CHAPTER 5

THE REES CONSTRUCTION FOR HODGE STRUCTURES

Summary. In this chapter, we revisit the notion of Hodge structure in order to adapt it to $\mathcal{D}$-modules. There are two major changes of point of view. On the one hand, we replace a vector space equipped with two filtrations with two free modules over the ring $\mathbb{C}[z]$ and we express oppositeness in this language as a gluing property. On the other hand, in order to handle singularities in the gluing properties for filtered $\mathcal{D}$-modules, we express the gluing as a nondegenerate pairing, in order to relax the nondegeneracy condition when necessary. The notion of sesquilinear pairing, which was mainly used for expressing the polarization in the previous chapters, is now used for expressing the oppositeness property. This leads to the general notion of *triples*, which form an abelian category, equipped with Hermitian duality. The polarization is now expressed as an isomorphism between a triple and its Hermitian dual, satisfying a suitable positivity condition. We will make clear the way to pass from one approach to the other one.

5.1. Filtered objects and the Rees construction

5.1.1. Convention. We denote with a *lower* index the *increasing* filtrations and with an *upper* index the *decreasing* ones. A standard rule is to pass from one type to the other one by changing the sign of the index. However, this rule is slightly modified for V-filtrations (see Chapter 9).

5.1.a. Filtered rings and modules

5.1.2. Definition. Let $(\mathcal{A}, F_\bullet)$ be a filtered $\mathbb{C}$-algebra. A *filtered $\mathcal{A}$-module* $(\mathcal{M}, F_\bullet\mathcal{M})$ is an $\mathcal{A}$-module $\mathcal{M}$ together with an increasing filtration indexed by $\mathbb{Z}$ satisfying (for left modules for instance)

$$F_k\mathcal{A} \cdot F_\ell\mathcal{M} \subset F_{k+\ell}\mathcal{M} \quad \forall\, k, \ell \in \mathbb{Z}.$$

We always assume that the filtration is *exhaustive*, i.e., $\bigcup_\ell F_\ell\mathcal{M} = \mathcal{M}$. We also say that $F_\bullet\mathcal{M}$ is an $F_\bullet\mathcal{A}$-filtration, or simply an *F-filtration*.

A filtered morphism between filtered $\mathcal{A}$-modules is a morphism of $\mathcal{A}$-modules which is compatible with the filtrations.

A simple way to treat a filtered module as a module is to consider the Rees object associated to any filtered object. Let us introduce a new variable z. We will replace the base field $\mathbb{C}$ with the polynomial ring $\mathbb{C}[z]$.

5.1.3. Rees ring and Rees module. If $(\mathcal{A}, F_\bullet)$ is a filtered $\mathbb{C}$-algebra, we denote by $\widetilde{\mathcal{A}}$ (or $R_F\mathcal{A}$ if we want to insist on the dependence with respect to the filtration) the *graded* subring $\bigoplus_p F_p\mathcal{A}\cdot z^p$ of $\mathcal{A}\otimes_{\mathbb{C}}\mathbb{C}[z,z^{-1}]$ (the term $F_p\mathcal{A}\cdot z^p$ is in degree p). For example, if $F_p\mathcal{A}=0$ for $p\leqslant -1$ and $F_p\mathcal{A}=\mathcal{A}$ for $p\geqslant 0$, we have $\widetilde{\mathcal{A}}=\mathcal{A}\otimes_{\mathbb{C}}\mathbb{C}[z]$.

It will be convenient to set $\widetilde{\mathbb{C}}=\mathbb{C}[z]$ (i.e., we apply the Rees construction to $\mathbb{C}$ equipped with its trivial filtration $F_0\mathbb{C}=\mathbb{C}$ and $F_{-1}\mathbb{C}=0$).

Any filtered module $(\mathcal{M}, F_\bullet)$ on the filtered ring $(\mathcal{A}, F_\bullet)$ gives rise similarly to a graded $\widetilde{\mathcal{A}}$-module $R_F\mathcal{M}=\bigoplus_p F_p\mathcal{M}\cdot z^p\subset\mathcal{M}\otimes_{\mathbb{C}}\mathbb{C}[z,z^{-1}]$, and a filtered morphism gives rise to a graded morphism (of degree zero) between the associated Rees modules.

Conversely, any graded $\widetilde{\mathcal{A}}$-module $\widetilde{\mathcal{M}}$ can be written as $\bigoplus_p \mathcal{M}_p z^p$ ($\widetilde{\mathcal{M}}_p=\mathcal{M}_p z^p$ is in degree p), where each $\mathcal{M}_p$ is an $\mathcal{A}$-module, and the $\mathbb{C}[z]$-structure is given by $\mathcal{A}$-linear morphisms $\mathcal{M}_p\to\mathcal{M}_{p+1}$. The $\mathcal{A}$-module $\mathcal{M}=\varinjlim_p\mathcal{M}_p$ is called the $\mathcal{A}$-module *associated with the graded* $\widetilde{\mathcal{A}}$*-module* $\widetilde{\mathcal{M}}$. The natural morphism

$$\mathbb{C}[z,z^{-1}]\otimes_{\mathbb{C}[z]}\widetilde{\mathcal{M}}\longrightarrow\mathbb{C}[z,z^{-1}]\otimes_{\mathbb{C}}\mathcal{M}=:\mathcal{M}[z,z^{-1}]$$

is an isomorphism of $\mathcal{A}[z,z^{-1}]$-modules.

The category $\mathsf{Modgr}(\widetilde{\mathcal{A}})$ is the category whose objects are graded $\widetilde{\mathcal{A}}$-modules and whose morphisms are graded morphisms of degree zero. It is an abelian category. It comes equipped with an automorphism σ: given an object of $\mathsf{Modgr}(\widetilde{\mathcal{A}})$ written as $\widetilde{\mathcal{M}}=\bigoplus_p\mathcal{M}_p z^p$, we set

$$\sigma(\widetilde{\mathcal{M}})=\widetilde{\mathcal{M}}(1)=z\widetilde{\mathcal{M}}\quad\text{so that}\quad\widetilde{\mathcal{M}}(1)_p=\mathcal{M}_{p-1}z^p. \tag{5.1.4}$$

In other words, we regard multiplication by z as an isomorphism $\widetilde{\mathcal{M}}\xrightarrow{\sim}\widetilde{\mathcal{M}}(1)$.

5.1.5. Remark **(Shift of the filtration and twist of the Rees module)**

(1) The shift $F[k]$ of an increasing filtration is defined by

$$F[k]_\bullet\mathcal{M}=F_{\bullet-k}\mathcal{M}\qquad(\text{hence }\mathrm{gr}_p^{F[k]}\mathcal{M}=\mathrm{gr}_{p-k}^F\mathcal{M},\ \forall\,p). \tag{5.1.5 $*$}$$

For example, if $F_\bullet\mathcal{M}$ only jumps at p_o, then $F[k]_\bullet\mathcal{M}$ only jumps at p_o+k.

(2) If $\widetilde{\mathcal{M}}=R_F\mathcal{M}$, then $F_p\mathcal{M}=\mathcal{M}_p$ and $\widetilde{\mathcal{M}}(k)_p=F_{p-k}\mathcal{M}z^p$. In other words,

$$\widetilde{\mathcal{M}}(k)\xrightarrow{\sim}R_{F[k]}\mathcal{M}. \tag{5.1.5 $**$}$$

5.1.b. Strictness. Strictness is a property which enables one to faithfully pass properties from a filtered object to the associated graded object.

5.1.6. Definition **(Strictness in $\mathsf{Mod}(\widetilde{\mathcal{A}})$ and $\mathsf{Modgr}(\widetilde{\mathcal{A}})$).**

(1) An object of $\mathsf{Mod}(\widetilde{\mathcal{A}})$ is said to be *strict* if it has no $\mathbb{C}[z]$-torsion.

(2) A morphism in $\mathsf{Mod}(\widetilde{\mathcal{A}})$ is said to be *strict* if its kernel and cokernel are strict (note that the composition of two strict morphisms need not be strict).

(3) A complex $\widetilde{\mathcal{M}}^\bullet$ of $\mathsf{Mod}(\widetilde{\mathcal{A}})$ is said to be *strict* if each of its cohomology modules is a strict object of $\mathsf{Mod}(\widetilde{\mathcal{A}})$.

An object, resp. morphism, resp. complex in $\mathsf{Modgr}(\widetilde{\mathcal{A}})$ is strict if it is so when considered in $\mathsf{Mod}(\widetilde{\mathcal{A}})$.

5.1.7. Caveat. The composition of strict morphisms between strict objects need not be strict. One cannot form a category (which would be abelian) by only considering these morphisms. On the other hand, the full subcategory $\mathsf{Modgr}(\widetilde{\mathcal{A}})_{\mathrm{st}}$ of $\mathsf{Modgr}(\widetilde{\mathcal{A}})$ whose objects are strict (morphisms need not be strict) is in general not abelian.

***5.1.8. Proposition* (Strict objects).**

(1) *An object of* $\mathsf{Modgr}(\widetilde{\mathcal{A}})$ *is strict if and only if it comes from a filtered* $\mathcal{A}$*-module by the Rees construction.*

(2) *The Rees construction induces an equivalence between the category of filtered* $\mathcal{A}$*-modules (and morphisms preserving filtrations) ant the category* $\mathsf{Modgr}(\widetilde{\mathcal{A}})_{\mathrm{st}}$*.*

(3) *The restriction functor* $\widetilde{\mathcal{M}} \mapsto \widetilde{\mathcal{M}}/(z-1)\widetilde{\mathcal{M}}$ *from* $\mathsf{Modgr}(\widetilde{\mathcal{A}})_{\mathrm{st}}$ *to* $\mathsf{Mod}(\mathcal{A})$ *is faithful.*

Proof.

(1) One checks that $\widetilde{\mathcal{M}}$ is strict if and only if the $\mathcal{A}$-linear morphisms $\mathcal{M}_p \to \mathcal{M}_{p+1}$ are all injective. In such a case, $\mathcal{M} = \varinjlim_p \mathcal{M}_p = \bigcup_p \mathcal{M}_p$ and the $\mathcal{M}_p$ form an increasing filtration $F_\bullet\mathcal{M}$, so that, by definition, $\widetilde{\mathcal{M}} = R_F\mathcal{M}$.

(2) We are left with considering morphisms. Let $\widetilde{\varphi} : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{N}}$ be a graded morphism of degree zero. Its restriction $\widetilde{\varphi}_p$ to $\widetilde{\mathcal{M}}_p$ satisfies $\widetilde{\varphi}_{p+1}zm_p = z\varphi_p m$ by $\widetilde{\mathbb{C}}[z]$-linearity. Therefore, $\widetilde{\varphi}_p$ is the restriction of $\widetilde{\varphi}_{p+1}$ by the inclusion $z\widetilde{\mathcal{M}}_p \hookrightarrow \widetilde{\mathcal{M}}_{p+1}$, hence the family $(\varphi_p)_p$ defines a morphism $\varphi : \mathcal{M} \to \mathcal{N}$, and we obviously have $\widetilde{\varphi} = R_F\varphi$.

(3) See Exercise 5.2. $\square$

We now consider the category WA (see Section 2.6.b) of W-filtered objects of the category $\mathsf{A} := \mathsf{Modgr}(\widetilde{\mathcal{A}})$ with $\widetilde{\mathcal{A}} = R_F\mathcal{A}$, and the notion of strictness is as in Definition 5.1.6.

5.1.9. Lemma. *We set* $\mathsf{Modgr}(\widetilde{\mathcal{A}}) = \mathsf{A}$*.*

(1) *Let* $\widetilde{\mathcal{M}}$ *be a an object of* WA*. If each* $\mathrm{gr}_k^W\widetilde{\mathcal{M}}$ *is strict, then* $\widetilde{\mathcal{M}}$ *is strict.*

(2) *Let* $\widetilde{\varphi} : \widetilde{\mathcal{M}}_1 \to \widetilde{\mathcal{M}}_2$ *be a morphism in* WA*. If* $\mathrm{gr}_k^W\widetilde{\mathcal{M}}_1, \mathrm{gr}_k^W\widetilde{\mathcal{M}}_2$ *are strict for all* k*, and if* $\widetilde{\varphi}$ *is strictly compatible with* W*, i.e., satisfies* $\widetilde{\varphi}(W_k\widetilde{\mathcal{M}}) = W_k\widetilde{\mathcal{N}} \cap \varphi(\widetilde{\mathcal{M}})$ *for all* k*, then* $\widetilde{\varphi}$ *is strict.*

Proof. The first point is treated in Exercise 5.1(2). Let us prove (2). Let $W_\bullet \operatorname{Ker}\widetilde{\varphi}$ and $W_\bullet \operatorname{Coker}\widetilde{\varphi}$ be the induced filtrations. By strict compatibility, the sequence

$$0 \longrightarrow \mathrm{gr}_k^W \operatorname{Ker}\widetilde{\varphi} \longrightarrow \mathrm{gr}_k^W\widetilde{\mathcal{M}} \xrightarrow{\mathrm{gr}_k^W\widetilde{\varphi}} \mathrm{gr}_k^W\widetilde{\mathcal{N}} \longrightarrow \mathrm{gr}_k^W \operatorname{Coker}\widetilde{\varphi} \longrightarrow 0$$

is exact. By strictness of $\mathrm{gr}_k^W\widetilde{\varphi}$, and applying (1) to $\operatorname{Ker}\widetilde{\varphi}$ and $\operatorname{Coker}\widetilde{\varphi}$, one gets that $\operatorname{Ker}\widetilde{\varphi}$ and $\operatorname{Coker}\widetilde{\varphi}$ are strict, i.e., $\widetilde{\varphi}$ is strict. $\square$

Strictness with respect to the monodromy filtration. We consider the setup of Chapter 3 on Lefschetz structures and take up the notation of Remark 3.3.4. We equip the category $\mathsf{Modgr}(\widetilde{\mathcal{A}})$ of graded $\widetilde{\mathcal{A}}$-modules with the automorphism σ shifting the grading by 1, so that $\sigma(\widetilde{\mathcal{M}}) = \widetilde{\mathcal{M}}(1)$ (see (5.1.4)). Let $\widetilde{\mathcal{M}}$ be an object of $\mathsf{Modgr}(\widetilde{\mathcal{A}})$ and let $\mathrm{N} : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}(-1) = \sigma^{-1}\widetilde{\mathcal{M}}$ be a nilpotent endomorphism.

5.1.10. Proposition. *Let* $\mathrm{M}_\bullet(\mathrm{N})\widetilde{\mathcal{M}}$ *be the monodromy filtration of* $(\widetilde{\mathcal{M}}, \mathrm{N})$ *in the abelian category* $\mathsf{Modgr}(\widetilde{\mathcal{A}})$ *(see Lemma* 3.3.1*). Assume that* $\widetilde{\mathcal{M}}$ *is strict. Then the following properties are equivalent:*

(1) *For every* $\ell \geqslant 1$, $\mathrm{N}^\ell : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}(-\ell)$ *is a strict morphism.*
(2) *For every* $\ell \in \mathbb{Z}$, $\mathrm{gr}^{\mathrm{M}}_\ell \widetilde{\mathcal{M}}$ *is strict.*
(3) *For every* $\ell \geqslant 0$, $\mathrm{P}_\ell \widetilde{\mathcal{M}}$ *is strict.*

Proof. The equivalence between (2) and (3) comes from the Lefschetz decomposition in the category $\mathsf{Modgr}(\widetilde{\mathcal{A}})$.

(2) $\Longrightarrow$ (1) Assume $\ell \geqslant 1$. The Lefschetz decomposition implies that each morphism $\mathrm{gr}_{-2\ell}\mathrm{N}^\ell$ on $\mathrm{gr}^{\mathrm{M}}_\bullet \widetilde{\mathcal{M}}$ is strict. Since N^ℓ, regarded as a filtered morphism

$$(\widetilde{\mathcal{M}}, \mathrm{M}_\bullet \widetilde{\mathcal{M}}) \longrightarrow (\widetilde{\mathcal{M}}(-\ell), \mathrm{M}[2\ell]_\bullet \widetilde{\mathcal{M}})$$

is strictly compatible with the filtrations M (Lemma 3.3.7), the result follows from Lemma 5.1.9(2).

(1) $\Longrightarrow$ (2) We will use the inductive construction of the monodromy filtration given in Exercise 3.10. We argue by induction on the order of nilpotence of N. Assume that $\mathrm{N}^{\ell+1} = 0$. The strictness of $\widetilde{\mathcal{M}}$ implies that $\mathrm{M}_\ell\widetilde{\mathcal{M}}$, $\mathrm{M}_{\ell-1}\widetilde{\mathcal{M}}$, $\mathrm{M}_{-\ell}\widetilde{\mathcal{M}} = \mathrm{gr}^{\mathrm{M}}_{-\ell}\widetilde{\mathcal{M}}$ and $\mathrm{P}_\ell\widetilde{\mathcal{M}} = \mathrm{gr}^{\mathrm{M}}_\ell\widetilde{\mathcal{M}} \simeq \mathrm{gr}^{\mathrm{M}}_{-\ell}\widetilde{\mathcal{M}}$ are strict. The strictness of $\widetilde{\mathcal{M}}' := \widetilde{\mathcal{M}}/\mathrm{M}_{-\ell}\widetilde{\mathcal{M}} = \operatorname{Coker} \mathrm{N}^\ell$ follows from the strictness of N^ℓ. Moreover, $(\widetilde{\mathcal{M}}', \mathrm{N}')$ satisfies (1) with $\mathrm{N}'^\ell = 0$, hence by induction each $\mathrm{gr}^{\mathrm{M}}_j\widetilde{\mathcal{M}}'$ is strict. Now, the relation between $\mathrm{gr}^{\mathrm{M}}_\bullet\widetilde{\mathcal{M}}'$ and $\mathrm{gr}^{\mathrm{M}}_\bullet\widetilde{\mathcal{M}}$ is easily seen from the Lefschetz decomposition (see Figure 3.1), and (2) for $\mathrm{gr}^{\mathrm{M}}_\bullet\widetilde{\mathcal{M}}$ follows. $\square$

5.1.c. Filtered holomorphic flat bundles. Let $(\mathcal{H}', \nabla)$ be an $\mathcal{O}_X$-module with connection on a complex manifold X, equipped with a decreasing filtration $F^\bullet\mathcal{H}'$ by $\mathcal{O}_X$-submodules (here, we do not make any coherence or local freeness assumption). The filtration on the sheaf of rings $\mathcal{O}_X$ is simply defined by $F^0\mathcal{O}_X = \mathcal{O}_X$ and $F^1\mathcal{O}_X = 0$, so that $\widetilde{\mathcal{O}}_X = \mathcal{O}_X[z]$ as a sheaf of graded rings. The Rees module attached to $F^\bullet\mathcal{H}'$ is the graded coherent $\mathcal{O}_X[z]$-module $\widetilde{\mathcal{H}}' := \bigoplus_p F^p\mathcal{H}'z^{-p}$.

By a *holomorphic z-connection* on $\widetilde{\mathcal{H}}'$, we mean a morphism

$$\widetilde{\nabla} : \widetilde{\mathcal{H}}' \longrightarrow \widetilde{\Omega}^1_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{H}}'(-1)$$

in the category $\mathsf{Modgr}(\widetilde{\mathbb{C}})$ (in particular it is $\widetilde{\mathbb{C}}$-linear) which satisfies the z-Leibniz rule

$$\widetilde{\nabla}(fv) = \widetilde{\mathrm{d}}f \otimes v + f\widetilde{\nabla}v, \quad \widetilde{\mathrm{d}} := z\mathrm{d},\ f \in \widetilde{\mathcal{O}}_X,\ v \in \widetilde{\mathcal{H}}'.$$

We say that $\widetilde{\nabla}$ is *flat* if its curvature $\widetilde{\nabla} \circ \widetilde{\nabla}$ (taken in the usual sense) is zero.

5.1.11. Lemma. *The connection ∇ on $\mathcal{H}'$ extends to a z-connection $\widetilde{\nabla}$ on $\widetilde{\mathcal{H}}'$ if and only if the filtration $F^\bullet\mathcal{H}'$ satisfies the Griffiths transversality property. In such a case, ∇ is flat if and only if $\widetilde{\nabla}$ is flat.*

Proof. For the "only if" part, define first $\widetilde{\nabla} : \mathcal{H}'[z, z^{-1}] \to \widetilde{\Omega}^1_X \otimes_{\widetilde{\mathcal{O}}_X} \mathcal{H}'[z, z^{-1}](-1)$ as $z\nabla$. This is a z-connection. Griffiths transversality implies that it sends $\widetilde{\mathcal{H}}'$ to $\widetilde{\Omega}^1_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{H}}'(-1)$, defining thus $\widetilde{\nabla}$. If ∇ is flat, then so is $z\nabla$, hence $\widetilde{\nabla}$.

Conversely, starting from $\widetilde{\nabla}$, one extends it by z-linearity to $\mathcal{H}'[z, z^{-1}]$ and, dividing by z and then restricting to $z = 1$, one obtains the desired ∇. □

We can now restate Proposition 5.1.8 in the present setting.

5.1.12. Proposition. *The Rees construction induces an equivalence between*

- *the category of filtered $\mathcal{O}_X$-modules with flat connection and with a filtration satisfying the Griffiths transversality property,*
- *and the full subcategory of strict objects in the category of graded $\widetilde{\mathcal{O}}_X$-modules with flat z-connection.* □

5.1.d. Filtrations discretely indexed by $\mathbb{R}$. We extend the Rees construction to filtrations indexed by $B + \mathbb{Z}$, where B is a finite subset of $[0, 1)$, in order to treat V-filtrations in Chapter 9. We fix a positive integer r (ramification order) and consider the ring $\mathbb{C}[u]$ with the subring $\mathbb{C}[z] \hookrightarrow \mathbb{C}[u]$ so that z is mapped to u^r. The variable u is given the degree $1/r$. We set $\widetilde{\mathcal{A}}^{(r)} = \mathbb{C}[u] \otimes_{\mathbb{C}[z]} \widetilde{\mathcal{A}}$. This is a $\frac{1}{r}\mathbb{Z}$-graded ring containing $\widetilde{\mathcal{A}}$ as a $\mathbb{Z}$-graded subring, with degree p/r term

$$(\widetilde{\mathcal{A}}^{(r)})_{p/r} = \widetilde{\mathcal{A}}_{\lfloor p/r \rfloor}.$$

5.1.13. Proposition. *Giving a $\frac{1}{r}\mathbb{Z}$-graded $\widetilde{\mathcal{A}}^{(r)}$-module $\widetilde{\mathcal{M}}^{(r)}$ is equivalent to giving a finite family $\widetilde{\mathcal{M}}_{i/r}$ $(i = 0, \dots, r-1)$ in $\mathsf{Modgr}(\widetilde{\mathcal{A}})$ together with morphisms $\widetilde{\mathcal{M}}_{(i-1)/r} \to \widetilde{\mathcal{M}}_{i/r}$ $(i = 1, \dots, r-1)$ and $\widetilde{\mathcal{M}}_{(r-1)/r} \to \widetilde{\mathcal{M}}_1(1)$ such that, for each $i = 0, \dots, r$, their composition $\widetilde{\mathcal{M}}_{i/r} \to \widetilde{\mathcal{M}}_{1+i/r}$ is equal to the multiplication by z. As an $\widetilde{\mathcal{A}}$-module, $\widetilde{\mathcal{M}}^{(r)}$ decomposes as $\bigoplus_{i=0}^{r-1} \widetilde{\mathcal{M}}_{i/r} \otimes u^i$.*

Furthermore, $\widetilde{\mathcal{M}}^{(r)}$ is strict (i.e., $\mathbb{C}[u]$-flat) if and only if each $\widetilde{\mathcal{M}}_{i/r}$ is strict (i.e., $\mathbb{C}[z]$-flat).

Lastly, if $\widetilde{\mathcal{A}}$ is Noetherian,[(1)] *then $\widetilde{\mathcal{M}}^{(r)}$ is $\widetilde{\mathcal{A}}^{(r)}$-coherent if and only if each $\widetilde{\mathcal{M}}_{i/r}$ is $\widetilde{\mathcal{A}}$-coherent.*

Proof. For $i = 0, \dots, r-1$, we consider the $\mathbb{Z}$-graded objects $\widetilde{\mathcal{M}}^{(r)}_{i+r\mathbb{Z}}$. These are $\mathbb{Z}$-graded $\widetilde{\mathcal{A}}$-modules, that we denote by $\widetilde{\mathcal{M}}_{i/r}$. The morphism $u : \widetilde{\mathcal{M}}^{(r)}_j \to \widetilde{\mathcal{M}}^{(r)}_{j+1}$ induces the desired family of morphisms.

Conversely, from the family $\widetilde{\mathcal{M}}_{i/r}$ and the morphisms, we set, for $p = qr + i$ with $i \in \{0, \dots, r-1\}$, $\widetilde{\mathcal{M}}^{(r)}_{p/r} = (\widetilde{\mathcal{M}}_{i/r})_q$ and the morphisms are interpreted as the multiplication by u.

(1) In the sense of [**Kas03**, Def. A.7], see Remark 8.8.3 in Chapter 8.

The flatness statement is then clear since u-torsion is equal to z-torsion in $\widetilde{\mathcal{M}}^{(r)}$, and the last statement follows e.g. from [**Kas03**, Prop. A.10]. □

If $\widetilde{\mathcal{A}}$ is the Rees ring of $\mathcal{A}$ with respect to the $\mathbb{Z}$-indexed filtration $G_\bullet\mathcal{A}$, then $\widetilde{\mathcal{A}}^{(r)}$ is the Rees ring of $\mathcal{A}$ with respect to the $\frac{1}{r}\mathbb{Z}$-indexed filtration $G^{(r)}_\bullet\mathcal{A}$ defined by

$$G^{(r)}_{p/r}\mathcal{A} := G_{\lfloor p/r\rfloor}\mathcal{A}.$$

If $\widetilde{\mathcal{M}}^{(r)}$ is $\mathbb{C}[u]$-flat, it is equal to the Rees module of some $\mathcal{A}$-module $\mathcal{M}$ with respect to a $\frac{1}{r}\mathbb{Z}$-indexed $G^{(r)}$-filtration $G^{(r)}_\bullet\mathcal{M}$, that we can consider as a family of nested $\mathbb{Z}$-indexed G-filtrations $G_{i/r+\bullet}\mathcal{M}$ with Rees module $\widetilde{\mathcal{M}}_{i/r}$, i.e., satisfying

$$p+i/r \leqslant q+j/r \implies G_{p+i/r}\mathcal{M}\subset G_{q+j/r}\mathcal{M} \quad \forall\, p,q\in\mathbb{Z},\ \forall\, i,j\in\{0,\dots,r-1\}.$$

Assuming that $\widetilde{\mathcal{A}}$ is a Noetherian sheaf of rings, we say that $G^{(r)}_\bullet\mathcal{M}$ is a coherent $G^{(r)}$-filtration if $\widetilde{\mathcal{M}}^{(r)}$ is $\widetilde{\mathcal{A}}^{(r)}$-coherent, and this property is equivalent to each $G_{i/r+\bullet}\mathcal{M}$ being a coherent G-filtration of $\mathcal{M}$.

Let now B be a finite subset of $[0,1)$ containing 0 and let $G_\bullet\mathcal{M}$ be a filtration indexed by $B+\mathbb{Z}$. In order to make such a filtrations enter the previous framework, we number $B=\{b_0,\dots,b_{r-1}\}$ with $b_0=0$ and we extend the numbering so that, if $p=qr+i$ with $i\in\{0,\dots,r-1\}$, we have $b_p=q+b_i$. We then set $G^{r)}_{\ell/r}\mathcal{M}=G_{b_\ell}\mathcal{M}$ ($\ell\in\mathbb{Z}$), which corresponds to the family of nested filtrations $G_{b_i+\mathbb{Z}}\mathcal{M}$.

We claim that $G^{r)}_\bullet\mathcal{M}$ is a $G^{(r)}\mathcal{A}$-filtration, so that the Rees module $R_{G^{(r)}}\mathcal{M}=\bigoplus_{k\in\mathbb{Z}}G^{(r)}_k\mathcal{M}u^k$ is an $\widetilde{\mathcal{A}}^{(r)}$-module. We need to prove that $G_{\lfloor k/r\rfloor}\mathcal{A}\cdot G^{r)}_{\ell/r}\mathcal{M}\subset G^{r)}_{(k+\ell)/r}\mathcal{M}$: this follows from the inequality $b_{k+\ell}\geqslant b_{r\lfloor k/r\rfloor+\ell}=\lfloor k/r\rfloor+b_\ell$.

5.2. The category of $\widetilde{\mathbb{C}}$-triples

5.2.a. A geometric interpretation of a bi-filtered vector space

Let $(\mathcal{H},F'^\bullet)$ be a filtered vector space. We introduce a new variable z and we consider, in the free $\mathbb{C}[z,z^{-1}]$-module $\widetilde{\mathcal{H}}:=\mathbb{C}[z,z^{-1}]\otimes_{\mathbb{C}}\mathcal{H}$, the $\mathbb{C}$-vector space $\widetilde{\mathcal{H}}':=\bigoplus_p F'^p\mathcal{H}z^{-p}$. Then $\widetilde{\mathcal{H}}'$ is a $\widetilde{\mathbb{C}}$-submodule of $\widetilde{\mathcal{H}}$ which generates $\widetilde{\mathcal{H}}$, that is, $\widetilde{\mathcal{H}}=\mathbb{C}[z,z^{-1}]\otimes_{\widetilde{\mathbb{C}}}\widetilde{\mathcal{H}}'$. It is a free $\widetilde{\mathbb{C}}$-module. Indeed, Let us choose for each p a family $\boldsymbol{v}^p$ in $F'^p\mathcal{H}$ inducing a basis of $\mathrm{gr}^p_{F'}\mathcal{H}$; then $\widetilde{\mathcal{H}}'=\bigoplus_p\widetilde{\mathbb{C}}z^{-p}\boldsymbol{v}^p$.

Similarly, denote by $\widetilde{\mathcal{H}}''$ the object $\bigoplus_q F''^q\mathcal{H}z^q$. Then $\widetilde{\mathcal{H}}''$ is a free $\mathbb{C}[z^{-1}]$-submodule of $\widetilde{\mathcal{H}}$ which generates $\widetilde{\mathcal{H}}$, that is, $\widetilde{\mathcal{H}}=\mathbb{C}[z,z^{-1}]\otimes_{\mathbb{C}[z^{-1}]}\widetilde{\mathcal{H}}''$. Using the gluing

$$\begin{tikzcd}\mathbb{C}[z,z^{-1}]\otimes_{\mathbb{C}[z]}\widetilde{\mathcal{H}}' \arrow[r, "\sim"] \arrow[rr, bend left, "\sim"] & \widetilde{\mathcal{H}} & \mathbb{C}[z,z^{-1}]\otimes_{\mathbb{C}[z^{-1}]}\widetilde{\mathcal{H}}'' \arrow[l, "\sim"']\end{tikzcd}$$

the pair $(\widetilde{\mathcal{H}}',\widetilde{\mathcal{H}}'')$ defines an algebraic vector bundle $\mathcal{F}$ on $\mathbb{P}^1$ of rank $\dim\mathcal{H}$. The properties 2.5(1a) (oppositeness) and (1b) (Hodge decomposition) are also equivalent to (see Exercise 5.5)

(c) The vector bundle $\mathcal{F}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(w)^{\dim\mathcal{H}}$.

The gluing isomorphism $\mathbb{C}[z,z^{-1}]\otimes_{\mathbb{C}}\mathcal{H}\xrightarrow{\sim}\mathbb{C}[z,z^{-1}]\otimes_{\mathbb{C}}\mathcal{H}$ described above has the property of being homogeneous of degree zero with respect to the z-grading, since it is induced by a constant isomorphism $\mathcal{H}\xrightarrow{\sim}\mathcal{H}$ (the identity).

5.2.b. $\widetilde{\mathbb{C}}$-Triples. We will introduce another language for dealing with polarizable complex Hodge structures. This is similar to the presentation given in Section 2.6.a, but compared with it, we replace $\mathcal{H}''$ with its dual $\mathcal{H}''^{\vee}$. This approach will be useful in higher dimensions.

***5.2.1. Definition* ($\widetilde{\mathbb{C}}$-Triples).** The category $\widetilde{\mathbb{C}}$-$\mathsf{Triples}$ is the category whose objects

$$T=(\widetilde{\mathcal{H}}',\widetilde{\mathcal{H}}'',\mathfrak{s})$$

consist of a pair of $\widetilde{\mathbb{C}}$-modules of finite type $\widetilde{\mathcal{H}}',\widetilde{\mathcal{H}}''$ and a sesquilinear pairing $\mathfrak{s}:\mathcal{H}'\otimes\overline{\mathcal{H}''}\to\mathbb{C}$ between the associated vector spaces (see Section 5.1.3) that we also regard as a morphism $\mathfrak{s}:\mathcal{H}''\to\mathcal{H}'^{*}$ (see Section 0.5), and whose morphisms $\varphi:T_1\to T_2$ are pairs $(\widetilde{\varphi}',\widetilde{\varphi}'')$ of morphisms (graded of degree zero)

$$\widetilde{\varphi}':\widetilde{\mathcal{H}}'_1\longrightarrow\widetilde{\mathcal{H}}'_2,\qquad\widetilde{\varphi}'':\widetilde{\mathcal{H}}''_2\longrightarrow\widetilde{\mathcal{H}}''_1\tag{5.2.1 $*$}$$

such that, for every $v'_1\in\mathcal{H}'_1$ and $v''_2\in\mathcal{H}''_2$, denoting by φ',φ'' the morphisms induced by $\widetilde{\varphi}',\widetilde{\varphi}''$ on $\mathcal{H}',\mathcal{H}''$, we have

$$\mathfrak{s}_1(v'_1,\overline{\varphi''(v''_2)})=\mathfrak{s}_2(\varphi'(v'_1),\overline{v''_2}),\tag{5.2.1 $**$}$$

or equivalently

$$\varphi'^{*}\circ\mathfrak{s}_2=\mathfrak{s}_1\circ\varphi'':\mathcal{H}''_2\longrightarrow\mathcal{H}'^{*}_1.$$

***5.2.2. Operations on the category* $\widetilde{\mathbb{C}}$-$\mathsf{Triples}$**

(1) The category $\widetilde{\mathbb{C}}$-$\mathsf{Triples}$ is abelian, the “prime” part is covariant, while the “double-prime” part is contravariant. In other words, the “prime” part is an object of the category of $\widetilde{\mathbb{C}}$-vector spaces, while the ‘double-prime” part is an object of the opposite category.

For example, the triple $\operatorname{Ker}\varphi$ is the triple $(\operatorname{Ker}\widetilde{\varphi}',\operatorname{Coker}\widetilde{\varphi}'',\mathfrak{s}_1^{\varphi})$, where $\mathfrak{s}_1^{\varphi}$ is the pairing between $\operatorname{Ker}\varphi'$ and $\operatorname{Coker}\varphi''$ induced by $\mathfrak{s}_1$, which is well-defined because of (5.2.1 $**$). Similarly, we have

$$\operatorname{Coker}\varphi=(\operatorname{Coker}\widetilde{\varphi}',\operatorname{Ker}\widetilde{\varphi}'',\mathfrak{s}_2^{\varphi}),\quad\operatorname{Im}\varphi=(\operatorname{Im}\widetilde{\varphi}',\widetilde{\mathcal{H}}''_2/\operatorname{Ker}\widetilde{\varphi}'',\mathfrak{s}_2^{\varphi}).$$

(2) An increasing filtration $W_{\bullet}T$ of a triple T consists of increasing filtrations $W_{\bullet}\widetilde{\mathcal{H}}',W_{\bullet}\widetilde{\mathcal{H}}''$ such that $\mathfrak{s}(W_{\ell}\mathcal{H}',\overline{W_{-\ell-1}\mathcal{H}''})=0$ for every ℓ. Then $\mathfrak{s}$ induces a pairing

$$\mathfrak{s}_{\ell}:W_{\ell}\mathcal{H}'\otimes\overline{\mathcal{H}''/W_{-\ell-1}\mathcal{H}''}\longrightarrow\mathbb{C}.$$

We set $W_{\ell}T=(W_{\ell}\widetilde{\mathcal{H}}',\widetilde{\mathcal{H}}''/W_{-\ell-1}\widetilde{\mathcal{H}}'',\mathfrak{s}_{\ell})$. We have

$$\operatorname{gr}_{\ell}^{W}T=(\operatorname{gr}_{\ell}^{W}\widetilde{\mathcal{H}}',\operatorname{gr}_{-\ell}^{W}\widetilde{\mathcal{H}}'',\mathfrak{s}_{\ell}).$$

(3) We say that a triple is *strict* if $\widetilde{\mathcal{H}}',\widetilde{\mathcal{H}}''$ are strict. Strict triples are in one-two-one correspondence with *filtered triples* $((\mathcal{H}',F^{\bullet}\mathcal{H}'),(\mathcal{H}'',F^{\bullet}\mathcal{H}''),\mathfrak{s})$. *We will not distinguish between $\widetilde{\mathbb{C}}$-triples and filtered $\mathbb{C}$-triples.*

We say that a morphism $\varphi : T_1 \to T_2$ is strict if its components $\widetilde{\varphi}', \widetilde{\varphi}''$ are strict. Strict morphisms between strict triples are in one-to-one correspondence with strict morphisms between filtered triples.

(4) The difference with the construction made in Section 2.6.a is that the "double prime" part is now contravariant, and the isomorphism γ is replaced with a pairing. This gives more flexibility since the pairing is not assumed to be non-degenerate a priori. We say that a $\widetilde{\mathbb{C}}$-triple T is *non-degenerate* if $\mathfrak{s}$ is so. If $T = (\widetilde{\mathcal{H}}', \widetilde{\mathcal{H}}'', \mathfrak{s})$ is strict and non-degenerate, one can associate a triple like in Section 2.6.a by replacing $(\mathcal{H}'', F^\bullet\mathcal{H}'')$ defined from $\widetilde{\mathcal{H}}''$ with $(\mathcal{H}''^*, F^\bullet\mathcal{H}''^*)$ and by defining γ as the isomorphism $\mathcal{H}' \to \mathcal{H}''^*$ obtained by Hermitian adjunction from $\mathfrak{s} : \mathcal{H}'' \to \mathcal{H}'^*$.

(5) Let $(T, W_\bullet T)$ be a W-filtered $\widetilde{\mathbb{C}}$-triple as in (2). Assume that T and all $\mathrm{gr}_\ell^W T$ are strict, i.e., all inclusions $W_\ell T \hookrightarrow W_{\ell+1}T$ are strict morphisms. Then $\mathrm{gr}_\ell^W \widetilde{\mathcal{H}}'$ is the Rees object attached with the filtered vector space

$$F^p \mathrm{gr}_\ell^W \mathcal{H}' := \frac{F^p\mathcal{H}' \cap W_\ell\mathcal{H}'}{F^p\mathcal{H}' \cap W_{\ell-1}\mathcal{H}'},$$

and a similar equality for $\mathrm{gr}_{-\ell}^W \widetilde{\mathcal{H}}''$.

(6) The *Hermitian dual* of a $\widetilde{\mathbb{C}}$-triple $T = (\widetilde{\mathcal{H}}', \widetilde{\mathcal{H}}'', \mathfrak{s})$ is the $\widetilde{\mathbb{C}}$-triple $T^* := (\widetilde{\mathcal{H}}'', \widetilde{\mathcal{H}}', \mathfrak{s}^*)$, where $\mathfrak{s}^*$ is defined by

$$\mathfrak{s}^*(v'', \overline{v'}) := \overline{\mathfrak{s}(v', \overline{v''})}.$$

We have $T^{**} = T$. If $\varphi = (\widetilde{\varphi}', \widetilde{\varphi}'') : T_1 \to T_2$ is a morphism, its Hermitian adjoint $\varphi^* : T_2^* \to T_1^*$ is the morphism $(\widetilde{\varphi}'', \widetilde{\varphi}')$.

The Hermitian dual of a strict $\widetilde{\mathbb{C}}$-triple T is also strict, and T^* corresponds to the filtered triple $T^* := ((\mathcal{H}'', F^\bullet\mathcal{H}''), (\mathcal{H}', F^\bullet\mathcal{H}'), \mathfrak{s}^*)$.

(7) Given a pair of integers (k, ℓ), the twist $T(k, \ell)$ is defined by

$$T(k, \ell) := (z^k\widetilde{\mathcal{H}}', z^{-\ell}\widetilde{\mathcal{H}}'', \mathfrak{s}).$$

We have $(T(k, \ell))^* = T^*(-\ell, -k)$. If $\varphi : T_1 \to T_2$ is a morphism, then it is also a morphism $T_1(k, \ell) \to T_2(k, \ell)$.

If T is strict with associated filtered triple T, the twisted object $T(k, \ell)$ is also strict and its associated filtered triple is

$$(F[k]^\bullet\mathcal{H}', F[-\ell]^\bullet\mathcal{H}'', \mathfrak{s}).$$

This is compatible with the twist as defined in Section 2.5.7, by means of the equivalence of Lemma 5.2.7 below. (Recall that $F[k]^p := F^{p+k}$.)

5.2.3. Notation. As in Definition 2.5.8, we simply use the notation (w) for the (symmetric) Tate twist (w, w): $T(w) = (z^w\widetilde{\mathcal{H}}', z^{-w}\widetilde{\mathcal{H}}'', \mathfrak{s})$.

***5.2.4. Definition** (w-**oppositeness condition**).* Let T be a filtered triple and let $w \in \mathbb{Z}$. The filtration $F^\bullet\mathcal{H}''$ naturally induces a filtration $F^\bullet\mathcal{H}''^*$ on the Hermitian dual space $\mathcal{H}''^* = \overline{\mathcal{H}''}^\vee$. We say that T satisfies the w-oppositeness condition if $\mathfrak{s}$ is non-degenerate and if the filtration $F^\bullet\mathcal{H}'$ is w-opposite to the filtration obtained from $F^\bullet\mathcal{H}''^*$ by means of the isomorphism $\mathcal{H}' \xrightarrow{\sim} \mathcal{H}''^*$ induced by $\mathfrak{s}^*$.

***5.2.5. Definition* ($\mathbb{C}$-Hodge triples).** The category of $\mathbb{C}$-Hodge triples of weight $w \in \mathbb{Z}$ is the full subcategory of $\widetilde{\mathbb{C}}$-$\mathsf{Triples}$ (i.e., morphisms are described by (5.2.1 $*$) and satisfying (5.2.1 $**$)) whose objects are strict and satisfy the w-oppositeness condition. In particular, $\mathfrak{s}$ is assumed to be non-degenerate. A $\mathbb{C}$-Hodge triple will be denoted by $H = (\widetilde{\mathcal{H}}', \widetilde{\mathcal{H}}'', \mathfrak{s})$ or $((\mathcal{H}', F^\bullet\mathcal{H}'), (\mathcal{H}'', F^\bullet\mathcal{H}''), \mathfrak{s})$.

***5.2.6. Remark* (Hermitian duality and twist).** The category of $\mathbb{C}$-Hodge triples of weight w is changed to that of $\mathbb{C}$-Hodge triples of weight $-w$ by the Hermitian duality functor 5.2.2(6) and, for a $\mathbb{C}$-Hodge triple H of weight w, the twisted $\widetilde{\mathbb{C}}$-triple $H(k,\ell)$ is a $\mathbb{C}$-Hodge triple H of weight $w-(k+\ell)$. In particular, the Tate twisted $\widetilde{\mathbb{C}}$-triple $H(k)$ is a $\mathbb{C}$-Hodge triple H of weight $w-2k$.

5.2.7. Lemma. *The correspondence*

$$H = (\mathcal{H}, F'^\bullet\mathcal{H}, F''^\bullet\mathcal{H}) \longmapsto H = \big(R_F\mathcal{H}', R_F\mathcal{H}'', \mathfrak{s}\big),$$

obtained by setting

$$(\mathcal{H}', F^\bullet\mathcal{H}') := (\mathcal{H}, F'^\bullet\mathcal{H}), \quad (\mathcal{H}'', F^\bullet\mathcal{H}'') := (\mathcal{H}^*, F''^\bullet\mathcal{H}^*), \quad \mathfrak{s} := \langle\bullet,\bullet\rangle : \mathcal{H}\otimes\mathcal{H}^\vee \to \mathbb{C}$$

(recall that $F''^\bullet\mathcal{H}^*$ is obtained by duality from $\overline{F''^\bullet\mathcal{H}}$) is an equivalence between $\mathsf{HS}(\mathbb{C}, w)$ *and the category of $\mathbb{C}$-Hodge triples of weight w.* □

From now on, we will not distinguish between $\mathbb{C}$-Hodge structures of weight w and $\mathbb{C}$-Hodge triples of weight w.

5.2.8. Lemma. *Assume we have a decomposition $H = H_1 \oplus H_2$ of $\widetilde{\mathbb{C}}$-triples. If H is $\mathbb{C}$-Hodge of weight w, so are H_1 and H_2.*

Proof. First, H_1 and H_2 must be strict, hence correspond to filtered triples. The non-degeneracy of $\mathfrak{s}_1$ and $\mathfrak{s}_2$ is also clear. Lastly, we use the interpretation 5.2.a(c) of w-oppositeness and the standard property that, if a vector bundle on $\mathbb{P}^1$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(w)^d$, then any direct summand is isomorphic to a power of $\mathcal{O}_{\mathbb{P}^1}(w)$. □

***5.2.9. Definition* (Pre-polarization of weight w of a $\widetilde{\mathbb{C}}$-triple)**

A *pre-polarization* of weight w of a $\widetilde{\mathbb{C}}$-triple T is an isomorphism

$$\mathrm{S} = (\widetilde{\mathcal{S}}', \widetilde{\mathcal{S}}'') : T \xrightarrow{\sim} T^*(-w)$$

which is Hermitian, in the sense that its Hermitian adjoint

$$\mathrm{S}^* = (\widetilde{\mathcal{S}}'', \widetilde{\mathcal{S}}') : (T^*(-w))^* = T(w) \xrightarrow{\sim} T^*,$$

which defines an isomorphism denoted in the same way $\mathrm{S}^* : T \xrightarrow{\sim} T^*(-w)$, satisfies

$$\mathrm{S}^* = \mathrm{S}.$$

The Tate twist acts on a pre-polarized $\widetilde{\mathbb{C}}$-triple (T, S) of weight w by the formula

$$(T,\mathrm{S})(\ell) = (T(\ell), (-1)^\ell\mathrm{S})$$

(see Notation 5.2.3 for the notation $H(\ell)$).

Let us make explicit this definition. Since the Hermitian dual T^* of a $\widetilde{\mathbb{C}}$-triple $T = (\widetilde{\mathcal{H}}', \widetilde{\mathcal{H}}'', \mathfrak{s})$ is nothing but the triple $(\widetilde{\mathcal{H}}'', \widetilde{\mathcal{H}}', \mathfrak{s}^*)$, we have

$$T^*(-w) = (z^{-w}\widetilde{\mathcal{H}}'', z^w\widetilde{\mathcal{H}}', \mathfrak{s}^*),$$

and a morphism $\mathrm{S} : T \to T^*(-w)$ is nothing but a pair $(\widetilde{\mathcal{S}}', \widetilde{\mathcal{S}}'')$, with $\widetilde{\mathcal{S}}' : \widetilde{\mathcal{H}}' \to z^{-w}\widetilde{\mathcal{H}}''$ and $\widetilde{\mathcal{S}}'' : z^w\widetilde{\mathcal{H}}' \to \widetilde{\mathcal{H}}''$ satisfying the compatibility property with $\mathfrak{s}$ and $\mathfrak{s}^*$, that is, for every $v_1', v_2' \in \mathcal{H}'$,

$$\mathfrak{s}(v_1', \overline{\mathcal{S}''v_2'}) = \mathfrak{s}^*(\mathcal{S}'v_1', \overline{v_2'}) =: \overline{\mathfrak{s}(v_2', \overline{\mathcal{S}'v_1'})}.$$

That S is Hermitian, i.e., $\mathrm{S}^* = \mathrm{S}$, means $\widetilde{\mathcal{S}}'' = z^w\widetilde{\mathcal{S}}'$. In other words, considering morphisms of filtered vector spaces, we have

$$\mathcal{S}' = \mathcal{S}'' : (\mathcal{H}', F^\bullet\mathcal{H}') \xrightarrow{\sim} (\mathcal{H}'', F^\bullet\mathcal{H}'')(-w). \tag{5.2.10}$$

As a consequence of $\mathcal{S}' = \mathcal{S}''$, the compatibility property reads

$$\mathfrak{s}(v_1', \overline{\mathcal{S}'v_2'}) = \overline{\mathfrak{s}(v_2', \overline{\mathcal{S}'v_1'})}. \tag{5.2.11}$$

This is equivalent to the property that the pairing $\mathcal{S}$ of $\mathbb{C}$-vector spaces defined by

$$\mathcal{S}(\bullet, \overline{\bullet}) := \mathfrak{s}(\bullet, \overline{\mathcal{S}'\bullet}) : \mathcal{H}' \otimes \overline{\mathcal{H}'} \longrightarrow \mathbb{C} \tag{5.2.12}$$

is Hermitian in the usual sense. We call $(\widetilde{\mathcal{H}}', \mathcal{S})$ the *Hermitian pair* attached to the pre-polarized $\widetilde{\mathbb{C}}$-triple (T, S) of weight w. Note that the weight w does not appear in the definition of a Hermitian pair. In fact, a Hermitian pair can give rise to a pre-polarized $\widetilde{\mathbb{C}}$-triple of any weight, as a consequence of the lemma below.

5.2.13. Lemma. *A pre-polarized $\widetilde{\mathbb{C}}$-triple (T, S) of weight w is isomorphic to the pre-polarized $\widetilde{\mathbb{C}}$-triple $((\widetilde{\mathcal{H}}', \widetilde{\mathcal{H}}'(w), \mathcal{S}), (\mathrm{Id}, \mathrm{Id}))$ of weight w. Two pre-polarized $\widetilde{\mathbb{C}}$-triples of the same weight w are isomorphic if and only if their associated Hermitian pairs are isomorphic.*

Proof. The second part follows from the first one. Let $\mathrm{S} = (\widetilde{\mathcal{S}}', \widetilde{\mathcal{S}}'' = \widetilde{\mathcal{S}}')$ be a pre-polarization of T of weight w. Then $(\mathrm{Id}, \widetilde{\mathcal{S}}')$ is an isomorphism $T \xrightarrow{\sim} (\widetilde{\mathcal{H}}', \widetilde{\mathcal{H}}'(w), \mathfrak{s}')$ with $\mathfrak{s}'(\bullet, \overline{\bullet}) = \mathfrak{s}(\bullet, \overline{\mathcal{S}'\bullet}) = \mathcal{S}$. □

We can now give the definition of a polarized $\mathbb{C}$-Hodge triple.

5.2.14. Definition **(Polarization of a $\mathbb{C}$-Hodge triple).** Let H be a $\mathbb{C}$-Hodge triple of weight w. A *polarization of H* is a pre-polarization $\mathrm{S} = (\widetilde{\mathcal{S}}', \widetilde{\mathcal{S}}'')$ of weight w of the underlying $\widetilde{\mathbb{C}}$-triple such that the associated filtered Hermitian pair $((\mathcal{H}, F^\bullet\mathcal{H}), \mathcal{S})$, with $(\mathcal{H}, F^\bullet\mathcal{H}) = (\mathcal{H}', F^\bullet\mathcal{H}')$ and $\mathcal{S}$ defined by (5.2.12), is a polarized Hodge structure of weight w in the sense of Section 2.5.18.

5.2.15. Tate twist of a Hermitian pair. The isomorphisms of Lemma 5.2.13 behave well with respect to Tate twist, that is, for a pre-polarized $\widetilde{\mathbb{C}}$-triple (T, S), we have

$$(T, \mathrm{S})(\ell) \simeq ((\widetilde{\mathcal{H}}'(\ell), \widetilde{\mathcal{H}}'(w-\ell), (-1)^{\ell}\mathcal{S}), (\mathrm{Id}, \mathrm{Id})),$$

and Tate twist reads as follows on the associated Hermitian pair $(\widetilde{\mathcal{H}}', \mathcal{S})$:

$$(\widetilde{\mathcal{H}}', \mathcal{S})(\ell) = (\widetilde{\mathcal{H}}'(\ell), (-1)^{\ell}\mathcal{S}).$$

If $(H, \mathrm{S}) = ((\widetilde{\mathcal{H}}', \widetilde{\mathcal{H}}'', \mathfrak{s}), \mathrm{S})$ is a polarized $\mathbb{C}$-Hodge triple of weight w, the pair $(H, \mathrm{S})(\ell) := (H(\ell), (-1)^{\ell}\mathrm{S})$ is a polarized $\mathbb{C}$-Hodge triple of weight $w - 2\ell$.

The relation with polarized $\mathbb{C}$-Hodge structures in the form of Hodge-Hermitian pairs (see Section 2.5.18) can now be expressed in a simpler way.

5.2.16. Proposition. *Let $T = (\widetilde{\mathcal{H}}', \widetilde{\mathcal{H}}'', \mathfrak{s})$ be an object of $\widetilde{\mathbb{C}}$-$\mathsf{Triples}$. It is a polarizable $\mathbb{C}$-Hodge triple of weight w if and only if it is isomorphic (in $\widetilde{\mathbb{C}}$-$\mathsf{Triples}$) to the object $(\widetilde{\mathcal{H}}', \widetilde{\mathcal{H}}'(w), \mathfrak{s}')$ for some suitable $\mathfrak{s}'$, such that $\widetilde{\mathcal{H}}'$ is strict and the corresponding filtered Hermitian pair $((\mathcal{H}, F^{\bullet}\mathcal{H}), \mathcal{S}) := ((\mathcal{H}', F^{\bullet}\mathcal{H}'), \mathfrak{s}')$ is a polarized Hodge structure of weight w (in particular, $\mathfrak{s}'$ is Hermitian).*

Proof. The "only if" part directly follows from Lemma 5.2.13 and the definition. Conversely, given a polarized Hodge structure $((\mathcal{H}, F^{\bullet}\mathcal{H}), \mathcal{S})$ of weight w, one checks that $((\mathcal{H}, F^{\bullet}\mathcal{H}), (\mathcal{H}, F[w]^{\bullet}\mathcal{H}), \mathfrak{s}' := \mathcal{S})$ is a $\mathbb{C}$-Hodge triple of weight w and that $(\mathrm{Id}, \mathrm{Id})$ is a polarization of it. If

$$\varphi = (\widetilde{\varphi}', \widetilde{\varphi}'') : T \xrightarrow{\ \sim\ } ((\mathcal{H}, F^{\bullet}\mathcal{H}), (\mathcal{H}, F[w]^{\bullet}\mathcal{H}), \mathfrak{s}')$$

is an isomorphism in $\widetilde{\mathbb{C}}$-$\mathsf{Triples}$, then T is a $\mathbb{C}$-Hodge triple and, setting $\widetilde{\mathcal{S}}' := \widetilde{\varphi}''^{-1}\widetilde{\varphi}'$, $\mathrm{S} := (\widetilde{\mathcal{S}}', \widetilde{\mathcal{S}}'' = \widetilde{\mathcal{S}}')$ is a polarization of T. $\square$

***5.2.17. Remark* (Two points of view on (pre-)polarized triples)**

The sesquilinear pairing $\mathfrak{s}$ is constitutive of the notion of a triple and is only used to reflect the oppositeness of filtrations (with no positivity involved). On the other hand, a (pre-)polarization can be regarded as a "sesquilinear pairing on triples". We thus have two distinct roles for a sesquilinear pairing, that we also distinguish with the notation.

Lemma 5.2.13 helps us to simplify the setting, by reducing the polarization S to identity, and transferring the positivity property to the sesquilinear pairing of the triple. There remains only one sesquilinear pairing involved.

While we can simplify in that way the presentation of *polarized* triples, we still have to keep the ordinary sesquilinear pairing $\mathfrak{s}$ for *polarizable* triples.

5.3. Hodge-Lefschetz triples

We now make explicit the notion of Hodge-Lefschetz structures, and $\mathfrak{sl}_2$-Hodge structure (Definitions 3.3.3, 3.4.3 and 3.2.7) in the language of $\widetilde{\mathbb{C}}$-triples of Section 5.2.

5.3.a. Lefschetz triples. The abelian category A is that of $\widetilde{\mathbb{C}}$-triples with its automorphism σ (see Section 3.3.4) given by the Tate twist (1). Let $H = (\widetilde{\mathcal{H}}', \widetilde{\mathcal{H}}'', \mathfrak{s})$ be a $\widetilde{\mathbb{C}}$-triple. Recall that $(\widetilde{\mathcal{H}}', \widetilde{\mathcal{H}}'', \mathfrak{s})(-1) = (\widetilde{\mathcal{H}}'(-1), \widetilde{\mathcal{H}}''(1), \mathfrak{s})$. Note also that giving a morphism $\widetilde{\mathcal{H}}''(1) \to \widetilde{\mathcal{H}}''$ is equivalent to giving a morphism $\widetilde{\mathcal{H}}'' \to \widetilde{\mathcal{H}}''(-1)$.

Assume that $H = (\widetilde{\mathcal{H}}', \widetilde{\mathcal{H}}'', \mathfrak{s})$ is equipped with a nilpotent endomorphism $\mathrm{N} = (\mathrm{N}', \mathrm{N}'') : H \to H(-1)$, that is,

$$\mathrm{N}' : \widetilde{\mathcal{H}}' \longrightarrow \widetilde{\mathcal{H}}'(-1) \qquad \text{and} \qquad \mathrm{N}'' : \widetilde{\mathcal{H}}''(1) \longrightarrow \widetilde{\mathcal{H}}''$$

which also reads, when H is *strict*,

$$\mathrm{N}' : (\mathcal{H}', F^\bullet\mathcal{H}') \longrightarrow (\mathcal{H}', F[-1]^\bullet\mathcal{H}') \quad \text{and} \quad \mathrm{N}'' : (\mathcal{H}'', F[1]^\bullet\mathcal{H}'') \longrightarrow (\mathcal{H}'', F^\bullet\mathcal{H}''),$$

are two nilpotent morphisms which satisfy, when forgetting the filtration,

$$\mathfrak{s}(v', \overline{\mathrm{N}''v''}) = \mathfrak{s}(\mathrm{N}'v', \overline{v''}), \quad v' \in \mathcal{H}',\ v'' \in \mathcal{H}''. \tag{5.3.1}$$

***5.3.2. Definition* (Hermitian dual of (H, N)).** The Hermitian dual $(H, \mathrm{N})^*$ of (H, N) is (H^*, N^*), where H^* is the Hermitian dual of H and N^* is the Hermitian adjoint of the morphism N, regarded as a morphism $H^* \to H^*(-1)$.

In other words, $H^* = (\widetilde{\mathcal{H}}'', \widetilde{\mathcal{H}}', \mathfrak{s}^*)$ and $\mathrm{N}^* = (\mathrm{N}'', \mathrm{N}')$. The monodromy filtration is defined in the abelian category $\widetilde{\mathbb{C}}$-$\mathsf{Triples}$. Let us make it explicit. The monodromy filtration $\mathrm{M}(\mathrm{N}')_\bullet\widetilde{\mathcal{H}}'$ exists in the abelian category of graded $\widetilde{\mathbb{C}}$-modules, as well as $\mathrm{M}(\mathrm{N}'')_\bullet\widetilde{\mathcal{H}}''$. By restricting to $z = 1$, it induces the monodromy filtration of $(\mathcal{H}', \mathrm{N}')$ resp. $(\mathcal{H}'', \mathrm{N}'')$. We note that, according to (5.3.1), $\mathfrak{s}$ induces zero on $\mathrm{M}_\ell\mathcal{H}' \otimes \overline{\mathrm{M}_{-\ell-1}\mathcal{H}''}$ for every ℓ, hence induces a sesquilinear pairing

$$\mathfrak{s} : \mathrm{M}_\ell\mathcal{H}' \otimes \overline{\mathcal{H}''/\mathrm{M}_{-\ell-1}\mathcal{H}''} \longrightarrow \mathbb{C}.$$

We then have

$$\mathrm{M}_\ell T = (\mathrm{M}_\ell\widetilde{\mathcal{H}}', \widetilde{\mathcal{H}}''/\mathrm{M}_{-\ell-1}\widetilde{\mathcal{H}}'', \mathfrak{s}).$$

Let us also consider the induced pairing

$$\mathfrak{s}_{\ell,-\ell} : \mathrm{gr}^{\mathrm{M}}_\ell\mathcal{H}' \otimes_{\widetilde{\mathbb{C}}} \overline{\mathrm{gr}^{\mathrm{M}}_{-\ell}\mathcal{H}''} \longrightarrow \mathbb{C}.$$

Then

$$\mathrm{gr}^{\mathrm{M}}_\ell H = (\mathrm{gr}^{\mathrm{M}}_\ell\widetilde{\mathcal{H}}', \mathrm{gr}^{\mathrm{M}}_{-\ell}\widetilde{\mathcal{H}}'', \mathfrak{s}_{\ell,-\ell}).$$

On the other hand, $\mathrm{gr}\mathrm{N} := (\mathrm{gr}\mathrm{N}', \mathrm{gr}\mathrm{N}'')$ induces a morphism

$$\begin{aligned}\mathrm{gr}^{\mathrm{M}}_\ell H = (\mathrm{gr}^{\mathrm{M}}_\ell\widetilde{\mathcal{H}}', \mathrm{gr}^{\mathrm{M}}_{-\ell}\widetilde{\mathcal{H}}'', \mathfrak{s}_{\ell,-\ell})& \\ \longrightarrow (\mathrm{gr}^{\mathrm{M}}_{\ell-2}\widetilde{\mathcal{H}}'(-1), \mathrm{gr}^{\mathrm{M}}_{-\ell+2}\widetilde{\mathcal{H}}''(1), \mathfrak{s}_{\ell-2,-\ell+2}) &= \mathrm{gr}^{\mathrm{M}}_{\ell-2}H(-1).\end{aligned}$$

5.3.1. ***Pre-polarization of weight*** w. A pre-polarization S of weight w of a Lefschetz triple (H,N) is a Hermitian isomorphism $(H,\mathrm{N})\xrightarrow{\sim}(H,\mathrm{N})^*(-w)$, i.e., an isomorphism $H\xrightarrow{\sim}H^*(-w)$ such that

$$\mathrm{S}\circ\mathrm{N}=\mathrm{N}^*\circ\mathrm{S}:H\longrightarrow H^*(-w-1).$$

More explicitly, setting $\mathrm{S}=(\widetilde{\mathcal{S}}',\widetilde{\mathcal{S}}'')$, we have $\widetilde{\mathcal{S}}''=z^w\widetilde{\mathcal{S}}'$ and

$$\widetilde{\mathcal{S}}'\circ\mathrm{N}'=\mathrm{N}''\circ\widetilde{\mathcal{S}}':\widetilde{\mathcal{H}}'\longrightarrow\widetilde{\mathcal{H}}''(-w-1).$$

In particular, the associated sesquilinear pairing $\mathcal{S}(\bullet,\overline{\bullet})=\mathfrak{s}(\bullet,\overline{\mathcal{S}'\bullet})$ on $\mathcal{H}'$ (see (5.2.12)) satisfies

$$\mathcal{S}(\mathrm{N}'\bullet,\overline{\bullet})=\mathcal{S}(\bullet,\overline{\mathrm{N}'\bullet})$$

because

$$\mathfrak{s}(\bullet,\overline{\mathcal{S}'\mathrm{N}'\bullet})=\mathfrak{s}(\bullet,\overline{\mathrm{N}''\mathcal{S}'\bullet})=\mathfrak{s}(\mathrm{N}'\bullet,\overline{\mathcal{S}'\bullet}).$$

Since S is a morphism, it is compatible with the monodromy filtrations and S induces a pre-polarization

$$\mathrm{gr}^{\mathrm{M}}\mathrm{S}:\mathrm{gr}^{\mathrm{M}}_{\bullet}H\longrightarrow\mathrm{gr}^{\mathrm{M}}_{\bullet}(H^*)(-w)=(\mathrm{gr}^{\mathrm{M}}_{-\bullet}H)^*(-w)$$

of the graded Lefschetz triple $(\mathrm{gr}^{\mathrm{M}}_{\bullet}H,\mathrm{grN})$.

5.3.b. Hodge-Lefschetz triples. Let (H,N) be a Lefschetz triple. We say that (H,N) is a Hodge-Lefschetz triple with central weight w if $\mathrm{gr}^{\mathrm{M}}_{\ell}H$ is a Hodge triple of weight $w+\ell$ for every ℓ. In such a case, for every $j,k\in\mathbb{Z}$,

$$(H,\mathrm{N})(j,k):=((\widetilde{\mathcal{H}}'(j),\mathrm{N}'),(\widetilde{\mathcal{H}}''(-k),\mathrm{N}''),\mathfrak{s})$$

is a Hodge-Lefschetz triple with central weight $w-(k+\ell)$ and $(H,\mathrm{N})^*$ is a Hodge-Lefschetz triple with central weight $-w$, with monodromy filtration satisfying $\mathrm{gr}^{\mathrm{M}}_{\ell}(H^*)=(\mathrm{gr}^{\mathrm{M}}_{-\ell}H)^*$.

Therefore, the data $(\mathrm{gr}^{\mathrm{M}}_{\bullet}H,\mathrm{grN})$ defined as

$$\bigoplus_{\ell}(\mathrm{gr}^{\mathrm{M}}_{\ell}\widetilde{\mathcal{H}}',\mathrm{gr}^{\mathrm{M}}_{-\ell}\widetilde{\mathcal{H}}'',\mathfrak{s}_{\ell,-\ell}),\qquad\mathrm{grN}:=(\mathrm{grN}',\mathrm{grN}'')$$

form an $\mathfrak{sl}_2$-Hodge triple. In particular, each $\mathfrak{s}_{\ell,-\ell}$ is non-degenerate, which implies that $\mathfrak{s}$ itself is non-degenerate. Its Hermitian dual $(\mathrm{gr}^{\mathrm{M}}_{\bullet}H,\mathrm{grN})^*$ is also an $\mathfrak{sl}_2$-Hodge triple.

5.3.3. Remark **(Stability by extension).** We consider the abelian category of graded $\widetilde{\mathbb{C}}$-triples $H=\bigoplus_{\ell}H_{\ell}$ equipped with a nilpotent endomorphism $\mathrm{N}:H_{\ell}\to H_{\ell-2}(-1)$. Let

$$0\longrightarrow(H_1,\mathrm{N}_1)\longrightarrow(H,\mathrm{N})\longrightarrow(H_2,\mathrm{N}_2)\longrightarrow 0$$

be an exact sequence in this category. Assume that $(H_1,\mathrm{N}_1),(H_2,\mathrm{N}_2)$ are $\mathfrak{sl}_2$-Hodge triples of the same weight w. Then (H,N) is of the same kind. Indeed, by Exercise 5.8, each summand H_{ℓ} is a $\mathbb{C}$-Hodge triple of weight $w+\ell$. It is then clear that N^{ℓ} is an isomorphism $H_{\ell}\xrightarrow{\sim}H_{-\ell}(-\ell)$ if this holds on H_1,H_2.

5.3.4. Polarization. Let (H, N) be a Hodge-Lefschetz triple of weight w. By a *polarization* S of (H, N) we mean a pre-polarization of weight w of the Lefschetz triple (Section 5.3.1) which satisfies the properties as in Definition 3.4.14.

Lastly, we remark as in Proposition 5.2.16 that any polarized Hodge-Lefschetz triple with central weight w is isomorphic to $((\widetilde{\mathcal{H}}, \widetilde{\mathcal{H}}(w), \mathcal{S}), \mathrm{N})$ for a suitable polarized Hodge-Lefschetz structure $((\mathcal{H}, F^\bullet\mathcal{H}), \mathrm{N}, \mathcal{S})$ with central weight w, as in Remark 3.4.16 (in particular, $\mathcal{S} : \mathcal{H} \to \mathcal{H}^*$ is Hermitian).

5.3.5. The polarized Hodge-Lefschetz triple **$\mathrm{Im}\,\mathrm{N}$.** Let $(H, \mathrm{N}, \mathrm{S})$ be a polarized Hodge-Lefschetz triple with central weight $w-1$. We have by definition a commutative diagram

$$\begin{tikzcd} H \arrow[r, "\mathrm{S}"] \arrow[d, "\mathrm{N}"'] & H^*(-w+1) \arrow[d, "\mathrm{N}^*"] \\ H(-1) \arrow[r, "\mathrm{S}"] & H^*(-w) \end{tikzcd}$$

Then $G = \mathrm{Im}\,\mathrm{N}$ equipped with the nilpotent endomorphism $\mathrm{N}_G = \mathrm{N}_{|\,\mathrm{Im}\,\mathrm{N}}$ is a Hodge-Lefschetz triple with central weight w polarized by S_G, which is the sesquilinear pairing

$$\mathrm{S}_G : \mathrm{Im}\,\mathrm{N} \longrightarrow (\mathrm{Im}\,\mathrm{N})^* = \mathrm{Coker}(\mathrm{N}^*)$$

induced by $-\mathrm{S}$. The image $\mathrm{Im}\,\mathrm{N} = G$ is expressed as follows:

$$\begin{aligned} (\mathcal{G}', F^\bullet\mathcal{G}') &= \big(\mathrm{N}'(\mathcal{H}'), \mathrm{N}'(F[-1]^\bullet\mathcal{H}') = F[-1]^\bullet\mathcal{H}' \cap \mathrm{N}'(\mathcal{H}')\big) \\ (\mathcal{G}'', F^\bullet\mathcal{G}'') &= \big((\mathcal{H}''/\,\mathrm{Ker}\,\mathrm{N}''), (F[1]^\bullet\mathcal{H}'')/(F[1]^\bullet\mathcal{H}'' \cap \mathrm{Ker}\,\mathrm{N}'')\big) \\ \mathfrak{s}_G &= \mathfrak{s}_{|\mathrm{N}'(\mathcal{H}') \otimes \overline{(\mathcal{H}''/\,\mathrm{Ker}\,\mathrm{N}'')}}, \end{aligned}$$

and $\mathrm{N}'_{\mathcal{G}}, \mathrm{N}''_{\mathcal{G}}$ are the naturally induced nilpotent endomorphisms. It can also be presented as a filtered Hermitian pair

$$\big((\mathcal{G}, F^\bullet\mathcal{G}), \mathrm{N}_{\mathcal{G}}, \mathcal{S}_{\mathcal{G}}\big)$$

obtained from the filtered Hermitian pair $\big((\mathcal{H}, F^\bullet\mathcal{H}), \mathrm{N}, \mathcal{S}\big)$ by setting

$$(\mathcal{G}, F^\bullet\mathcal{G}) = (\mathrm{N}(\mathcal{H}), F[-1]^\bullet\mathcal{H} \cap \mathrm{N}(\mathcal{H}))$$

with the induced action $\mathrm{N}_{\mathcal{G}}$ of N and by defining $\mathcal{S}_{\mathcal{G}}$ by (see Definition 3.2.12)

$$\mathcal{S}_{\mathcal{G}}(\mathrm{N}\bullet, \overline{\mathrm{N}\bullet}) = -\mathcal{S}(\mathrm{N}\bullet, \overline{\bullet}) = -\mathcal{S}(\bullet, \overline{\mathrm{N}\bullet}).$$

5.3.6. Polarized Hodge-Lefschetz quivers. The definition of a polarized Hodge-Lefschetz quiver in the setting of triples can be mimicked from that of a polarized Hodge-Lefschetz quiver of Section 3.4.d.

Given two Lefschetz quivers in the category of $\widetilde{\mathbb{C}}$-modules

$$\begin{tikzcd} \widetilde{\mathcal{H}}' \arrow[r, bend left, "\mathrm{c}'"] & \widetilde{\mathcal{G}}' \arrow[l, bend left, "\mathrm{v}'", "(-1)"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \widetilde{\mathcal{H}}'' \arrow[r, bend left, "\mathrm{c}''"] & \widetilde{\mathcal{G}}'' \arrow[l, bend left, "\mathrm{v}''", "(-1)"'] \end{tikzcd}$$

(notation of Remark 5.1.5) with $\mathrm{N}' = \mathrm{v}'\mathrm{c}'$, etc., and sesquilinear pairings $\mathfrak{s}_H, \mathfrak{s}_G$ giving rise to $\widetilde{\mathbb{C}}$-triples

$$H = (\widetilde{\mathcal{H}}', \widetilde{\mathcal{H}}''(-1), \mathfrak{s}_H) \quad \text{and} \quad G = (\widetilde{\mathcal{G}}', \widetilde{\mathcal{G}}'', \mathfrak{s}_G)$$

we can build up a Lefschetz quiver $(H, G, \mathrm{c}, \mathrm{v})$ in the category $\widetilde{\mathbb{C}}$-Triples by setting

$$\begin{aligned} \mathrm{c} &= (\mathrm{c}', -\mathrm{v}'') : (\widetilde{\mathcal{H}}', \widetilde{\mathcal{H}}''(-1), \mathfrak{s}_H) \longrightarrow (\widetilde{\mathcal{G}}', \widetilde{\mathcal{G}}'', \mathfrak{s}_G) \\ \mathrm{v} &= (\mathrm{v}', -\mathrm{c}'') : (\widetilde{\mathcal{G}}', \widetilde{\mathcal{G}}'', \mathfrak{s}_G) \longrightarrow (\widetilde{\mathcal{H}}', \widetilde{\mathcal{H}}''(-1), \mathfrak{s}_H)(-1) = (\widetilde{\mathcal{H}}'(-1), \widetilde{\mathcal{H}}'', \mathfrak{s}_H), \end{aligned} \tag{5.3.7}$$

provided the compatibility relations of $\mathrm{c}', \dots, \mathrm{v}''$ with the sesquilinear pairings $\mathfrak{s}_H, \mathfrak{s}_G$ hold, that is,

$$\begin{aligned} \mathfrak{s}_G(\mathrm{c}'x', \overline{y''}) &= -\mathfrak{s}_H(x', \overline{\mathrm{v}''y''}) \quad x' \in \mathcal{H}',\ y'' \in \mathcal{G}'', \\ \mathfrak{s}_G(\mathrm{v}'y', \overline{x''}) &= -\mathfrak{s}_H(y', \overline{\mathrm{c}''x''}) \quad x'' \in \mathcal{H}'',\ y' \in \mathcal{G}'. \end{aligned} \tag{5.3.8}$$

The choice of signs is made to ensure later compatibility with the signs occurring in Definition 3.4.19. The signs cancel out when defining $\mathrm{N}_H = \mathrm{vc}$ and $\mathrm{N}_G = \mathrm{cv}$. Furthermore, they are compatible with the definition of the Hermitian dual of a Lefschetz quiver given in Remark 3.2.13. Indeed, working now in the category of $\mathbb{C}$-vector spaces, and recalling that the Hermitian dual of $(\mathcal{H}'', \mathcal{G}'', \mathrm{c}'', \mathrm{v}'')$ is $(\mathcal{H}'', \mathcal{G}'', \mathrm{c}'', \mathrm{v}'')^* = (\mathcal{H}''^*, \mathcal{G}''^*, -\mathrm{v}'', -\mathrm{c}'')$, the relations (5.3.8) amount to the property that the pair $(\mathfrak{s}_H, \mathfrak{s}_G)$ is a morphism of Lefschetz quivers

$$(\mathfrak{s}_H, \mathfrak{s}_G) : (\mathcal{H}', \mathcal{G}', \mathrm{c}', \mathrm{v}') \longrightarrow (\mathcal{H}'', \mathcal{G}'', \mathrm{c}'', \mathrm{v}'')^*.$$

One can check that, conversely, any Lefschetz quiver $(H, G, \mathrm{c}, \mathrm{v})$ in the category $\widetilde{\mathbb{C}}$-Triples is obtained by the previous construction.

A Hodge-Lefschetz quiver with central weight w is defined as a Lefschetz quiver in $\widetilde{\mathbb{C}}$-Triples such that $(H, \mathrm{N}_H), (G, \mathrm{N}_G)$ are Hodge-Lefschetz triples with respective weights $w-1$ and w (Section 5.3.b), and $\mathrm{c} : (H, \mathrm{N}_H) \to (G, \mathrm{N}_G)$ and $\mathrm{v} : (G, \mathrm{N}_G) \to (H(-1), \mathrm{N}_H)$ are morphisms in the category of Hodge-Lefschetz triples, so that they are morphisms of mixed Hodge structures.

Defining the Hermitian dual $(H, G, \mathrm{c}, \mathrm{v})^* = (H^*(1), G^*, -\mathrm{v}^*, -\mathrm{c}^*)$ as in Remark 3.2.13, a polarization of a Hodge-Lefschetz quiver $(H, G, \mathrm{c}, \mathrm{v})$ is a pair of polarizations $\mathrm{S} = (\mathrm{S}_H, \mathrm{S}_G)$ of H and G respectively, defining an isomorphism

$$\mathrm{S} : (H, G, \mathrm{c}, \mathrm{v}) \xrightarrow{\sim} (H, G, \mathrm{c}, \mathrm{v})^*(-w). \tag{5.3.9}$$

Assume now that H and G are presented as Hermitian pairs

$$H = (\widetilde{\mathcal{H}}', \widetilde{\mathcal{H}}'(-w+1), \mathcal{S}_H) \quad \text{and} \quad G = (\widetilde{\mathcal{G}}', \widetilde{\mathcal{G}}'(-w), \mathcal{S}_G),$$

so that $\mathrm{S}_H = (\mathrm{Id}, \mathrm{Id})$ and $\mathrm{S}_G = (\mathrm{Id}, \mathrm{Id})$, and $(\widetilde{\mathcal{H}}', \mathcal{S}_H), (\widetilde{\mathcal{G}}', \mathcal{S}_G)$ are polarized Hodge structures of respective weights $w-1, w$ in the sense of Section 2.5.18. In this setting, (5.3.9) implies that $\mathrm{c} = -\mathrm{v}^*$, equivalently $\mathrm{v} = -\mathrm{c}^*$, so that, by their definition (5.3.7), we obtain $\mathrm{c}' = \mathrm{c}''$ and $\mathrm{v}' = \mathrm{v}''$. Therefore, (5.3.8) reads

$$\mathcal{S}_G(\mathrm{c}'x', \overline{y'}) = -\mathcal{S}_H(x', \overline{\mathrm{v}'y'}) \quad \text{and} \quad \mathcal{S}_G(\mathrm{v}'y', \overline{x'}) = -\mathcal{S}_H(y', \overline{\mathrm{c}'x'}).$$

5.4. Variations of Hodge triple

5.4.a. Variations of Hodge structure as triples. We now revisit the notion of variation of Hodge structure, by the using the language of $\widetilde{\mathbb{C}}$-triples of Section 5.2. It enables us to keep holomorphy for both filtrations, by putting the non-holomorphic behaviour in the sesquilinear pairing $\mathfrak{s}$. This approach will be convenient in presence of singularities.

When working with a pairing $\mathfrak{s}$, we start by introducing a larger category, which can be enlarged to an abelian category.

We denote by $\widetilde{\mathcal{O}}_X$ the sheaf of graded rings $\mathcal{O}_X[z]$ and by an $\widetilde{\mathcal{O}}_X$-module we mean a *graded* $\mathcal{O}_X[z]$-module. By a locally free $\widetilde{\mathcal{O}}_X$-module of rank $r<\infty$ we mean an $\widetilde{\mathcal{O}}_X$-module locally isomorphic to the direct sum (see Exercise 5.4)

$$\bigoplus_{i=1}^r \widetilde{\mathcal{O}}_X(k_i) \quad (k_i\in\mathbb{Z}).$$

We replace the filtered flat bundles $(\mathcal{H}',\nabla,F^\bullet\mathcal{H}')$ and $(\mathcal{H}'',\nabla,F^\bullet\mathcal{H}'')$ by graded $\widetilde{\mathcal{O}}_X$-modules with a flat z-connection (see Section 5.1.c). (This point of view will be expanded in Section 8.1.)

***5.4.1. Definition* (Flat $\widetilde{\mathcal{O}}$-triples with pairing $\mathfrak{s}$).** A *flat* $\widetilde{\mathcal{O}}$*-triple* on X consists of the data of

- a pair of $\widetilde{\mathcal{O}}_X$-modules $\widetilde{\mathcal{H}}',\widetilde{\mathcal{H}}''$ equipped with a flat z-connection $\widetilde{\nabla}$,
- a flat $\mathcal{O}_X\otimes_{\mathbb{C}}\mathcal{O}_{\overline{X}}$-linear morphism $\mathfrak{s}:\mathcal{H}'\otimes_{\mathbb{C}}\overline{\mathcal{H}''}\to\mathcal{C}^\infty_X$, i.e., for local holomorphic sections m',m'' of $\mathcal{H}',\mathcal{H}''$, we have

$$\begin{aligned}\partial\mathfrak{s}(m',\overline{m''})&=\mathfrak{s}(\nabla m',\overline{m''}),\\ \overline{\partial}\mathfrak{s}(m',\overline{m''})&=\mathfrak{s}(m',\overline{\nabla m''}).\end{aligned}$$

***5.4.2. Remark* (Flatness of $\mathfrak{s}$).** The restriction $\underline{\mathfrak{s}}$ of $\mathfrak{s}$ to the local system $\underline{\mathcal{H}}'\otimes_{\mathbb{C}}\overline{\underline{\mathcal{H}}''}$ takes values in the constant sheaf $\mathbb{C}_X$ since for local sections m' of $\underline{\mathcal{H}}'$ and m'' of $\underline{\mathcal{H}}''$, we have, by the previous formulas, $\partial\mathfrak{s}(m',\overline{m''})=\overline{\partial}\mathfrak{s}(m',\overline{m''})=0$. Moreover, we can recover $\mathfrak{s}$ from its restriction $\underline{\mathfrak{s}}$ by $\mathcal{O}_X\otimes_{\mathbb{C}}\mathcal{O}_{\overline{X}}$-linearity. As a consequence, we see that if X is connected, $\mathfrak{s}$ is non-degenerate if and only if its restriction at some point $x\in X$ is a non-degenerate pairing $\mathcal{H}'_x\otimes_{\mathbb{C}}\overline{\mathcal{H}''_x}\to\mathbb{C}$, since this obviously holds for $\underline{\mathfrak{s}}$.

***5.4.3. Definition* (Variation of $\mathbb{C}$-Hodge structure, third definition)**

A variation of $\mathbb{C}$-Hodge structure of weight w is a flat $\widetilde{\mathcal{O}}_X$-triple

$$H=((\widetilde{\mathcal{H}}',\widetilde{\nabla}),(\widetilde{\mathcal{H}}'',\widetilde{\nabla}),\mathfrak{s})$$

such that $\widetilde{\mathcal{H}}',\widetilde{\mathcal{H}}''$ are $\widetilde{\mathcal{O}}_X$-locally free of finite rank and whose restriction $H_x=(\widetilde{\mathcal{H}}'_x,\widetilde{\mathcal{H}}''_x,\mathfrak{s}_x)$ at each $x\in X$ is a $\mathbb{C}$-Hodge triple of weight w. In particular, $\mathfrak{s}$ is non-degenerate.

A *polarization* is a *flat* morphism $\mathrm{S}:H\to H^*(-w)$ inducing a polarization at each $x\in X$. Equivalently (see Section 2.5.18), a polarized variation of $\mathbb{C}$-Hodge structure of weight w consists of the data $((\widetilde{\mathcal{H}},\widetilde{\nabla}),\mathcal{S})$, where $\mathcal{S}$ is a flat sesquilinear pairing on $(\mathcal{H},\nabla)$, inducing a polarized $\mathbb{C}$-Hodge structure at every $x\in X$.

5.4.4. Example. The triple

$$_{\mathrm{T}}\mathcal{O}_X := ((\widetilde{\mathcal{O}}_X, \widetilde{\mathrm{d}}), (\widetilde{\mathcal{O}}_X, \widetilde{\mathrm{d}}), \mathfrak{s}_n), \qquad \mathfrak{s}_n(1,1) := 1,$$

is a variation of $\mathbb{C}$-Hodge triple of weight 0. It is polarized by $\mathrm{S} = (\mathrm{Id}, \mathrm{Id})$. The associated Hodge-Hermitian pair is $(\widetilde{\mathcal{O}}_X, \mathfrak{s}_n)$.

5.4.5. Remarks.

(1) One can also define the category $\mathsf{VHS}(X, \mathbb{C}, w)$ as the full subcategory of that of filtered flat triples whose objects are variations of $\mathbb{C}$-Hodge structures of weight w on X. The category $\mathsf{pVHS}(X, \mathbb{C}, w)$ of polarizable objects is defined correspondingly.

The category VHS can be naturally equipped with the operations Hom, tensor product, duality, and conjugation. The full subcategory pVHS is stable by these operations, since the polarization can be constructed in a natural way in each of these operations (see Section 2.5.19).

(2) Let $f : X \to Y$ be a holomorphic map between smooth complex manifolds. The pullback $_{\mathrm{T}}f^*H$ of a triple H is defined as $(f^*(\widetilde{\mathcal{H}}', \widetilde{\nabla}), f^*(\widetilde{\mathcal{H}}'', \widetilde{\nabla}), f^*\mathfrak{s})$, where $f^*\mathfrak{s} : f^*\mathcal{H}' \otimes \overline{f^*\mathcal{H}''} \to \mathcal{C}^\infty_X$ is defined by $f^*\mathfrak{s}(1 \otimes m', \overline{1 \otimes m''}) := \mathfrak{s}(m', \overline{m''}) \circ f$. If H is a variation of $\mathbb{C}$-Hodge triple of weight w, then so is $_{\mathrm{T}}f^*H$.

5.4.b. Smooth $\mathbb{C}$-Hodge triples. Let us now introduce a different normalization of the objects, in order to fit with the notion of polarizable Hodge module developed in Chapters 7 and 14. Recall that we set $n = \dim X$.

5.4.6. Definition **(The polarized $\mathbb{C}$-Hodge triple $_{\mathrm{H}}\mathcal{O}_X$).** We denote by $_{\mathrm{H}}\mathcal{O}_X$ the triple $_{\mathrm{T}}\mathcal{O}_X(0, n)$ (note the half-twist), that is,

$$_{\mathrm{H}}\mathcal{O}_X = ((\widetilde{\mathcal{O}}_X, \widetilde{\mathrm{d}}), (\widetilde{\mathcal{O}}_X(n), \widetilde{\mathrm{d}}), \mathfrak{s}_n), \qquad \mathfrak{s}_n(1,1) := 1.$$

It is a $\mathbb{C}$-Hodge triple of weight $n = \dim X$ with polarization

$$_{\mathrm{H}}\mathrm{S} = (\mathrm{Id}, \mathrm{Id}) : {_{\mathrm{H}}\mathcal{O}_X} \longrightarrow {_{\mathrm{H}}\mathcal{O}_X}(-n).$$

The associated Hermitian pair is $((\mathcal{O}_X, F^\bullet\mathcal{O}_X), \mathrm{d}), \mathfrak{s}_n)$ where $F^0\mathcal{O}_X = \mathcal{O}_X$ and $F^1\mathcal{O}_X = 0$.

A smooth $\mathbb{C}$-Hodge triple of weight w is defined to be a variation of Hodge triple of weight $w - \dim X$ (Definition 5.4.3) twisted by $_{\mathrm{H}}\mathcal{O}_X$.

5.4.7. Definition **(Smooth $\mathbb{C}$-Hodge triples).**

(1) A smooth $\mathbb{C}$-Hodge triple of weight w is a triple

$$_{\mathrm{H}}H := ((\widetilde{\mathcal{H}}', \widetilde{\nabla}), (\widetilde{\mathcal{H}}'', \widetilde{\nabla}), \mathfrak{s}),$$

such that the triple

$$H = {_{\mathrm{H}}H}(0, -n)$$

is a variation of $\mathbb{C}$-Hodge structure of weight $w - n$ on X.

(2) A polarization $_{\mathrm{H}}\mathrm{S}$ of $_{\mathrm{H}}H$ is a Hermitian morphism $_{\mathrm{H}}\mathrm{S} : {_{\mathrm{H}}H} \to {_{\mathrm{H}}H^*}(-w)$ such that, when regarded as a morphism $H \to H^*(-(w-n))$, $\mathrm{S} := {_{\mathrm{H}}\mathrm{S}}$ is a polarization of H.

(3) A smooth polarized $\mathbb{C}$-Hodge triple of weight w on X consists of the data $((\widetilde{\mathcal{H}},\widetilde{\nabla}),{}_{\mathrm{H}}\mathcal{S})$, where ${}_{\mathrm{H}}\mathcal{S}$ is a non-degenerate Hermitian pairing on $\mathcal{H}$ and ${}_{\mathrm{H}}H := ((\widetilde{\mathcal{H}},\widetilde{\nabla}),(\widetilde{\mathcal{H}},\widetilde{\nabla})(w),{}_{\mathrm{H}}\mathcal{S})$ is a smooth $\mathbb{C}$-Hodge triple of weight w polarized by ${}_{\mathrm{H}}\mathrm{S} = (\mathrm{Id},\mathrm{Id})$. Tate twist reads $((\widetilde{\mathcal{H}},\widetilde{\nabla}),{}_{\mathrm{H}}\mathcal{S})(\ell) = ((\widetilde{\mathcal{H}}(\ell),\widetilde{\nabla}),(-1)^{\ell}{}_{\mathrm{H}}\mathcal{S})$.

***5.4.8. Definition* (Pullback of a smooth $\mathbb{C}$-Hodge triple).** Let $f : X \to Y$ be a holomorphic map between smooth manifolds of relative dimension $p = n - m$, and let ${}_{\mathrm{H}}H = ((\widetilde{\mathcal{H}}',\widetilde{\nabla}),(\widetilde{\mathcal{H}}'',\widetilde{\nabla}),\mathfrak{s})$ be a smooth $\mathbb{C}$-Hodge triple of weight w on Y. The pullback ${}_{\mathrm{T}}f^*{}_{\mathrm{H}}H$ is the triple defined as

$${}_{\mathrm{T}}f^*{}_{\mathrm{H}}H := (f^*(\widetilde{\mathcal{H}}',\widetilde{\nabla}),f^*(\widetilde{\mathcal{H}}'',\widetilde{\nabla})(p),f^*\mathfrak{s}).$$

Since ${}_{\mathrm{T}}f^*{}_{\mathrm{H}}\mathcal{O}_Y = {}_{\mathrm{H}}\mathcal{O}_X$, we see that ${}_{\mathrm{T}}f^*{}_{\mathrm{H}}H = {}_{\mathrm{H}}({}_{\mathrm{T}}f^*H)$ is a smooth $\mathbb{C}$-Hodge triple of weight $w{+}p$. Moreover, the pullback ${}_{\mathrm{T}}f^*{}_{\mathrm{H}}\mathrm{S} := f^*{}_{\mathrm{H}}\mathrm{S}$ of a polarization is a polarization. Correspondingly, the pullback of a smooth polarized $\mathbb{C}$-Hodge triple $((\widetilde{\mathcal{H}},\widetilde{\nabla}),{}_{\mathrm{H}}\mathcal{S})$ of weight w is the smooth polarized $\mathbb{C}$-Hodge triple $(f^*(\widetilde{\mathcal{H}},\widetilde{\nabla}),f^*\mathcal{S})$ of weight $w+p$.

***5.4.9. Remark* (on the symmetry breaking).** Definition 5.4.7 clearly breaks the symmetry between the “prime” (or holomorphic) part and the “double prime” (or anti-holomorphic) part of a triple, in order to obtain a formalism of weights similar to that of the theory of mixed Hodge modules of M. Saito. Similarly, the definition of the pullback functor is not symmetric, and the same will occur for other functors in Chapter 12. However, *pre-polarized* triples can be reduced to Hermitian pairs, for which the problem disappears since the behaviour of weights by functors is reflected by simply changing the sign of the pre-polarization.

5.5. Exercises

Exercise 5.1. Show the following properties in $\mathsf{Mod}(\widetilde{\mathcal{A}})$ or in $\mathsf{Modgr}(\widetilde{\mathcal{A}})$.

(1) A subobject of a strict object is strict.

(2) An extension in of two strict objects is strict.

(3) A morphism between two strict objects is strict if and only if its cokernel is strict.

(4) A complex which consists of strict objects and which is bounded from above is a strict complex if and only if each differential is a strict morphism.

Exercise 5.2.

(1) If $\widetilde{\mathcal{M}}$ is a graded $\widetilde{\mathcal{A}}$-module, show that its $\widetilde{\mathbb{C}}$-torsion is also graded and each torsion element is annihilated by some power of z.

(2) Conclude that $(z-a) : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}$ is injective for every $a \in \mathbb{C} \smallsetminus \{0\}$, equivalently that $\widetilde{\mathcal{M}}[z^{-1}] := \mathbb{C}[z,z^{-1}] \otimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{M}}$ is $\mathbb{C}[z,z^{-1}]$-flat, and that a graded $\widetilde{\mathcal{A}}$-module is $\widetilde{\mathbb{C}}$-flat if and only if it has no z-torsion.

(3) Let $\varphi : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{N}}$ be a morphism in $\mathsf{Modgr}(\widetilde{\mathcal{A}})$. Assume that φ is injective. Show that the induced morphism $\varphi_a : \widetilde{\mathcal{M}}/(z-a)\widetilde{\mathcal{M}} \to \widetilde{\mathcal{N}}/(z-a)\widetilde{\mathcal{N}}$ is injective

- if $a \neq 0$,
- and also if $a = 0$ provided that $\widetilde{\mathcal{M}}$ is strict.

[*Hint*: Use (2) for $\operatorname{Coker}\varphi$.]

(4) Let $\widetilde{\mathcal{M}}^\bullet$ be a complex in $\mathsf{Modgr}(\widetilde{\mathcal{A}})$. Show that, for every i, there is a natural isomorphism

$$H^i(\widetilde{\mathcal{M}}^\bullet/(z-a)\widetilde{\mathcal{M}}^\bullet) \simeq H^i\widetilde{\mathcal{M}}^\bullet/(z-a)H^i\widetilde{\mathcal{M}}^\bullet,$$

- if $a \neq 0$,
- and also if $a = 0$ provided that $\widetilde{\mathcal{M}}^\bullet$ is strict (see Definition 5.1.6(3)).

[*Hint*: Consider the long exact sequence

$$\cdots H^i\widetilde{\mathcal{M}}^\bullet \xrightarrow{\ z-a\ } H^i\widetilde{\mathcal{M}}^\bullet \longrightarrow H^i(\widetilde{\mathcal{M}}^\bullet/(z-a)\widetilde{\mathcal{M}}^\bullet) \longrightarrow \cdots$$

attached to the exact sequence of complexes (according to (3))

$$0 \longrightarrow \widetilde{\mathcal{M}}^\bullet \xrightarrow{\ z-a\ } \widetilde{\mathcal{M}}^\bullet \longrightarrow \widetilde{\mathcal{M}}^\bullet/(z-a)\widetilde{\mathcal{M}}^\bullet \longrightarrow 0$$

and apply (3).]

(5) Recover the associated $\mathcal{A}$-module $\mathcal{M}$ of $\widetilde{\mathcal{M}}$ as $\widetilde{\mathcal{M}}/(z-1)\widetilde{\mathcal{M}}$ and, if $\widetilde{\mathcal{M}} = R_F\mathcal{M}$ is strict, $\mathrm{gr}^F\mathcal{M}$ as $R_F\mathcal{M}/zR_F\mathcal{M}$ (as a graded $\mathrm{gr}^F\mathcal{A}$-module).

(6) Let $\varphi : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{N}}$ be a morphism in $\mathsf{Modgr}(\widetilde{\mathcal{A}})_{\mathrm{st}}$ (i.e., between strict objects). Assume that $\varphi_{|z=1}$ is zero. Show that $\varphi = 0$. Deduce the faithfulness of the restriction functor $\mathsf{Modgr}(\widetilde{\mathcal{A}})_{\mathrm{st}} \mapsto \mathsf{Mod}(\mathcal{A})$ given by $\widetilde{\mathcal{M}} \mapsto \widetilde{\mathcal{M}}/(z-1)\widetilde{\mathcal{M}}$.

(7) Let $\varphi : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{N}}$ be a morphism in $\mathsf{Modgr}(\widetilde{\mathcal{A}})_{\mathrm{st}}$. Show that φ is strict if and only if the associated morphism $\varphi : (\mathcal{M}, F_\bullet\mathcal{M}) \to (\mathcal{N}, F_\bullet\mathcal{N})$ is strict, i.e., satisfies $\varphi(F_k\mathcal{M}) = \varphi(\mathcal{M}) \cap F_k\mathcal{N}$ for every index k. [*Hint*: Use Exercise 5.1(3).]

Exercise 5.3. If $(\mathcal{M}, F_\bullet\mathcal{M})$ is a filtered object of $\mathsf{Mod}(\mathcal{A})$, then a subobject $\mathcal{M}'$ of $\mathcal{M}$ carries the induced filtration $(F_p\mathcal{M} \cap \mathcal{M}')_{p\in\mathbb{Z}}$, while a quotient object $\mathcal{M}/\mathcal{M}''$ carries the induced filtration $((F_p\mathcal{M} + \mathcal{M}'')/\mathcal{M}'')_{p\in\mathbb{Z}}$. Show the following properties.

(1) $R_F\mathcal{M}' = R_F\mathcal{M} \cap \mathcal{M}'[z, z^{-1}]$ and $R_F(\mathcal{M}/\mathcal{M}'') = R_F\mathcal{M} \cap \mathcal{M}''[z, z^{-1}]/\mathcal{M}''[z, z^{-1}]$.

(2) The two possible induced filtrations on a subquotient $\mathcal{M}' \cap \mathcal{M}''/\mathcal{M}''$ of $\mathcal{M}$ agree.

(3) For every filtered complex $(\mathcal{M}^\bullet, F)$, the i-th cohomology of the complex is a subquotient of $\mathcal{M}^i$, hence it carries an induced filtration. Then there is a canonical morphism $H^i(F_p\mathcal{M}^\bullet) \to H^i(\mathcal{M}^\bullet)$, whose image is denoted by $F_pH^i(\mathcal{M}^\bullet)$.

***Exercise 5.4* (Locally free $\widetilde{\mathcal{O}}_X$-modules and filtrations by sub-bundles)**

Let $\widetilde{\mathcal{H}}$ be a locally free $\widetilde{\mathcal{O}}_X$-module of rank r. Show that the corresponding filtration $F^\bullet\mathcal{H}$ is a filtration by sub-bundles, i.e., $F^p\mathcal{H}/F^{p+1}\mathcal{H}$ is $\widetilde{\mathcal{O}}_X$-locally free for each $p \in \mathbb{Z}$. [*Hint*: reduce the statement to the case where $r = 1$.]

Conversely, let $(\mathcal{H}, F^\bullet\mathcal{H})$ be a filtered holomorphic bundle. Show that $\widetilde{\mathcal{H}} := R_F\mathcal{H}$ is $\widetilde{\mathcal{O}}_X$-locally free. [*Hint*: use local bases of $F^p\mathcal{H}/F^{p+1}\mathcal{H}$ for each p.]

Exercise 5.5. We take the notation of Section 5.2.a.

(1) Let $F'^\bullet\mathcal{H}$ and $F''^\bullet\mathcal{H}$ be 0-opposite filtrations of $\mathcal{H}$ in the sense of Definition 2.5.1. Show that $\widetilde{\mathcal{H}}' \simeq \bigoplus_p \mathcal{H}^{p,-p}z^{-p}\mathbb{C}[z]$ (where the sum is finite), and similarly $\widetilde{\mathcal{H}}'' \simeq \bigoplus_p \mathcal{H}^{p,-p}z^p\mathbb{C}[z^{-1}]$. Using that the gluing of $z^{-p}\mathbb{C}[z]$ with $z^p\mathbb{C}[z^{-1}]$ in $\mathbb{C}[z, z^{-1}]$ gives rise to the trivial bundle $\mathcal{O}_{\mathbb{P}^1}$, conclude that the bundle $\widetilde{\mathcal{F}}$ of (c) in Section 5.2.a is isomorphic to the trivial bundle on $\mathbb{P}^1$.

(2) Argue similarly in weight w.

(3) In order to prove that Condition (c) on $\mathcal{F}$ in Section 5.2.a implies oppositeness, reduce first to the case where $w = 0$ by identifying the effect of tensoring with $\mathcal{O}_{\mathbb{P}^1}(-w)$ with a shift of one filtration.

(4) Assume that $\mathcal{F}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}^{\dim \mathcal{H}}$. Show that the two filtrations giving rise to $\mathcal{F}$ are opposite.

Exercise 5.6.

(1) (Another proof of 2.5.6(2)) Show that a morphism in $\mathsf{HS}(\mathbb{C})$ induces a morphism between the associated vector bundles on $\mathbb{P}^1$ (see Section 5.2.a). Conclude that there is no non-zero morphism if $w_1 > w_2$. [*Hint*: Use standard properties of vector bundles on $\mathbb{P}^1$.]

(2) Let H_1 and H_2 be objects of $\mathsf{HS}(\mathbb{C}, w)$, let $\mathcal{F}_1, \mathcal{F}_2$ be the associated $\mathcal{O}_{\mathbb{P}^1}$-modules (see Section 5.2.a) and let H be a bi-filtered vector space whose associated $\mathcal{O}_{\mathbb{P}^1}$-module $\widetilde{\mathcal{F}}$ is an extension of $\mathcal{F}_1, \mathcal{F}_2$ in the category of $\mathcal{O}_{\mathbb{P}^1}$-modules. Show that H is an object of $\mathsf{HS}(\mathbb{C}, w)$. [*Hint*: Use standard properties of vector bundles on $\mathbb{P}^1$.]

***Exercise 5.7* (Another proof of Exercise 2.7).** Use the geometric interpretation of a Hodge structure in Section 5.2.a to prove the existence of operations as in Exercise 2.7 (e.g. use that $\mathcal{O}_{\mathbb{P}^1}(w_1) \otimes \mathcal{O}_{\mathbb{P}^1}(w_2) \simeq \mathcal{O}_{\mathbb{P}^1}(w_1 + w_2)$).

***Exercise 5.8* (Stability by extension).** Let $0 \to T_1 \to T \to T_2 \to 0$ be a short exact sequence of $\widetilde{\mathbb{C}}$-triples. Show that, if T_1, T_2 are $\mathbb{C}$-Hodge triples of weight w, then so is T. [*Hint*: By using the interpretation (c) of w-oppositeness in Section5.2.a, reduce the question to showing that, if a locally free $\mathcal{O}_{\mathbb{P}^1}$-module is an extension of two trivial bundles $\mathcal{O}_{\mathbb{P}^1}^{d_1}$ and $\mathcal{O}_{\mathbb{P}^1}^{d_2}$, then it is itself a trivial bundle.]

Exercise 5.9. Show that the category $\mathsf{VHS}(X, \mathbb{C}, w)$ as defined by 5.4.3 is equivalent to $\mathsf{VHS}(X, \mathbb{C}, w)$ as defined by 4.1.4, and hence to $\mathsf{VHS}(X, \mathbb{C}, w)$ as defined by 4.1.5. Show a similar result for $\mathsf{pVHS}(X, \mathbb{C}, w)$.

5.6. Comments

The Rees construction for filtered objects, embedding the non-abelian category of filtered objects into the abelian category of modules over a ring, is a well-known trick to treat filtered objects. The main application has been the proof of the Artin-Rees lemma, that we will reproduce in the context of filtered $\mathcal{D}$-modules in Chapters 7, 8 and 9.

The notion of triple has been instrumental in defining generalizations of the categories of Hodge modules, called twistor $\mathcal{D}$-modules (see **[Sab05, Moc07, Moc11a]**. When working with the Hodge metric (or more generally a harmonic metric) as a primary object on flat vector bundles, one is lead to the problem of extending the notion at the singularities of the vector bundle. The notion of metric is difficult to extend, because it contains in it the property of being non-degenerate. Similarly, the notion of C^∞ vector bundle does not extend across singularities, because the sheaf $\mathcal{C}_X^\infty$ is not coherent. The notion of sesquilinear pairing with values in distributions is a good replacement of the C^∞ isomorphism between a holomorphic vector bundle and its conjugate, as it allows “degenerate” gluings.

This new point of view, which will be present all along this book, is explained at all levels of Hodge theory, starting from classical Hodge theory.