CHAPTER 16

THE STRUCTURE THEOREM FOR POLARIZABLE HODGE MODULES

Summary. The structure theorem proved in this chapter allows for a more accessible approach to polarizable Hodge modules: they can be obtained from polarizable variations of complex Hodge structures. The correspondence in one direction has been proved in Chapter 14 and the other direction involves the extension to arbitrary dimensions of the results of Chapter 6 together with the detailed analysis in the normal crossing cas made in Chapter 15. Various applications follow, the Kodaira-Saito vanishing theorem among others.

16.1. Introduction

The definition of a polarizable Hodge module allows for proving various properties by an inductive procedure, but makes it difficult to check that a given object of $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$ equipped with a pre-polarization is actually a polarizable Hodge module. For example, proving that a polarizable variation of Hodge structure is a polarizable Hodge module is already non trivial (see Theorem 14.6.1) and there is an equivalence between such variations and smooth Hodge modules. It is therefore desirable to provide a similar criterion for any polarizable Hodge module. This is realized by the structure theorem 16.2.1. This new characterization of polarizable Hodge modules allows for various applications:

- the stability of WHM by smooth pullback which, together with Proposition 14.7.5, implies the stability of WHM by strictly non-characteristic pullback;
- the Kodaira-Saito vanishing theorem for objects of $\mathsf{WHM}(X)$.

16.2. The structure theorem

This is the converse of Proposition 14.2.10. Let X be a complex manifold and let Z be an irreducible closed analytic subset of X. Let $\mathsf{VHS}_{\mathrm{gen}}(Z,w)$ the category of "generically defined variations of Hodge structure of weight w on Z", as defined now.

We say that a pair $(Z^o, {}_{\mathrm{H}}H)$ consisting of a smooth Zariski-dense open subset Z^o of Z and of a variation of Hodge structure ${}_{\mathrm{H}}H$ of weight w on Z^o is equivalent to a similar pair $(Z'^o, {}_{\mathrm{H}}H')$ if ${}_{\mathrm{H}}H$ and ${}_{\mathrm{H}}H'$ coincide on $Z^o \cap Z'^o$. An object of $\mathsf{VHS}_{\mathrm{gen}}(Z,w)$ is such an equivalence class. Note that it has a maximal representative (by considering the union of the domains of all the representatives). A morphism between objects of $\mathsf{VHS}_{\mathrm{gen}}(Z,w)$ is defined similarly.

We also denote by $\mathsf{pVHS}_{\mathrm{gen}}(Z,w)$ the full subcategory of $\mathsf{VHS}_{\mathrm{gen}}(Z,w)$ consisting of objects which are polarizable, i.e., have a polarizable representative.

By Proposition 14.2.10, there is a restriction functor

$$\mathsf{pHM}_Z(X,w) \longmapsto \mathsf{pVHS}_{\mathrm{gen}}(Z, w - \operatorname{codim} Z).$$

16.2.1. Theorem **(Structure theorem).** *Under these assumptions, the restriction functor $\mathsf{pHM}_Z(X,w) \mapsto \mathsf{pVHS}_{\mathrm{gen}}(Z, w - \operatorname{codim} Z)$ is an equivalence of categories.*

Since each polarizable Hodge module has a unique decomposition with respect to the irreducible components of its pure support, the structure theorem gives a complete description of the category $\mathsf{pHM}(X,w)$. The remaining part of this section is devoted to the proof of the structure theorem.

16.2.a. Reduction to the normal crossing situation. We first notice that the restriction functor $\mathsf{pHM}_Z(X,w) \to \mathsf{pVHS}_{\mathrm{gen}}(Z, w - \operatorname{codim} Z)$ is faithful. Indeed, let M_1, M_2 be objects of $\mathsf{pHM}_Z(X,w)$ and let $\varphi, \varphi' : M_1 \to M_2$ be morphisms between them, which coincide on some Z^o. Then the image of $\varphi - \varphi'$ is an object of $\mathsf{pHM}(X,w)$, according to Corollary 14.2.19, and is supported on $Z \smallsetminus Z^o$, hence is zero according to the definition of the pure support. It follows that $\varphi = \varphi'$.

Due to the faithfulness, we note that the question is local: for fullness, if a morphism between the restriction to some Z^o of two polarized Hodge modules locally extends on Z, then it globally extends by uniqueness of the extension; for essential surjectivity, we note that two local extensions as polarized Hodge modules of a polarized variation of Hodge structure coincide, by extending the identity morphism on some Z^o according to local fullness, and we can thus glue local extensions into a global one.

For the essential surjectivity we start from a polarized variation of Hodge structure on some smooth Zariski-dense open subset $Z^o \subset Z$. We choose a projective morphism $\pi : Z' \to X$ with Z' smooth and connected, such that π is an isomorphism $Z'^o := \pi^{-1}(Z^o) \to Z^o$, and such that $Z' \smallsetminus Z'^o$ is a divisor with normal crossing. Assuming we have extended the variation on Z'^o as a polarized Hodge module on Z' with pure support Z', we apply to the latter the direct image theorem 14.3.1 for π, and get the desired polarized Hodge module as the component of this direct image ${}_{\mathrm{T}}\pi_*^0$ having pure support Z. We argue similarly for the fullness: if any morphism defined on some Z^o can be extended as a morphism between the extended objects on Z', we push it forward by π and restrict it as a morphism between the corresponding components. We are thus reduced to the case where $Z = X$ and the variation exists on $X^o := X \smallsetminus D$, where D is a divisor with normal crossings.

16.2.b. The normal crossing case. We consider a normal crossing pair (X, D) and a polarized variation of Hodge structure $({}_{\mathrm{H}}H, \mathrm{S})$ of weight $w - \dim X$ on $X^o := X \smallsetminus D$. The theorem is a consequence of the next proposition.

16.2.2. Proposition. *With these assumptions,*

(1) *the polarized variation of Hodge structure $({}_{\mathrm{H}}H, \mathrm{S})$ extends as a pre-polarized triple $(\widetilde{\mathfrak{T}}, \mathrm{S})$ of weight w on X, which is of normal crossing type and middle extension on (X, D);*

(2) *the pre-polarized triple $(\widetilde{\mathfrak{T}}, \mathrm{S})$ obtained in* (1) *satisfies the properties of Definition* 14.2.2 *with respect to any germ of holomorphic function g such that $g^{-1}(0) \subset D$;*

(3) *the pre-polarized triple $(\widetilde{\mathfrak{T}}, \mathrm{S})$ obtained in* (1) *satisfies the properties of Definition* 14.2.2 *with respect to any germ of holomorphic function g.*

Let us start with simple observations.

***16.2.3. Fullness and locality of the extension property* 16.2.2(1).** We first show:

Let $(\widetilde{\mathcal{M}}, F_\bullet\widetilde{\mathcal{M}})$ and $(\widetilde{\mathcal{M}}', F_\bullet\widetilde{\mathcal{M}}')$ be coherent filtered $\widetilde{\mathcal{D}}_X$-modules of normal crossing type and middle extension along each component of D (hence along D, by Corollary 15.11.2*). Any morphism $\varphi^o : (\widetilde{\mathcal{M}}, F_\bullet\widetilde{\mathcal{M}})|_{X^o} \xrightarrow{\sim} (\widetilde{\mathcal{M}}', F_\bullet\widetilde{\mathcal{M}}')|_{X^o}$ extends in a unique way as a morphism $\varphi : (\widetilde{\mathcal{M}}, F_\bullet\widetilde{\mathcal{M}}) \xrightarrow{\sim} (\widetilde{\mathcal{M}}', F_\bullet\widetilde{\mathcal{M}}')$. In particular, if φ^o is an isomorphism, then so is φ.*

The question is local and we can argue with coordinates $(x_1, \dots, x_n)$ as in Part 2 of Chapter 15. By considering Deligne's canonical meromorphic extension, one first checks that φ^o extends in a unique way as a morphism $\varphi : \widetilde{\mathcal{M}}(*D) \to \widetilde{\mathcal{M}}'(*D)$ which sends $V_{<\mathbf{0}}^{(\boldsymbol{n})}\widetilde{\mathcal{M}}$ to $V_{<\mathbf{0}}^{(\boldsymbol{n})}\widetilde{\mathcal{M}}'$. The question is to check that it is strictly compatible with the filtrations. Then, denoting by $j : X^o \hookrightarrow X$ the open inclusion, φ induces a morphism

$$j_*j^{-1}F_p\widetilde{\mathcal{M}} \cap V_{<\mathbf{0}}^{(\boldsymbol{n})}\widetilde{\mathcal{M}} \longrightarrow j_*j^{-1}F_p\widetilde{\mathcal{M}}' \cap V_{<\mathbf{0}}^{(\boldsymbol{n})}\widetilde{\mathcal{M}}'$$

for each $p \in \mathbb{Z}$. Then, Proposition 15.9.11 together with Remark 15.9.13 yield the conclusion.

We now check compatibility of φ with the sesquilinear pairings in $\widetilde{\mathfrak{T}}, \widetilde{\mathfrak{T}}'$. For that, we first observe that φ is compatible with the associated moderate pre-polarizations $\mathfrak{s}^{\mathrm{mod}\, D}, \mathfrak{s}'^{\mathrm{mod}\, D}$ (see Section 12.5.f). As $\widetilde{\mathcal{M}}$ is a middle extension along D (see Corollary 15.11.2), it follows from Corollary 12.5.41 that φ is compatible with $\mathfrak{s}, \mathfrak{s}'$. At this point, we have shown fullness in the structure theorem.

It remains to check that, in 16.2.2(1), the extension of S is unique. This also follows from the middle extension property of Corollary 15.11.2. □

***16.2.4. Property* 16.2.2(2) *implies* 16.2.2(3).** Let g be any germ of holomorphic function at $x \in X$. If the germ is taken at a point $x \in X^o$, then the properties hold, since we already know that a polarized variation of Hodge structure is a polarized Hodge module (Theorem 14.6.1). Therefore, we only need to consider germs g at a point $x \in D$. To reduce to the case of a monomial, we argue as in the proof when $(\widetilde{\mathfrak{T}}, \mathrm{S})$ is a polarized variation of Hodge structure (see Step one of the proof of Theorem 14.6.1).

Given any g, we can find a projective modification $\pi : X' \to X$ such that $D' := \pi^{-1}(D \cup g^{-1}(0))$ is a divisor with normal crossing in the complex manifold X', so that $g \circ \pi$ can be expressed in local coordinates as a monomial, and such that π is an isomorphism above $X \smallsetminus g^{-1}(0)$. By the first step, we extend the variation as a pre-polarized triple $(\widetilde{\mathcal{T}}', \mathrm{S}')$ of normal crossing type and middle extension on (X', D') and the properties of Definition 14.2.2 are satisfied for $g\circ\pi$. We can then apply Proposition 14.4.2 to obtain the desired properties for g with respect to ${}_{\mathrm{T}}\pi_*^{(0)}(\widetilde{\mathcal{T}}', \mathrm{S}')$, which is S-decomposable along $D \cup g^{-1}(0)$. Let $(\widetilde{\mathcal{T}}'_0, \mathrm{S}'_0)$ be its pure component supported on X. On the other hand, let $(\widetilde{\mathcal{T}}, \mathrm{S})$ be the object obtained from $({}_{\mathrm{H}}H, \mathrm{S})$ at the first step. Then $\widetilde{\mathcal{T}}$ also has pure support equal to X.

16.2.c. Polarized Hodge modules in the normal crossing case. Let X be a complex manifold and let $D = \bigcup_{i\in I} D_i$ be a reduced divisor with normal crossings. Let (M, S) be a polarized Hodge module with pure support X and singularities on D, so that $(M, \mathrm{S})_{|X\smallsetminus D}$ is a polarized variation of Hodge structure.

16.2.5. Theorem. *With these assumptions, the filtered $\mathcal{D}_X$-module $(\mathcal{M}, F^\bullet\mathcal{M})$ underlying M is of normal crossing type and a middle extension along $D_{i\in I}$ (Definitions* 15.9.1 *and* 15.9.10*).*

We first check the property for $\mathcal{M}$.

16.2.6. Lemma. *With these assumptions, the underlying $\mathcal{D}_X$-module $\mathcal{M}$ is of normal crossing type (Definition* 15.7.11*) and a middle extension along $D_{i\in I}$ (Definition* 15.7.8*).*

Let us recall the local setting of Chapter 15. The space X is a polydisc in $\mathbb{C}^n$ with analytic coordinates $x_1, \dots, x_n$, we fix $\ell \leqslant n$ and we denote by D the divisor $\{x_1 \cdots x_\ell = 0\}$. We also denote by D_i $(i \in I)$ the smooth components of D and by $D_{(\ell)}$ their intersection $D_1 \cap \cdots \cap D_\ell$. We will shorten the notation $\mathcal{O}_{D_{(\ell)}}[x_1, \dots, x_\ell]$ into $\mathcal{O}_{D_{(\ell)}}[x]$ and $\mathcal{D}_{D_{(\ell)}}[x_1, \dots, x_\ell]\langle\partial_{x_1}, \dots, \partial_{x_\ell}\rangle$ into $\mathcal{D}_{D_{(\ell)}}[x]\langle\partial_x\rangle$.

***Proof of Lemma* 16.2.6.** Since $\mathcal{M}$ is holonomic, and smooth on $X \smallsetminus D$, $\mathcal{M}(*D)$ is a coherent $\mathcal{O}_X(*D)$-module, according to Example 11.3.14.

On the smooth open subset of D, we can apply the same argument as for Proposition 7.4.12 and conclude that for each p, we have the equality

$$F_p\mathcal{M} \cap V_{<0}\mathcal{M} = (j_*j^{-1}F_p\mathcal{M}) \cap V_{<0}\mathcal{M}.$$

In particular, for $p \gg 0$ we obtain that $V_{<0}\mathcal{M} = F_p\mathcal{M} \cap V_{<0}\mathcal{M}$ is $\mathcal{O}_X$-coherent. This means that the $\mathcal{O}_X(*D)$-module with flat connection $\mathcal{M}(*D)$ has regular singularities along the smooth open subset of D. It follows from [**Del70**, Th. 4.1 p. 88] that $\mathcal{M}(*D)$ is $\mathcal{O}_X(*D)$-locally free and has regular singularities along D, so $\mathcal{M}(*D)$ is of normal crossing type along D. Moreover, $\mathcal{M}$ is its middle extension along $D_{i\in I}$, hence is also of normal crossing type. $\square$

16.2.7. Lemma. *Assume there exists a coherent filtered $\mathcal{D}_X$-module $(\mathcal{M}', F_\bullet\mathcal{M}')$ of normal crossing type along D and a middle extension along $D_{i\in I}$ such that the restrictions $j^{-1}(\mathcal{M}, F_\bullet\mathcal{M})$ and $j^{-1}(\mathcal{M}', F_\bullet\mathcal{M}')$ are isomorphic. Then $(\mathcal{M}, F_\bullet\mathcal{M}) \simeq (\mathcal{M}', F_\bullet\mathcal{M}')$.*

This lemma reduces the proof of Theorem 16.2.5 to the construction of $(\mathcal{M}', F_\bullet\mathcal{M}')$.

Proof. By Lemma 16.2.6 we have $\mathcal{M} \simeq \mathcal{M}'$, so we identify these modules, and we set $F_\bullet\mathcal{M}' = F'_\bullet\mathcal{M}$. Let g be a reduced defining equation for D and let ι_g be the corresponding graph embedding. Then $(\mathcal{M}, F'_\bullet\mathcal{M})$ is strictly $\mathbb{R}$-specializable along (g) and a middle extension along $D_{i\in I}$, according to Theorem 15.11.1. By definition, the same property holds for $(\mathcal{M}, F_\bullet\mathcal{M})$. Applying Remark 10.5.2 to $F_\bullet\mathcal{M}_g$ and $F'_\bullet\mathcal{M}_g$ leads to $F_\bullet\mathcal{M}_g = F'_\bullet\mathcal{M}_g$, hence $F_\bullet\mathcal{M} = F'_\bullet\mathcal{M}$. □

According to Proposition 15.9.24, the proof of Theorem 16.2.5 will be achieved if we prove the higher dimensional analogue of Theorem 6.7.3:

16.2.8. Theorem. *If $(M, \mathrm{S})_{X\smallsetminus D}$ is a polarized variation of Hodge structure on $X \smallsetminus D$, and if we set $F_p\mathcal{M}_{<0} = j_*F_p\mathcal{M}_{|X\smallsetminus D}$, then for each $\boldsymbol{\alpha} < \boldsymbol{0}$ the sheaves $\mathrm{gr}_p^F V_{\boldsymbol{\alpha}}^{(\boldsymbol{\ell})}\mathcal{M}$ are (coherent and) locally free $\mathcal{O}_X$-modules.*

16.3. Applications of the structure theorem

16.3.a. Semi-simple components. Let X be a smooth projective variety and M be a polarizable Hodge module of weight w. The underlying $\mathcal{D}_X$-module $\mathcal{M}$ is semi-simple, according to Theorem 14.7.7.

***16.3.1. Proposition* (Semi-simple components).** *Any simple component $\mathcal{M}_\alpha$ of $\mathcal{M}$ underlies a unique (up to equivalence) polarized Hodge module $(M_\alpha, \mathrm{S}_\alpha)$ of the same weight w and there exists a polarized Hodge structure $(H_\alpha^o, \mathrm{S}_\alpha^o)$ of weight* 0 *such that $(M, \mathrm{S}) \simeq \bigoplus_\alpha((H_\alpha^o, \mathrm{S}_\alpha^o) \otimes (M_\alpha, \mathrm{S}_\alpha))$.*

(See Section 4.3.c for the notion of equivalence.)

Proof. We can assume that M has pure support a closed irreducible analytic subset Z of X and that the restriction of M to a Zariski dense open subset $Z^o \subset Z$ is a polarizable variation of Hodge structure.

We will apply the same argument as for Theorem 4.3.13(2). For that purpose, we need to know that the space of global sections of a polarized variation of Hodge structure on Z^o is a polarized Hodge structure. Since we have the choice of a compactification Z^o, we can assume that Z is smooth and $D = Z \smallsetminus Z^o$ is a divisor with normal crossings. By the structure theorem, the variation extends as a polarizable Hodge module M with pure support Z. By the Hodge-Saito theorem 14.3.1 applied to the constant map $a_Z : Z \to \mathrm{pt}$, the hypercohomology $\boldsymbol{H}^{-\dim Z}{}_{\mathrm{H}}a_{Z*}M$ is a polarizable Hodge structure. Its underlying vector space is $\boldsymbol{H}^{-n}(Z, {}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}) = H^0(Z, \mathcal{H}^{-n}({}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}))$ (since all differentials d_r $(r \geqslant 2)$ in the spectral sequence starting with $E_2^{i,j} = H^i(Z, \mathcal{H}^j({}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}))$ vanish on $E_2^{0,-n}$). We are thus left with proving $H^0(Z, \mathcal{H}^{-n}({}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M})) = H^0(Z^o, \underline{\mathcal{H}})$, with $\underline{\mathcal{H}} = \mathcal{H}^{-n}({}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M})_{|Z^o}$. This amounts to the equality of sheaves $\mathcal{H}^{-n}({}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}) = j_*\underline{\mathcal{H}}$, where $j : Z^o \hookrightarrow Z$ denotes the inclusion, and this is a local question in the neighbourhood of each point of D.

By Lemma 16.2.6, the $\mathcal{D}_X$-module $\mathcal{M}$ is of normal crossing type and a middle extension along $D_{i\in I}$. Let us work in the local setting with the simplifying assumption 15.6.2. We will show the equality of germs $\mathcal{H}^{-n}({}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M})_0 = (j_*\underline{\mathcal{H}})_0$. The germ at 0 of the de Rham complex ${}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}$ is the simple complex associated to the n-complex having vertices equal to $\bigoplus_{\boldsymbol{\alpha}\in[-1,0)^n} M_{\boldsymbol{\alpha}+\boldsymbol{k}}$ with $\boldsymbol{k} \in \{0,1\}^n$ and arrows in the i-th direction induced by $\widetilde{\partial}_{x_i}$. The latter is an isomorphism on each $M_{\boldsymbol{\alpha}+\boldsymbol{k}}$ with $\alpha_i \neq -1$ and $k_i = 0$. This complex is thus isomorphic to its subcomplex with vertices $M_{-\mathbf{1}+\boldsymbol{k}}$, so that $\widetilde{\partial}_{x_i}$ reads can_i, and $\mathcal{H}^0(\mathrm{DR}\,\mathcal{M})_0 = \bigcap_i \mathrm{Ker}\,\mathrm{can}\,i \subset M_{-\mathbf{1}}$. A similar analysis shows that $(j_*\underline{\mathcal{H}})_0 = \bigcap_i \mathrm{Ker}\,\mathrm{N}_i \subset M_{-\mathbf{1}}$. Recall now that $\mathcal{M}$ is a middle extension along $D_{i\in I}$. This means that can_i is onto and var_i is injective, so $\mathrm{Ker}\,\mathrm{N}_i = \mathrm{Ker}\,\mathrm{can}_i$, and this concludes the proof. □

16.3.b. Smooth and strictly non-characteristic pullbacks

16.3.2. Proposition. *Let $f : X \to Y$ be a smooth morphism of complex analytic manifolds and let $(M, W_\bullet M)$ be an object of* $\mathsf{WHM}(Y)$. *Then $({}_{\mathrm{T}}f^*M, {}_{\mathrm{T}}f^*W_\bullet M)$ is an object of* $\mathsf{WHM}(X)$.

Proof. Since f is flat, we can reduce to the case where M is pure, and it is enough to consider the case where it has pure support a closed irreducible analytic subset Z of Y. The question is local, so that we can assume that f is the projection of a product $X = Y \times Z \to Y$. The result amounts then to the property that the equivalence given by the structure theorem 16.2.1 is compatible with the external product by ${}_{\mathrm{H}}\widetilde{\mathcal{O}}_Z$ (see Example 14.6.2). This property is straightforward from the construction. □

***16.3.3. Corollary* (of Propositions 14.7.5 and 16.3.2).** *Let $(M, W_\bullet M)$ be an object of* $\mathsf{WHM}(Y)$ *and let $f : X \to Y$ be a morphism of complex analytic manifolds which is non-characteristic with respect to $\mathrm{gr}_\ell^W M$ for each $\ell \in \mathbb{Z}$. Then $({}_{\mathrm{T}}f^*M, {}_{\mathrm{T}}f^*W_\bullet M)$ is an object of* $\mathsf{WHM}(X)$. □

16.3.c. Duality. For M underlying an object of $\mathsf{WHM}(X)$, the dual $\boldsymbol{D}M$ is well-defined in $\widetilde{\mathcal{D}}$-$\mathsf{Triples}(X)$, according to Sections 14.7.b and 14.7.c. The same property applies to each $W_\ell M$ and $M/W_\ell M$.

16.3.4. Proposition. *Let $(M, W_\bullet M)$ be an object of* $\mathsf{WHM}(X)$. *Then the W-filtered triple $(\boldsymbol{D}M, W_\bullet \boldsymbol{D}M)$, with $W_\ell \boldsymbol{D}M := \boldsymbol{D}(M/W_{-\ell-1}M)$, is an object of* $\mathsf{WHM}(X)$.

16.3.d. The Kodaira-Saito vanishing theorem

16.3.5. Theorem. *Let X be a smooth projective variety, let L be an ample line bundle on X and let $\widetilde{\mathcal{M}}$ be any of the $\widetilde{\mathcal{D}}_X$-module components of an object $(M, W_\bullet M)$ of* $\mathsf{WHM}(X)$. *Then $\widetilde{\mathcal{M}}$ satisfies the Kodaira-Saito vanishing property (Definition* 11.9.1*),* *that is*

$$\boldsymbol{H}^k(X, L^{-1}\otimes \operatorname{gr}^F {}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}) = 0 \quad \textit{for } k<0,$$
$$\boldsymbol{H}^k(X, L\otimes \operatorname{gr}^F {}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}) = 0 \quad \textit{for } k>0.$$

Proof. We will check the criteria of Theorem 11.9.5 for $\widetilde{\mathcal{M}}$ and $\boldsymbol{D}\widetilde{\mathcal{M}}$. As $\boldsymbol{D}\widetilde{\mathcal{M}}$ also underlies an object of $\mathsf{WHM}(X)$ by Proposition 16.3.4, it is enough to argue with $\widetilde{\mathcal{M}}$.

Firstly, by arguing by induction on the maximal weight, we can assume that M is pure of weight w as we have an exact sequence

$$0 \longrightarrow W_{\ell-1}\widetilde{\mathcal{M}} \longrightarrow W_\ell\widetilde{\mathcal{M}} \longrightarrow \operatorname{gr}_\ell^W\widetilde{\mathcal{M}} \longrightarrow 0$$

which leads to an exact sequence of complexes after applying $\operatorname{gr}^F {}^{\mathrm{p}}\mathrm{DR}_X$.

We then argue by induction on the dimension of the support Z of M. The case where $\dim Z = 0$ is clear due to Property 14.2.2(0). We can assume that Z is of dimension $d\geqslant 1$, that M has support Z and that the theorem holds if $\dim\operatorname{Supp} M \leqslant d-1$.

Instead of checking 11.9.5(1), we check (1′) of Remark 11.9.6. Let H be any hyperplane section (with respect to the embedding defined by $L^{\otimes m}$) which is non-characteristic with respect to M. By Proposition 9.5.2, since by definition M is strictly $\mathbb{R}$-specializable along H, it is also strictly non-characteristic along H. The cyclic covering morphism $f: X'\to X$ considered in Section 11.9.2 is thus strictly non-characteristic with respect to M. By Corollary 16.3.3, ${}_{\mathrm{D}}f^{*(0)}M$ belongs to $\mathsf{pHM}(X', w)$. The Hodge-Saito theorem 14.3.1 for the constant map $a_{X'}$ implies the strictness required in 11.9.6(1′).

For 11.9.5(2), by definition of the category $\mathsf{pHM}(X, w)$, the object ${}_{\mathrm{T}}\iota_{H*}({}_{\mathrm{T}}\iota_H^{*(0)}M)$ also belongs to $\mathsf{pHM}(X, w)$ by the strictly non-characteristic condition, and has support contained in $Z\cap H$, hence of dimension $\leqslant d-1$. The induction assumption ensures that 11.9.5(2) is satisfied. □

16.4. Comments

Here come the references to the existing work which has been the source of inspiration for this chapter.