullet-1}\mathrm{gr}_\alpha^V\mathcal{M}\bigr)$$

comes with a nilpotent operator $N_\alpha = t\partial_t - \alpha$ and a Hermitian pairing

$$\mathrm{S}_\alpha\colon \mathrm{gr}_\alpha^V\mathcal{M} \otimes_{\mathbb{C}} \overline{\mathrm{gr}_\alpha^V\mathcal{M}} \longrightarrow \mathfrak{C}_X, \quad \mathrm{S}_\alpha \circ (N_\alpha \otimes \mathrm{Id}) = \mathrm{S}_\alpha \circ (\mathrm{Id} \otimes N_\alpha).$$

If we denote by $W_\bullet$ the weight filtration of N_α, then

$$\mathrm{gr}^W(\psi_{t,\lambda}M) = \bigoplus_{\ell\in\mathbb{Z}} \mathrm{gr}_\ell^W(\psi_{t,\lambda}M)$$

is a polarized $\mathfrak{sl}_2$-Hodge module of weight $w-1$; the element $\mathrm{Y} \in \mathfrak{sl}_2(\mathbb{C})$ acts as $N_\alpha = t\partial_t - \alpha$, and the polarization is induced by S_α.

(2) For $\alpha = 0$, one has the *unipotent vanishing cycles* $\phi_{t,1}M$. This is again an object on X. The underlying filtered $\mathcal{D}_X$-module

$$\big(\mathrm{gr}_0^V \mathcal{M}, F_\bullet \mathrm{gr}_0^V \mathcal{M}\big)$$

comes with a nilpotent operator $N_0 = t\partial_t$ and a Hermitian pairing

$$\mathrm{S}_0 \colon \mathrm{gr}_0^V \mathcal{M} \otimes_{\mathbb{C}} \overline{\mathrm{gr}_0^V \mathcal{M}} \longrightarrow \mathfrak{C}_X, \quad \mathrm{S}_0 \circ (N_0 \otimes \mathrm{Id}) = \mathrm{S}_0 \circ (\mathrm{Id} \otimes N_0).$$

If we denote by $W_\bullet$ the weight filtration of N_0, then

$$\mathrm{gr}^W(\phi_{t,1}M) = \bigoplus_{\ell \in \mathbb{Z}} \mathrm{gr}_\ell^W \phi_{t,1}M$$

is a polarized $\mathfrak{sl}_2$-Hodge module of weight w; the element $\mathrm{Y} \in \mathfrak{sl}_2(\mathbb{C})$ acts as $N_0 = t\partial_t$, and the polarization is induced by S_0.

Note that the Hodge filtration and the weight of the $\mathfrak{sl}_2$-Hodge module are different in both cases; this is forced on us by the following two examples:

A.10.1. Example. A polarized variation of Hodge structure of weight n on the punctured disk Δ^* gives rise to a polarized Hodge module $M \in \mathsf{HM}(\Delta, n+1)$, with $F_\bullet \mathcal{M} = \omega_\Delta \otimes_{\mathcal{O}_\Delta} F^{-\bullet-1}\mathcal{E}$. In this case, $\psi_{t,\lambda}M \cong E_\lambda(T_s)$, and we have seen in the previous section that $\mathrm{gr}^W(\psi_{t,\lambda}M)$ is a polarized $\mathfrak{sl}_2$-Hodge structure of weight $n = (n+1) - 1$. To get back the correct Hodge filtration, we also need to undo the shift that is built into the definition of $F_\bullet \mathcal{M}$.

A.10.2. Example. A polarized Hodge structure H of weight n gives rise to a polarized Hodge module $i_* H \in \mathsf{HM}(\Delta, n)$, where $i \colon \{0\} \hookrightarrow \Delta$ is the embedding of the origin. In this case, $\phi_{t,1}(i_* H) \cong H$ clearly has weight n, and there is no shift in the Hodge filtration.

This is the general picture, but we still need to figure what signs to use in the construction of the pairings S_α. Let us begin by treating the nearby cycles, because that case is slightly easier to explain. Fix a real number $\alpha \in [-1, 0)$. Consider local sections $m', m'' \in V_\alpha \mathcal{M}$ and the current $\mathrm{S}_M(m', m'') \in \mathfrak{C}_{X \times \mathbb{C}}$. Ideally, $\mathrm{S}_M(m', m'')$ would have an asymptotic expansion in t, in terms of the basic currents from (A.1.4), and the coefficient in front of $C_{\alpha,0}$ would be a current on X that could be used to define the pairing between $[m'], [m''] \in \mathrm{gr}_\alpha^V \mathcal{M}$. Fortunately, we can accomplish the same thing, without having the asymptotic expansion, by working with Mellin transforms.

More precisely, suppose that $m', m'' \in H^0(U, V_\alpha \mathcal{M})$. Let $\varphi(x)$ be a test function on X, and let $\eta(t)$ be a cutoff function on $\mathbb{C}$, such that the product $\eta(t)\varphi(x)$ has compact support inside U. The Mellin transform

$$F_{m',m''}(s) = \big\langle \mathrm{S}_M(m', m''), |t|^{2s} \eta(t) \varphi(x) \big\rangle$$

is holomorphic for $\mathrm{Re}\, s \gg 0$, and has a meromorphic extension to $\mathbb{C}$ with poles contained in the interval $(-\infty, \alpha]$. One can show that the residue at $s = \alpha$ depends continuously on φ, and that the formula

$$\big\langle \mathrm{S}_\alpha\big([m'],[m'']\big),\varphi\big\rangle = \mathrm{Res}_{s=\alpha}\big\langle \mathrm{S}_M(m',m''),|t|^{2s}\eta(t)\varphi(x)\big\rangle$$

defines the desired Hermitian pairing S_α. Let us check in several examples that this definition (with no extra sign factors) is the correct one.

A.10.3. Example. The first example explains how the Mellin transform can be used to pick up individual terms in a (hypothetical) asymptotic expansion. On the unit disk Δ, fix a test function $\varphi(t)$. Because the function $|t|^{2s-2} = e^{-(s-1)\mathrm{L}(t)}$ is locally integrable for $\mathrm{Re}\, s > 0$, the expression

$$F(s) = \frac{\varepsilon(2)}{2\pi\mathsf{i}}\int_{\mathbb{C}} |t|^{2s-2}\varphi \mathrm{d}t\wedge \mathrm{d}\bar{t} = \big\langle C_{-1,0},|t|^{2s-2}\varphi\big\rangle$$

defines a holomorphic function on the halfplane $\mathrm{Re}\, s > 0$. To understand its behavior near $s = 0$, one can use integration by parts to prove the identity

$$s^2F(s) = \frac{\varepsilon(2)}{2\pi\mathsf{i}}\int_{\Delta} |t|^{2s}\,\frac{\partial^2\varphi}{\partial t\partial\bar{t}}\,\mathrm{d}t\wedge\mathrm{d}\bar{t},$$

valid for $\mathrm{Re}\, s > 0$. The function on the right-hand side is holomorphic for $\mathrm{Re}\, s > -1$, and so $F(s)$ extends to a meromorphic function on this larger halfplane. From the power series expansion of the exponential function, we get

$$s^2F(s) = \sum_{j=0}^{\infty}(-1)^j s^j\,\frac{\varepsilon(2)}{2\pi\mathsf{i}}\int_{\Delta}\frac{\mathrm{L}(t)^j}{j!}\,\frac{\partial^2\varphi}{\partial t\partial\bar{t}}\mathrm{d}t\wedge\mathrm{d}\bar{t} = \sum_{j=1}^{\infty}(-1)^j s^j\big\langle C_{-1,j}\partial_t\partial_{\bar{t}},\varphi\big\rangle.$$

Using the identity $\delta_0 = -C_{-1,1}\partial_t\partial_{\bar{t}}$, we can rewrite this as

$$F(s) = \frac{\varphi(0)}{s} + \sum_{j=0}^{\infty}(-1)^j s^j\big\langle C_{-1,j+2}\partial_t\partial_{\bar{t}},\varphi\big\rangle. \tag{A.10.4}$$

Differentiating under the integral sign p times gives

$$\big\langle C_{-1,p},|t|^{2s-2}\varphi\big\rangle = \frac{\varepsilon(2)}{2\pi\mathsf{i}}\int_{\mathbb{C}}|t|^{2s-2}\frac{\mathrm{L}(t)^p}{p!}\varphi\mathrm{d}t\wedge\mathrm{d}\bar{t} = \frac{(-1)^p}{p!}F^{(p)}(s) \equiv \frac{\varphi(0)}{s^{p+1}}$$

modulo entire functions. Consequently, the Mellin transform of the basic current $C_{-1,p}$ has a pole of order exactly $p+1$ at the point $s=0$; the residue is $\varphi(0)$ for $p = 0$, and trivial for $p \geqslant 1$.

A.10.5. Example. Now let us go back to polarized variations of Hodge structure on Δ^*, and compute the nearby cycles with respect to $t = 0$, using the notation from the previous section. Let $v', v'' \in E_{e^{-2\pi\mathsf{i}\alpha}}(T_s)$ be two multivalued flat sections, for some $\alpha \in [-1,0)$. The formula for the pairing in (A.9.3) shows that

$$\mathrm{S}_M(\mathrm{d}t\otimes v',\mathrm{d}t\otimes v'') = \sum_{j=0}^{\infty}(-1)^j\mathrm{S}\big(v',N^jv''\big)\cdot C_{\alpha,j}.$$

According to the calculations in the preceding example, the Mellin transform

$$\big\langle \mathrm{S}_M(\mathrm{d}t\otimes v',\mathrm{d}t\otimes v''),|t|^{2s}\varphi(t)\big\rangle$$

is holomorphic on the halfplane $\operatorname{Re} s > \alpha$, and the polar part at $s = \alpha$ equals

$$\sum_{j=0}^{\infty} (-1)^j \mathrm{S}\big(v', N^j v''\big) \frac{\varphi(0)}{(s-\alpha)^{j+1}}.$$

In particular, the residue

$$\operatorname{Res}_{s=\alpha} \big\langle \mathrm{S}_M(\mathrm{d}t \otimes v', \mathrm{d}t \otimes v''), |t|^{2s} \varphi(t) \big\rangle = \mathrm{S}(v', v'') \cdot \varphi(0)$$

recovers the restriction of S to the eigenspace $E_{e^{-2\pi i \alpha}}(T_s)$; we saw in the previous section that this pairing gives the polarization on the $\mathfrak{sl}_2$-Hodge structure.

Now we turn to the unipotent nearby cycles, which are the boundary case $\alpha = 0$. The general idea is the same, but the construction needs to be modified slightly. As before, let $m', m'' \in V_0\mathcal{M}$ be two local sections, and consider the current $\mathrm{S}_M(m', m'')$. In the hypothetical asymptotic expansion of $\mathrm{S}_M(m', m'')$, we should take the coefficient of the delta function δ_0; recall that

$$\langle \delta_0, \varphi \rangle = \varphi(0).$$

The problem is that the Mellin transform of the delta function is trivial, and so a small trick is required. It is based on the identity

$$\delta_0 = -C_{-1,1} \partial_t \partial_{\bar{t}},$$

that has already appeared in the example above. Because ∂_t and $\partial_{\bar{t}}$ are surjective on the level of currents, we can extract the term with δ_0 from the hypothetical asymptotic expansion by writing our current in the form $-T\partial_t\partial_{\bar{t}}$, and then looking at the Mellin transform of T.

To make this precise, let $m', m'' \in H^0(U, V_0\mathcal{M})$ be two sections. Choose a current $T_{m',m''} \in H^0(U, \mathfrak{C}_{X \times \mathbb{C}})$ with the property that

$$\mathrm{S}_M(m', m'') = -T_{m',m''} \partial_t \partial_{\bar{t}};$$

such a current always exists, and is unique up to adding harmonic functions. With $\varphi(x)$ and $\eta(t)$ as above, the Mellin transform

$$G_{m',m''}(s) = \big\langle T_{m',m''}, |t|^{2s-2} \eta(t) \varphi(x) \big\rangle$$

is holomorphic for $\operatorname{Re} s \gg 0$, and extends to a meromorphic function on $\mathbb{C}$ with poles contained in the interval $(-\infty, 0]$. Integration by parts shows that, modulo entire functions, one has

$$F_{m',m''}(s) \equiv -s^2 G_{m',m''}(s),$$

and so the the quantity of interest is now the coefficient in front of $1/s^2$, hence the residue of $sG_{m',m''}(s)$ at $s = 0$. This observation suggests defining the Hermitian pairing S_0 by the formula

$$\big\langle \mathrm{S}_0\big([m'], [m'']\big), \varphi \big\rangle = \operatorname{Res}_{s=0} \big\langle T_{m',m''}, s \cdot |t|^{2s-2} \eta(t) \varphi(x) \big\rangle.$$

The following example explains why this definition is the correct one.

A.10.6. Example. For direct images along the closed embedding $i\colon X \hookrightarrow X \times \mathbb{C}$, we recover the pairing on the original $\mathcal{D}_X$-module. Indeed, suppose $\mathcal{N}$ is a coherent right $\mathcal{D}_X$-module, and $\mathrm{S}_N : \mathcal{N} \otimes_{\mathbb{C}} \overline{\mathcal{N}} \to \mathfrak{C}_X$ a Hermitian pairing. Then

$$\mathcal{M} = i_+\mathcal{N} \cong \mathcal{N} \otimes_{\mathbb{C}} \mathbb{C}[\partial_t], \quad \mathrm{S}_M = i_+\mathrm{S}_N,$$

and under this isomorphism, we have $V_0\mathcal{M} \cong \mathcal{N} \otimes 1$, hence $\mathrm{gr}_0^V \mathcal{M} \cong \mathcal{N}$. For two local sections $n', n'' \in \mathcal{N}$, the current

$$\mathrm{S}_M(n' \otimes 1, n'' \otimes 1) = i_*\mathrm{S}_N(n', n'')$$

is a multiple of δ_0; under the isomorphism $\mathrm{gr}_0^V \mathcal{M} \cong \mathcal{N}$, the construction above therefore recovers the original pairing: $\mathrm{S}_0 = \mathrm{S}_N$

We close this section with a brief discussion of the sign conventions for the "canonical morphism" and the "variation morphism",

$$\mathrm{can}\colon \psi_{t,1}M \longrightarrow \phi_{t,1}M \quad \text{and} \quad \mathrm{var}\colon \phi_{t,1}M \longrightarrow \psi_{t,1}M(-1).$$

The underlying morphisms of filtered $\mathcal{D}_X$-modules are the obvious ones:

$$\begin{aligned} \mathrm{can}\colon \big(\mathrm{gr}_{-1}^V \mathcal{M}, F_{\bullet-1}\mathrm{gr}_{-1}^V \mathcal{M}\big) &\longrightarrow \big(\mathrm{gr}_0^V \mathcal{M}, F_\bullet \mathrm{gr}_0^V \mathcal{M}\big), \quad \mathrm{can}(m) = m\partial_t \\ \mathrm{var}\colon \big(\mathrm{gr}_0^V \mathcal{M}, F_\bullet \mathrm{gr}_0^V \mathcal{M}\big) &\longrightarrow \big(\mathrm{gr}_{-1}^V \mathcal{M}, F_\bullet \mathrm{gr}_{-1}^V \mathcal{M}\big), \quad \mathrm{var}(m) = mt \end{aligned}$$

In particular, we have $\mathrm{can} \circ \mathrm{var} = N_0$ and $\mathrm{var} \circ \mathrm{can} = N_{-1}$. In the proof that every polarized Hodge module admits a decomposition by pure support, the following identity for can and var plays a crucial role:

$$\mathrm{S}_{-1} \circ (\mathrm{var} \otimes \mathrm{Id}) + \mathrm{S}_0 \circ (\mathrm{Id} \otimes \mathrm{can}) = 0 \tag{A.10.7}$$

With our construction of the pairings S_{-1} and S_0, this identity is easily proved using integration by parts. It can be shown that both can and var reduce the index in the weight filtration by 1; consequently,

$$\mathrm{can}\colon \mathrm{gr}_\ell^W(\psi_{t,1}M) \longrightarrow \mathrm{gr}_{\ell-1}^W(\phi_{t,1}M)$$

is a morphism of Hodge structures of weight $w + \ell - 1$, and

$$\mathrm{var}\colon \mathrm{gr}_\ell^W(\phi_{t,1}M) \longrightarrow \mathrm{gr}_{\ell-1}^W(\psi_{t,1}M)(-1)$$

is a morphism of Hodge structures of weight $w + \ell$ (for every $\ell \in \mathbb{Z}$).