CHAPTER 15

$\widetilde{\mathcal{D}}$-MODULES OF NORMAL CROSSING TYPE
PART 1: DISTRIBUTIVE FILTRATIONS AND STRICTNESS

Summary. This chapter, although somewhat technical, is nevertheless essential to understand the behaviour of Hodge modules when the singularities form a normal crossing divisor. It analyzes the compatibility properties, on a given $\mathbb{R}$-specializable $\mathcal{D}$-module, between the F-filtration and the V-filtrations attached to various functions, when these functions form part of a coordinate system. The results of this chapter will therefore be of a local nature. In this part, we introduce the general notion of distributivity or compatibilty of a family of filtrations, and we relate it to flatness properties of the associated multi-Rees modules. These will be our main tools for Parts 2 and 3.

We recall:

10.2.1. Convention. We work in the abelian category A of sheaves of vector spaces (over some fixed field, that will be the field of complex numbers for our purposes) on some topological space T. In such a category, all filtered direct limits exist and are exact. Given an object A in this category, we only consider increasing filtrations $F_\bullet A$ that are indexed by $\mathbb{Z}$ and satisfy $\varinjlim_k F_kA = A$. We write a filtered object in A as (A, F), where $F = (F_kA)_{k\in\mathbb{Z}}$.

15.1. Distributive filtrations

The results of this section being well-known, complete proofs will not be given and we refer to Sections 1.6 and 1.7 of [**PP05**] and Section 1 of [**Kas85**] (for the case of finite filtrations) for details.

Suppose that A is an object of our category A, and $A_1, \dots, A_n \subseteq A$ are finitely many subobjects. When $n = 3$, the inclusion

$$(A_1 \cap A_2) + (A_1 \cap A_3) \subset A_1 \cap (A_2 + A_3)$$

is strict in general. For example,

- assume that the three non-zero objects A_1, A_2, A_3 behave like three lines in $\mathbb{C}^2$ having zero pairwise intersections, i.e., $A_i \cap A_j = 0$ for all $i \neq j$, and $A = A_i + A_j$ for all $i \neq j$; then the above inclusion is strict;
- on the other hand, if $A_2 \subset A_1$ or $A_3 \subset A_1$, this inclusion is an equality.

When this inclusion is an equality, we say that A_1, A_2, A_3 form a *distributive family* of objects of A, i.e., the equivalent equalities, or any other obtained by permuting A_1, A_2, A_3, are satisfied (see [**PP05**, Lem. 6.1]):

$$(15.1.1)\qquad \begin{aligned}(A_1 \cap A_2) + (A_1 \cap A_3) &= A_1 \cap (A_2 + A_3),\\ (A_1 + A_2) \cap (A_1 + A_3) &= A_1 + (A_2 \cap A_3).\end{aligned}$$

We will interpret the distributivity property in terms of exact sequence. For one subobject A_1 of A, we have a short exact sequence of the form

$$A_1 \longrightarrow A \longrightarrow *$$

where $*$ is of course just an abbreviation for the quotient A/A_1. For two subobjects A_1, A_2, we similarly have a commutative diagram of the form

$$(15.1.2)\qquad \begin{array}{ccccc} * & \longrightarrow & * & \longrightarrow & * \\ \uparrow & & \uparrow & & \uparrow \\ A_2 & \longrightarrow & A & \longrightarrow & * \\ \uparrow & & \uparrow & & \uparrow \\ * & \longrightarrow & A_1 & \longrightarrow & * \end{array}$$

in which all rows and all columns are short exact sequences. (For example, the entry in the upper-right corner is $A/(A_1 + A_2)$, the entry in the lower-left corner $A_1 \cap A_2$.) For three subobjects A_1, A_2, A_3, such a diagram no longer exists in general; if it does exist, one says that A_1, A_2, A_3 define a *compatible family* of objects of A.

15.1.3. Lemma. *A family of three subobjects of A is distributive if and only if it is compatible.*

Proof. We consider a cubical diagram having vertices in $\{-1, 0, 1\}^3 \subset \mathbb{R}^3$, and we identify each vertex with a subquotient object of A such that

$$A = (0, 0, 0), \quad A_1 = (-1, 0, 0), \quad A_2 = (0, -1, 0), \quad A_3 = (0, 0, -1).$$

We assume that all rows and columns are exact (compatibility). One first easily checks, as in the case of two objects, that the vertices with i entries -1 and $3 - i$ entries 0 ($i = 1, 2, 3$) are the intersections of the corresponding vertices with only one entry -1 and two entries 0. We then find $(-1, -1, -1) = A_1 \cap A_2 \cap A_3$ and, since $(A_2 \cap A_3)/(A_1 \cap A_2 \cap A_3) = (A_1 + (A_2 \cap A_3))/A_1$ the exact sequence $(1, -1, -1) \to (1, 0, -1) \to (1, 1, -1)$ reads

$$(A_1 + (A_2 \cap A_3))/A_1 \longrightarrow (A_1 + A_3)/A_1 \longrightarrow (A_1 + A_3)/(A_1 + (A_2 \cap A_3)),$$

while the exact sequence $(1, -1, 0) \to (1, 0, 0) \to (1, 1, 0)$ reads

$$(A_1 + A_2)/A_1 \longrightarrow A/A_1 \longrightarrow A/(A_1 + A_2).$$

The morphism $(1, 1, -1) \to (1, 1, 0)$, that is,

$$(A_1 + A_3)/(A_1 + (A_2 \cap A_3)) \longrightarrow A/(A_1 + A_2)$$

should be injective, that is, the second equality in (15.1.1) should hold.

Conversely, assuming distributivity, we obtain similarly the exactness of the rows and the columns of the cubical diagram. □

When $n \geqslant 4$, the definition uses the case $n = 3$ for many 3-terms subfamilies obtained from $A_1, \dots, A_n$.

***15.1.4. Definition* (Distributivity).** A family $A_1, \dots, A_n$ of subobjects of A is *distributive* if for any partition $\{1, \dots, n\} = I_1 \sqcup I_2 \sqcup I_3$, the subobjects

$$A'_1 = \sum_{i \in I_1} A_i, \quad A'_2 = \sum_{i \in I_2} A_i, \quad A'_3 = \bigcap_{i \in I_3} A_i$$

form a distributive family (with the convention that the sum over the empty set is zero and the intersection over the empty set is A), i.e.,

$$\Big((\textstyle\sum_{i \in I_1} A_i) \cap (\sum_{i \in I_2} A_i)\Big) + \Big((\sum_{i \in I_1} A_i) \cap (\bigcap_{i \in I_3} A_i)\Big) = (\textstyle\sum_{i \in I_1} A_i) \cap \Big((\sum_{i \in I_2} A_i) + (\bigcap_{i \in I_3} A_i)\Big),$$

equivalently,

$$\Big((\textstyle\sum_{i \in I_1} A_i) + (\sum_{i \in I_2} A_i)\Big) \cap \Big((\sum_{i \in I_1} A_i) + (\bigcap_{i \in I_3} A_i)\Big) = (\textstyle\sum_{i \in I_1} A_i) + \Big((\sum_{i \in I_2} A_i) \cap (\bigcap_{i \in I_3} A_i)\Big).$$

It is equivalent to asking, for any partition (I_1, I_2, I_3) of $\{1, \dots, n\}$, distributivity of the three objects

$$A''_1 = \sum_{i \in I_1} A_i, \quad A''_2 = \bigcap_{i \in I_2} A_i, \quad A''_3 = \bigcap_{i \in I_3} A_i.$$

Let us state a few main properties.

***15.1.5. Proposition* (see [PP05, Cor. 6.4 & 6.5]).**

(1) *A family $A_1, \dots, A_n$ is distributive if and only if any subfamily containing no pair A_i, A_j with $A_i \subset A_j$ is distributive.*

(2) *A family $A_0, \dots, A_n$ is distributive if and only if*

(a) *the induced families on A_0 and A/A_0 are distributive, equivalently, the families $A_0 \cap A_1, \dots, A_0 \cap A_n$ and $A_0 + A_1, \dots, A_0 + A_n$ are distributive, and*

(b) *any three objects A_0, A_i, A_j are distributive.*

(3) *A family $A_0, \dots, A_n$ is distributive if and only if*

- *the families $A_1, \dots, A_n$ and $A_0 \cap A_1, \dots, A_0 \cap A_n$ are distributive and the following identity holds for any subset $I \subset \{1, \dots, n\}$:*

$$A_0 \cap \Big(\sum_{i \in I} A_i\Big) = \sum_{i \in I} (A_0 \cap A_i),$$

- *or the similar condition obtained by exchanging everywhere $+$ and $\cap$.* □

15.1.6. Example. Let $A_1, \dots, A_n$ be a distributive family and let $I_1, \dots, I_r$ be subsets of $\{1, \dots, n\}$. Then the following families are distributive:

- $A_1, \dots, A_n, (\bigcap_{k\in I_1} A_k), \dots, (\bigcap_{k\in I_r} A_k)$;
- $(A_1 \cap \bigcap_{k\in I_j} A_k), \dots, (A_n \cap \bigcap_{k\in I_j} A_k)$ for any $j \in \{1, \dots, r\}$;
- $\big(\sum_j (A_i \cap \bigcap_{k\in I_j} A_k)\big)_{i=1,\dots,n} = \big(A_i \cap \sum_j (\bigcap_{k\in I_j} A_k)\big)_{i=1,\dots,n}$.

Let us now consider increasing filtrations $F^{(1)}_\bullet A, \dots, F^{(n)}_\bullet A$ of A.

***15.1.7. Definition* (Distributive filtrations).** Given finitely many increasing filtrations $F^{(1)}_\bullet A, \dots, F^{(n)}_\bullet A$ of an object A in the abelian category, we call them *distributive* if

$$F^{(1)}_{k_1} A, \dots, F^{(n)}_{k_n} A \subseteq A$$

are distributive sub-objects for every choice of $k_1, \dots, k_n \in \mathbb{Z}$.

15.1.8. Remark. Assume that $F^{(1)}_\bullet A, \dots, F^{(n)}_\bullet A$ is a distributive family of filtrations of A.

(1) As a consequence of Definition 15.1.4, any sub-family of filtrations of a distributive family remains distributive. Moreover, any finite family of sub-objects consisting of terms of the filtrations $F^{(1)}_\bullet A, \dots, F^{(n)}_\bullet A$ is distributive, and Proposition 15.1.5(2) implies that the induced filtrations $F^{(1)}_\bullet, \dots, F^{n-1}_\bullet$ on each $\mathrm{gr}^{F^{(n)}}_\ell A$ are distributive.

(2) Let $B = F^{(1)}_{j_1} A \cap \cdots \cap F^{(n)}_{j_n} A$ for some $j_1, \dots, j_n$. Then the family of filtrations $F^{(1)}_\bullet B, \dots, F^{(n)}_\bullet B$ naturally induced on B is distributive, as follows from the distributivity of the family of $2n$ sub-objects $F^{(1)}_{k_1} A, \dots, F^{(n)}_{k_n} A, F^{(1)}_{j_1} A, \dots, F^{(n)}_{j_n} A$ and that of the induced family on B.

(3) One can interpret distributivity of filtrations as distributivity of subobjects as in Definition 15.1.4. For that purpose, we consider the ring $\widetilde{R} = \mathbb{C}[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$ of Laurent polynomials in n variables. Recall (see Convention 10.2.1) that A is a sheaf of $\mathbb{C}$-vector spaces on some topological space T. We consider the object $\widetilde{A} = \widetilde{R} \otimes_{\mathbb{C}} A$. To each filtration $F^{(i)}_\bullet A$ we associate the subobject of $\widetilde{A}$:

$$\widetilde{A}_i = \mathbb{C}[z_1^{\pm 1}, \dots, \widehat{z_i^{\pm 1}}, \dots, z_n^{\pm 1}] \otimes_{\mathbb{C}} \Big(\bigoplus_{k\in\mathbb{Z}} F^{(i)}_k A \cdot z_i^k\Big).$$

Then the distributivity of $F^{(1)}_\bullet A, \dots, F^{(n)}_\bullet A$ is equivalent to that of $\widetilde{A}_1, \dots, \widetilde{A}_n$.

15.2. Distributivity and flatness

15.2.a. Reformulation of distributivity in terms of flatness. Let A be an object with n filtrations $F^{(1)}_\bullet A, \dots, F^{(n)}_\bullet A$. As usual, we can pass from filtered to graded objects by the Rees construction. Let $R = \mathbb{C}[z_1, \dots, z_n]$ denote the polynomial ring in n variables, with the $\mathbb{Z}^n$-grading that gives z_i the weight $\mathbf{1}_1 = (0, \dots, 1, \dots, 0)$. For $\boldsymbol{k} \in \mathbb{Z}^n$, we define

$$M_{\boldsymbol{k}} = M_{k_1,\dots,k_n} = F^{(1)}_{k_1} A \cap \cdots \cap F^{(n)}_{k_n} A \subseteq A.$$

We then obtain a $\mathbb{Z}^n$-graded sheaf of modules M over the constant sheaf of rings R_T on the topological space T (recall Convention 10.2.1) by taking the direct sum

$$R_{F^{(1)},\dots,F^{(n)}} A := M = \bigoplus_{\boldsymbol{k}\in\mathbb{Z}^n} M_{\boldsymbol{k}},$$

with the obvious $\mathbb{Z}^n$-grading: for $m \in M_{\boldsymbol{k}}$, the product $z_i m$ is simply the image of m under the inclusion $M_{\boldsymbol{k}} \subseteq M_{\boldsymbol{k}+\mathbf{1}_i}$. From now on, we use the term "graded" to mean "$\mathbb{Z}^n$-graded".

15.2.1. Dictionary. There is a dictionary between operations on R_T-modules and operations on filtrations. Let us keep the notation of Remark 15.1.8(3) (see also Exercise 15.1).

(a) We consider the graded components $M_{\boldsymbol{k}}$ as forming a directed system, indexed by $\boldsymbol{k} \in \mathbb{Z}^n$, with morphisms given by multiplication by $z_1, \dots, z_n$. Since we are working in an abelian category in which all filtered direct limits exist and are exact, we can define

$$A = \varinjlim_{k \in \mathbb{Z}^n} M_{\boldsymbol{k}}.$$

Then M and A are related by

$$\widetilde{R} \otimes_R M \simeq \widetilde{A} := \widetilde{R} \otimes_{\mathbb{C}} A \quad \text{and} \quad A \simeq M/(z_1 - 1, \dots, z_n - 1)M.$$

(b) Let I be a subset of $\{1, \dots, n\}$ and let I^{c} denote its complementary subset. If we hold, for each $i \in I$, the i-th index fixed, the resulting direct limit determines a $\mathbb{Z}^I$-graded object $M^{(I)}$, which is a $\mathbb{Z}^I$-graded module over the ring $R_I = \mathbb{C}[z_I] = R/((z_i - 1)_{i \in I^{\mathrm{c}}})R$:

$$M^{(I)} = \bigoplus_{\boldsymbol{k}_I \in \mathbb{Z}^I} M^{(I)}_{\boldsymbol{k}_I}, \quad \text{with} \quad M^{(I)}_{\boldsymbol{k}_I} = \varinjlim_{\boldsymbol{k}_{I^{\mathrm{c}}} \in \mathbb{Z}^{I^{\mathrm{c}}}} M_{\boldsymbol{k}}.$$

We then have

$$M^{(I)} \simeq R_I \otimes_R M.$$

Let $\widetilde{R}_I \subset \widetilde{R}$ be the subring of Laurent polynomials whose z_i-degree is non-negative for $i \in I$. Then M and $M^{(I)}$ are also related by

$$\widetilde{M}^{(I)} := \widetilde{R}_I \otimes_R M \simeq \mathbb{C}[z_{I^{\mathrm{c}}}^{\pm 1}] \otimes_{\mathbb{C}} M^{(I)}.$$

15.2.2. Theorem. *A graded R_T-module comes from an object with n distributive filtrations if and only if it is flat over R_T.*

We note that both the distributivity property and the flatness property for sheaves of $\mathbb{C}$-vector spaces or of R-modules can be checked stalkwise on the topological space T, so that the statement concerns multi-filtered $\mathbb{C}$-vector spaces and R-modules. This remark will be used implicitly in the following.

Before giving the proof, we recall a few general facts about flatness. For any commutative ring R, flatness of an R-module M is equivalent to the condition that

$$\mathrm{Tor}_1^R(M, R/I) = 0$$

for every finitely generated ideal $I \subseteq R$; when R is Noetherian, it is enough to check this for all prime ideals $P \subseteq R$. In our setting, the ring R is graded, and by a similar argument as in the ungraded case, flatness is equivalent to

$$\mathrm{Tor}_1^R(M, R/P) = 0$$

for every *graded* prime ideal $P \subseteq R$. Of course, there are only finitely many graded prime ideals in $R = \mathbb{C}[z_1, \dots, z_n]$, namely those that are generated by the 2^n possible subsets of the set $\{z_1, \dots, z_n\}$. Moreover, the quotient R/P always has a canonical free resolution given by the Koszul complex.

We conclude:

15.2.3. Proposition. *A graded R-module is R-flat if and only if, for any subset J of $\{1, \dots, n\}$, the Koszul complex $K(M; (z_j)_{j\in J})$ is exact in negative degrees, i.e., is a resolution of $M/\sum_{j\in J} z_j M$.* □

15.2.4. Example. For $n = 1$, a graded R-module M is flat if and only if $z_1 \colon M \to M$ is injective. For $n = 2$, a graded R-module M is flat if and only if $z_1 \colon M \to M$ and $z_2 \colon M \to M$ are both injective and the Koszul complex $K(M; z_1, z_2)$:

$$M \xrightarrow{(-z_2, z_1)} M \oplus M \xrightarrow{z_1\bullet + z_2\bullet} M$$

is exact in the middle. (Here we are ignoring the grading in the notation.) The Koszul complex is just the simple complex associated to the double complex

$$\begin{array}{ccc} M & \xrightarrow{z_1} & M \\ {\scriptstyle z_2}\big\downarrow & & \big\downarrow{\scriptstyle z_2} \\ M & \xrightarrow{z_1} & M \end{array}$$

with Deligne's sign conventions, and the right-most term is in degree zero. The exactness of the Koszul complex in the middle can be read on each graded term as $M_{k_1-1,k_2} \cap M_{k_1,k_2-1} = M_{k_1-1,k_2-1}$. In this way, it is clear that two filtrations give rise to a flat R-module, illustrating thereby Theorem 15.2.2.

Exactness of the Koszul complex is closely related to the concept of regular sequences. Recall that $z_1, \dots, z_n$ form a *regular sequence* on M if multiplication by z_1 is injective on M, multiplication by z_2 is injective on M/z_1M, multiplication by z_3 is injective on $M/(z_1, z_2)M$, and so on.

***15.2.5. Corollary* (A flatness criterion).** *A graded R-module M is flat over R if and only if any permutation of $z_1, \dots, z_n$ is a regular sequence on M.*

Proof. This is one of the basic properties of the Koszul complex. The point is that multiplication by z_1 is injective on M if and only if the Koszul complex

$$M \xrightarrow{z_1} M$$

is a resolution of M/z_1M. If this is the case, multiplication by z_2 is injective on M/z_1M if and only if the Koszul complex

$$M \xrightarrow{(-z_2, z_1)} M \oplus M \xrightarrow{z_1\bullet + z_2\bullet} M$$

is a resolution of $M/(z_1, z_2)M$, etc. In general, the equivalence is obtained in Exercise 15.2 together with the flatness criterion of Proposition 15.2.3. □

***15.2.6. Proposition* (Another flatness criterion).** *Let M be an R-graded module. Assume that*

(1) *$z_1 : M \to M$ is injective and M/z_1M is flat over R/z_1R,*
(2) *$M_{\boldsymbol{k}} = 0$ if $k_1 < 0$.*

Then M is R-flat.

Proof. We apply the criterion of Proposition 15.2.3. Let J be a subset of $\{2, \dots, n\}$ and set $I = \{1\} \cup J$. On the one hand, since z_1 is injective on M, we have an exact sequence of complexes

$$0 \longrightarrow K(M; (z_j)_{j\in J}) \xrightarrow{z_1} K(M; (z_j)_{j\in J}) \longrightarrow K(M/z_1M; (z_j)_{j\in J}) \longrightarrow 0.$$

On the other hand, by definition, $K(M; (z_i)_{i\in I})$ is the cone of the morphism $K(M; (z_j)_{j\in J}) \xrightarrow{z_1} K(M; (z_j)_{j\in J})$. We deduce a quasi-isomorphism $K(M; (z_i)_{i\in I}) \simeq K(M/z_1M; (z_j)_{j\in J})$, and thus the cohomology of $K(M; (z_i)_{i\in I})$ is zero in negative degrees.

Let us now consider $K_J := K(M; (z_j)_{j\in J})$ with differential denoted by δ and show that its cohomology vanishes in negative degrees. The long exact sequence attached to the short exact sequence above shows that $z_1 : H^k(K_J) \to H^k(K_j)$ is an isomorphism for $k < 0$. Let $m \in K_J^k$ be such that $\delta m = 0$. Modulo a coboundary $\delta m''$ it is thus divisible by z_1, that is, $m = z_1m' + \delta m''$, and $\delta m' = 0$. Considering the graded components, this reads $m_{\boldsymbol{k}} = z_1 m'_{\boldsymbol{k}-\boldsymbol{1}_1} + (\delta m'')_{\boldsymbol{k}}$. Continuing this way, we write $m_{\boldsymbol{k}} = z_1^N \mu'_{\boldsymbol{k}-N\boldsymbol{1}_1} + (\delta\mu'')_{\boldsymbol{k}}$ for N large enough so that all nonzero graded components $m_{\boldsymbol{k}}$ of m satisfy $k_1 < N$. The second assumption implies that $\mu'_{\boldsymbol{k}-N\boldsymbol{1}_1} = 0$ for each $\boldsymbol{k}$, and thus the class of m in $H^k(K_J)$ is zero, as desired. □

Under certain conditions on the graded R-module M, one can deduce flatness from the vanishing of the single R-module

$$\mathrm{Tor}_1^R\big(M, R/(z_1, \dots, z_n)R\big).$$

In the case of local rings, this kind of result is usually called the "local criterion for flatness". The simplest example is when M is finitely generated as an R-module, which is to say that all the filtrations are bounded from below.

15.2.7. Proposition. *If M is a finitely generated graded R-module, then the vanishing of $\mathrm{Tor}_1^R\big(M, R/(z_1, \dots, z_n)R\big)$ implies that M is flat.*

Proof. This is a general result in commutative algebra. To show what is going on, let us give a direct proof in the case $n = 2$. By assumption, the Koszul complex

$$M \xrightarrow{(-z_2, z_1)} M \oplus M \xrightarrow{z_1\bullet + z_2\bullet} M$$

is exact in the middle. It follows quite easily that multiplication by z_1 is injective. Indeed, if there is an element $m \in M_{i,j}$ with $z_1m = 0$, then the pair $(m, 0)$ is in the kernel of the differential (z_1, z_2), and therefore $m = -z_2m'$ and $0 = z_1m'$ for some $m' \in M_{i,j-1}$. Continuing in this way, we eventually arrive at the conclusion that $m = 0$, because $M_{i,j} = 0$ for $j \ll 0$. For the same reason, multiplication by z_2 is injective; but now we have checked the condition in the definition of flatness for all graded prime ideals in R. □

***Proof of Theorem* 15.2.2.** Let us first show that if $F_\bullet^{(1)}A, \dots, F_\bullet^{(n)}A$ are distributive filtrations, then the associated Rees module M is flat over R. Because of the inherent symmetry, it is enough to prove that $z_n, \dots, z_1$ form a regular sequence on M. Because M comes from a filtered object, multiplication by z_n is injective and

$$M/z_nM = \bigoplus_{k\in\mathbb{Z}^n} M_{k_1,\dots,k_n}/M_{k_1,\dots,k_{n-1},k_n-1}.$$

This is now a $\mathbb{Z}^n$-graded module over the polynomial ring $\mathbb{C}[z_1, \dots, z_{n-1}]$. We remarked, after Definition 15.1.7, that for every $\ell \in \mathbb{Z}$, the $n-1$ induced filtrations on

$$A_\ell = \operatorname{gr}_\ell^{F^{(n)}} A = F_\ell^{(n)}A/F_{\ell-1}^{(n)}A$$

are still distributive, and by definition,

$$F_{k_1}^{(1)}A_\ell \cap \dots \cap F_{k_{n-1}}^{n-1}A_\ell \simeq M_{k_1,\cdots,k_{n-1},\ell}/M_{k_1,\dots,k_{n-1},\ell-1}.$$

By induction, this implies that $z_{n-1}, \dots, z_1$ form a regular sequence on M/z_nM, which is what we wanted to show.

For the converse, suppose that M is now an arbitrary graded R-module that is flat over R. We need to construct from M an object A with n distributive filtrations. We take $A = \varinjlim_{\boldsymbol{k}\in\mathbb{Z}^n} M_{\boldsymbol{k}}$ as defined by the dictionary 15.2.1(a). Setting $I = \{i\}$ in 15.2.1(b), we obtain a graded R_i-module $M^{(i)}$ which is R_i-flat (flatness is preserved by base change), hence of the form $R_{F^{(i)}}A$ for some filtration $F_\bullet^{(i)}A$. We will also use the flatness of M to prove that these n filtrations are distributive, and that

$$M_{k_1,\dots,k_n} = F_{k_1}^{(1)}A \cap \dots \cap F_{k_n}^{(n)}A, \tag{15.2.8}$$

as subobjects of A. We will argue by induction on n, by checking the criterion of Proposition 15.1.5(2) for the objects $\widetilde{A}_i$ (notation of Remark 15.1.8(3)). The case $n = 1$ is clear, and the case $n = 2$ is reduced to checking (15.2.8): the diagram of exact sequences (15.1.2) exists for the subobjects $\widetilde{R}_1, \widetilde{R}_2$ of $\widetilde{R}$ with lower left corner equal to $R = \widetilde{R}_1 \cap \widetilde{R}_2$; all sequences remain exact after tensoring by M over R, and the lower left corner is $M = \widetilde{M}^{(1)} \cap \widetilde{M}^{(2)}$, as desired. We now assume $n \geqslant 3$.

Let us start with 15.1.5(2b). If $n = 3$, the family $(\widetilde{R}_i)_{i=1,2,3}$ of subobjects of $\widetilde{R}$ is clearly distributive. Since M is R-flat, each $\widetilde{R}_i \otimes_R M$ is a subobject of $\widetilde{R} \otimes_R M = \widetilde{A}$ and the criterion of Lemma 15.1.3 implies that the family $(\widetilde{R}_i \otimes_R M)$ of subobjects of $\widetilde{A}$ is also distributive. As seen in 15.2.1(b) and since $M^{(i)} = R_{F^{(i)}}A$, this is nothing but the family $(\widetilde{A}_i)$, so that 15.1.5(2b) holds if $n = 3$. If $n \geqslant 4$, we apply the previous result to $M^{(I)}$ for any subset of three elements in $\{1, \dots, n\}$. This is possible since $M^{(I)}$ is R_I-flat, as flatness is preserved by base change. Therefore, 15.1.5(2b) holds for all $n \geqslant 3$.

For 15.1.5(2a), we set $z' = (z_2, \dots, z_n)$. We regard $R' := \mathbb{C}[z']$ as a subalgebra of R and M as an R'-module, that we write as the direct sum of R'-modules $\bigoplus_{k_1\in\mathbb{Z}} M'_{k_1}$. As such, it is still flat, and therefore each $M'_{k_1} = \bigoplus_{\boldsymbol{k}'\in\mathbb{Z}^{n-1}} M_{k_1,\boldsymbol{k}'}$ is also R'-flat. The limit $\varinjlim_{\boldsymbol{k}'\in\mathbb{Z}^{n-1}} M_{k_1,\boldsymbol{k}'}$ is $M_{k_1}^{(1)}$ that we have identified with $F_{k_1}^{(1)}A$. For $i \geqslant 2$, we write

$(M'_{k_1})^{(i)} = R_{F^{(i)}}(F^{(1)}_{k_1}A)$. By induction, the family $(F^{(i)}_\bullet(F^{(1)}_{k_1}A))_{i\geqslant 2}$ is distributive and for each $\boldsymbol{k}' \in \mathbb{Z}^{n-1}$, we have

$$M_{k_1,\boldsymbol{k}'} = F^{(2)}_{k_2}(F^{(1)}_{k_1}A) \cap \cdots \cap F^{(n)}_{k_n}(F^{(1)}_{k_1}A).$$

On the other hand, we have for each $k_i \in \mathbb{Z}$ the equality $(M'_{k_1})^{(i)}_{k_i} = M^{\{1,i\}}_{k_1,k_i}$ and, by the case $n = 2$ treated above, we conclude that $F^{(i)}_{k_i}(F^{(1)}_{k_1}A) = (F^{(i)}_{k_i}A) \cap (F^{(1)}_{k_1}A)$, so that the first part of 15.1.5(2a) holds, as well as (15.2.8).

The second part of 15.1.5(2a) amounts to asking that, for any $k_1 \in \mathbb{Z}$, the induced filtrations $F^{(i)}_\bullet(A/F^{(1)}_{k_1}A)$ $(i \geqslant 2)$ form a distributive family. We set $z' = (z_2, \dots, z_n)$ and $R' = \mathbb{C}[z']$ that we regard as R/z_1R. Since M is R-flat, $R' \otimes_R M$ is R'-flat. We write

$$R' \otimes_R M = \bigoplus_{k_1\in\mathbb{Z}} \bigoplus_{\boldsymbol{k}'\in\mathbb{Z}^{n-1}} M_{k_1+1,\boldsymbol{k}'}/M_{k_1,\boldsymbol{k}'},$$

so that $\bigoplus_{\boldsymbol{k}'\in\mathbb{Z}^{n-1}} M_{k_1+1,\boldsymbol{k}'}/M_{k_1,\boldsymbol{k}'}$ is R'-flat for any k_1. By induction on $\ell \geqslant 0$, we deduce that $\bigoplus_{\boldsymbol{k}'\in\mathbb{Z}^{n-1}} M_{k_1+\ell,\boldsymbol{k}'}/M_{k_1,\boldsymbol{k}'}$ is R'-flat for any k_1, ℓ and taking inductive limit on ℓ leads to the R'-flatness of $M'_{\boldsymbol{k}'}/M_{k_1,\boldsymbol{k}'}$ for any k_1. By induction on n, the filtrations on $\varinjlim_{\boldsymbol{k}'} M'_{\boldsymbol{k}'}/M_{k_1,\boldsymbol{k}'} = A/F^{(1)}_{k_1}A$ obtained by taking inductive limit with k_i fixed are distributive. They read, for $i \geqslant 2$, $M^{(i)}_{k_i}/M^{(\{1,i\})}_{k_1,k_i}$. By the first part of the proof of 15.1.5(2a), this expression writes $F^{(i)}_{k_i}A/(F^{(1)}_{k_1}A \cap F^{(i)}_{k_i}A)$, so that the desired distributivity is obtained. □

15.2.9. Remark **(Interpretation of flatness in terms of multi-grading)**

Corollary 15.2.5 has the following practical consequence: for distributive filtrations $F^{(1)}_\bullet A, \dots, F^{(n)}_\bullet A$, the n-graded object obtained by inducing iteratively the filtrations on the j-graded object $\mathrm{gr}^{F^{(i_j)}}_{k_{i_j}} \cdots \mathrm{gr}^{F^{(i_1)}}_{k_{i_1}} A$ $(j = 1, \dots, n)$ does not depend on the order $\{i_1, \dots, i_n\} = \{1, \dots, n\}$, and is equal to

$$\frac{F^{(1)}_{k_1}A \cap \cdots \cap F^{(n)}_{k_n}A}{\sum_j F^{(1)}_{k_1}A \cap \cdots \cap F^{(j)}_{k_j-1}A \cap \cdots \cap F^{(n)}_{k_n}A}.$$

When one filtration is bounded from below, the inductive property of Proposition 15.1.5(2) takes a more accessible form.

15.2.10. Proposition **(Distributivity by induction).** *Let $(A, F^{(1)}_\bullet A, \dots, F^{(n)}_\bullet A)$ be a multi-filtered object of* A. *Assume the following properties:*

(a) $F^{(1)}_p A = 0$ *for* $p \ll 0$*;*

(b) *for each p, the induced filtrations $F^{(2)}, \dots, F^{(n)}$ on $\mathrm{gr}^{F^{(1)}}_p A$ are distributive;*

(c) *for each p, the natural morphism $R_{F'}(F^{(1)}_p A) \to R_{F'}(\mathrm{gr}^{F^{(1)}}_p A)$ is an epimorphism.*

Then the filtrations $F^{(1)}, F^{(2)}, \dots, F^{(n)}$ on A are distributive.

Proof. We will apply Proposition 15.2.6 to the Rees module $M = R_F A$. It is clear that multiplication by z_1 is injective on M, so we only need to check the $\mathbb{C}[z_2, \dots, z_n]$-flatness of M/z_1M.

Set $\boldsymbol{p}' = (p_2, \dots, p_n)$, $F'_{\boldsymbol{p}'}A = \bigcap_{i=2}^n F^{(i)}_{p_i}A$ and, for each p,

$$F'_{\boldsymbol{p}'}(\mathrm{gr}_p^{F^{(1)}}A) = \bigcap_{i=2}^n F^{(i)}_{p_i}\mathrm{gr}_p^{F^{(1)}}A = \frac{\bigcap_{i=2}^n\big((F^{(i)}_{p_i}A \cap F^{(1)}_pA) + F^{(1)}_{p-1}A\big)}{F^{(1)}_{p-1}A}.$$

Then (c) amounts to

$$(F'_{\boldsymbol{p}'}A \cap F^{(1)}_pA) + F^{(1)}_{p-1}A = \bigcap_{i=2}^n\big((F^{(i)}_{p_i}A \cap F^{(1)}_pA) + F^{(1)}_{p-1}A\big) \quad \forall\, p. \tag{15.2.11}$$

On the other hand, M/z_1M is the direct sum indexed by p of the terms

$$(F'_{\boldsymbol{p}'}A \cap F^{(1)}_pA) + F^{(1)}_{p-1}A/F^{(1)}_{p-1}A = (F'_{\boldsymbol{p}'}A \cap F^{(1)}_pA)/(F'_{\boldsymbol{p}'}A \cap F^{(1)}_{p-1}A).$$

Therefore, (c) amounts to the equality

$$M/z_1M = R_{F'}\mathrm{gr}^{F^{(1)}}A,$$

and (b) yields it $\mathbb{C}[z_2, \dots, z_n]$-flatness. □

15.2.12. Remark **(Multi-filtered morphisms).** Given two multi-filtered objects

$$(A, (F^{(i)}_\bullet A)_{i=1,\dots,n}) \quad\text{and}\quad (B, (F^{(i)}_\bullet B)_{i=1,\dots,n})$$

in A, let $\varphi : A \to B$ be a morphism compatible with the filtrations. It induces various morphisms $\mathrm{gr}^{F^{(i_j)}}_{k_{i_j}} \cdots \mathrm{gr}^{F^{(i_1)}}_{k_{i_1}}\varphi$. Assume that the filtrations in A and in B are distributive. Then the source and the target of these morphisms are independent of the order of multi-grading, as remarked above. We claim that *the morphisms $\mathrm{gr}^{F^{(i_j)}}_{k_{i_j}} \cdots \mathrm{gr}^{F^{(i_1)}}_{k_{i_1}}\varphi$ are also independent of the order of multi-grading.* Indeed, φ induces a graded morphism $R_F\varphi : M \to N$ between the associated Rees objects, and due to the distributivity assumption, we are led to checking that the restriction of $R_F\varphi$ to $M/(z_{k_{i_1}}, \dots, z_{k_{i_j}})M$ is independent of the order, which is clear.

15.2.b. Application of the flatness criterion. We will make more explicit the general notion of distributive filtrations in the case of x_i-adic filtrations on a coherent $\mathcal{O}_X$-module. For such a module $\mathcal{E}$, assume we are given, for each $i = 1, \dots, n$, an increasing filtration $V^{(i)}_\bullet\mathcal{E}$ indexed by $[-1, 0)$ by coherent submodules, such that $\mathcal{E} = \bigcup_{\alpha_i \in [-1,0)} V^{(i)}_{\alpha_i}\mathcal{E}$ and the set of jumps $A_i \subset [-1, 0)$ is finite. We extend the filtration as a filtration indexed by $A_i + \mathbb{Z}$ by setting

$$V^{(i)}_{\alpha_i+k}\mathcal{E} = \begin{cases} x_i^{-k}V^{(i)}_{\alpha_i}\mathcal{E} & \text{if } k \leqslant 0,\\ V^{(i)}_{\alpha_i}\mathcal{E} & \text{if } k \geqslant 0.\end{cases}$$

We define $V^{(\boldsymbol{n})}_{\boldsymbol{a}}\mathcal{E} = \bigcap_i V^{(i)}_{a_i}\mathcal{E}$ for any $\boldsymbol{a} \in \prod_i(A_i + \mathbb{Z}_i)$.

15.2.13. Example **(Rank-one objects).** Assume that $\mathcal{E}$ is $\mathcal{O}_X$-locally free of rank 1. Then, for each i, A_i is reduced to one element $\alpha_i \in [-1, 0)$ and we have for any $\boldsymbol{a} \in \prod_i(A_i + \mathbb{Z})$

$$V^{(\boldsymbol{n})}_{\boldsymbol{a}}\mathcal{E} = \mathcal{E}(\textstyle\sum_{i|a_i \leqslant \alpha_i}[a_i - \alpha_i]D_i).$$

We claim that the family $(V^{(1)}_\bullet\mathcal{E}, \dots, V^{(n)}_\bullet\mathcal{E})$ is distributive. Indeed, the multi-Rees module $R_V\mathcal{E}$ (see Section 15.2.a) reads

$$\bigoplus_{\boldsymbol{k}\in\mathbb{Z}^n} x^{\boldsymbol{k}}\mathcal{E}z^{-\boldsymbol{k}}, \quad \text{with } x_i^{k_i} := 1 \text{ if } k_i \leqslant 0.$$

We have to check that each permutation of $(z_1, \dots, z_n)$ is a regular sequence on $R_{V^{(n)}}\mathcal{E}$. This is obtained by induction, noticing that z_i is injective on $R_{V^{(n)}}\mathcal{E}$ and $R_{V^{(n)}}\mathcal{E}/z_i R_{V^{(n)}}\mathcal{E}$ is the Rees module of $\mathrm{gr}^{V^{(i)}}_\bullet \mathcal{E}$ equipped with the similar filtrations $V^{(j)}_\bullet \mathrm{gr}^{V^{(i)}}_\bullet \mathcal{E}$ $(j \neq i)$.

15.2.14. Proposition. *Let $\mathcal{E}$ be a coherent $\mathcal{O}_X$-module and let $(V^{(1)}_\bullet \mathcal{E}, \dots, V^{(n)}_\bullet \mathcal{E})$ be filtrations as defined above. Let us assume that, for each $\boldsymbol{a} \in \prod_i (A_i + \mathbb{Z})$,*

(1) *the $\mathcal{O}_X$-module $V^{(\boldsymbol{n})}_{\boldsymbol{a}}\mathcal{E}$ is locally free,*

(2) *if $a_i < 0$, then $x_i V^{(\boldsymbol{n})}_{\boldsymbol{a}}\mathcal{E} = V^{(\boldsymbol{n})}_{\boldsymbol{a}-\boldsymbol{1}_i}\mathcal{E}$.*

Then the filtrations $(V^{(1)}_\bullet \mathcal{E}, \dots, V^{(n)}_\bullet \mathcal{E})$ are distributive.

Proof. Note that the assumption implies that $\mathcal{E}$ itself is $\mathcal{O}_X$-locally free. The multi-Rees module $R_{V^{(n)}}\mathcal{E}$ is the direct sum over $\boldsymbol{\alpha} \in \prod_i A_i$ of multi-Rees modules associated with the multi-filtrations $V^{(\boldsymbol{n})}_{\boldsymbol{\alpha}+\mathbb{Z}}\mathcal{E}$. To check its $\mathbb{C}[z_1, \dots, z_n]$-flatness, it is enough to check that of each summand. We can therefore assume that $\prod_i A_i = \{\boldsymbol{\alpha}\}$. We then simply write $V^{(\boldsymbol{n})}_{\boldsymbol{\alpha}+\boldsymbol{k}}\mathcal{E} = V^{(\boldsymbol{n})}_{\boldsymbol{k}}\mathcal{E}$. By (2),

$$V^{(\boldsymbol{n})}_{\boldsymbol{k}}\mathcal{E} = \mathcal{E}\big(\textstyle\sum_{i | k_i \leqslant 0} k_i D_i\big)$$

and we argue as in the example to conclude. □

15.3. Strictness of morphisms

Let A and B be two objects in our abelian category A, each with n distributive filtrations

$$F^{(1)}_\bullet A, \dots, F^{(n)}_\bullet A, \quad \text{respectively} \quad F^{(1)}_\bullet B, \dots, F^{(n)}_\bullet B.$$

Denote by M and N the graded R-modules that are obtained by the Rees construction; both are flat by Theorem 15.2.2. Now consider a filtered morphism $\varphi\colon A \to B$. It induces an R-linear morphism $R_F\varphi\colon M \to N$ between the two Rees modules.

15.3.1. Definition. We say that $\varphi\colon A \to B$ is *n-strict* if $\operatorname{Coker} R_F\varphi$ is again a flat R-module.

Flatness of $\operatorname{Coker} R_F\varphi$ also implies that $\operatorname{Ker} R_F\varphi$ and $\operatorname{Im} R_F\varphi$ are flat: the reason is that we have two short exact sequences

$$0 \longrightarrow \operatorname{Ker} R_F\varphi \longrightarrow M \longrightarrow \operatorname{Im} R_F\varphi \longrightarrow 0$$

and

$$0 \longrightarrow \operatorname{Im} R_F\varphi \longrightarrow N \longrightarrow \operatorname{Coker} R_F\varphi \longrightarrow 0,$$

and because M and N are both flat, flatness of $\operatorname{Coker} R_F\varphi$ implies that of $\operatorname{Im} R_F\varphi$, which implies that of $\operatorname{Ker} R_F\varphi$. Note that $\operatorname{Ker}\varphi$ and $\operatorname{Coker}\varphi$ are equipped with filtrations $F^{(1)}_\bullet \operatorname{Ker}\varphi, \dots, F^{(n)}_\bullet \operatorname{Ker}\varphi$, respectively $F^{(1)}_\bullet \operatorname{Coker}\varphi, \dots, F^{(n)}_\bullet \operatorname{Coker}\varphi$ naturally induced from those on A, respectively B. On the other hand, $\operatorname{Im}\varphi$ has two possible natural families of filtrations: those induced from M and those from N. If φ is n-strict, both coincide and we have

$$\operatorname{Ker} R_F\varphi = R_F \operatorname{Ker}\varphi, \quad \operatorname{Im} R_F\varphi = R_F \operatorname{Im}\varphi, \quad \operatorname{Coker} R_F\varphi = R_F \operatorname{Coker}\varphi.$$

Indeed, we know by Theorem 15.2.2 that the graded modules $\operatorname{Ker} R_F\varphi$, $\operatorname{Im} R_F\varphi$ and $\operatorname{Coker} R_F\varphi$ are attached to distributive filtrations, and (for $\operatorname{Coker}\varphi$ for example) the term in degree $k \in \mathbb{Z}^n$ is $(F^{(1)}_{k_1}B \cap \cdots \cap F^{(n)}_{k_n}B) + \operatorname{Im}\varphi/\operatorname{Im}\varphi$, so that the distributive filtrations on $\operatorname{Coker}\varphi$ given by the theorem are nothing but the filtrations induced by $F^i_\bullet B$.

For example, in the case of two filtrations F', F'' as considered in Definition 10.2.4, the last equality in bi-degree k, ℓ gives

$$F'_k F''_\ell B/\varphi(F'_k F''_\ell A) = (F'_k B + \operatorname{Im}\varphi) \cap (F''_\ell B + \operatorname{Im}\varphi)/\operatorname{Im}\varphi,$$

which corresponds to the condition of Definition 10.2.4.

15.3.2. Caveat. The strictness of φ implies that the induced filtrations (on $\operatorname{Ker}\varphi$, $\operatorname{Im}\varphi$ and $\operatorname{Coker}\varphi$) are distributive. However, the latter condition is not enough for ensuring strictness of φ. For example, two filtrations are always distributive, while a morphism between bi-filtered objects need not be strict.

15.3.3. Example **(Strict inclusions).** The composition of strict morphisms need not be strict in general. However, the composition of strict monomorphisms i_1, i_2 between objects with distributive filtrations remains a strict monomorphism since $\operatorname{Coker} R_F(i_2 \circ i_1) = \operatorname{Coker}(R_F i_2 \circ R_F i_1)$ is an extension of $\operatorname{Coker} R_F i_2$ by $\operatorname{Coker} R_F i_1$, and flatness is preserved by extensions.

Given n distributive filtrations $F^{(1)}_\bullet A, \dots, F^{(n)}_\bullet A$, they induce distributive filtrations on $A_{\boldsymbol{k}} := F^{(1)}_{k_1}A \cap \cdots \cap F^{(n)}_{k_n}A$ for every $\boldsymbol{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n$ (see Remark 15.1.8). Moreover, for $\boldsymbol{k} \leqslant \boldsymbol{\ell} \in \mathbb{Z}^n$ (i.e., $k_i \leqslant \ell_i$ for all $i = 1, \dots, n$), the inclusion $A_{\boldsymbol{k}} \hookrightarrow A_{\boldsymbol{\ell}}$ is n-strict. Indeed, by the preliminary remark, it is enough to show that the inclusion $A_{\boldsymbol{k}-\boldsymbol{1}_i} \hookrightarrow A_{\boldsymbol{k}}$ is strict for all i. This has been explained in the first part of the proof of Theorem 15.2.2.

15.3.4. Proposition **(A criterion for strictness of inclusions).** *Let $(A, F^{(1)}_\bullet A, \dots, F^{(n)}_\bullet A)$ and $(B, F^{(1)}_\bullet B, \dots, F^{(n)}_\bullet B)$ be multi-filtered objects of A with distributive filtrations, and let φ be a multi-filtered monomorphism between them. Assume the following properties:*

(a) *$F^{(1)}_p B = 0$ for $p \ll 0$;*
(b) *φ is $F^{(i)}$-strict for $i = 1, \dots, n$ (i.e., $F^{(i)}_p A = F^{(i)}_p B \cap A$),*
(c) *for each p, $\operatorname{gr}^{F^{(1)}}_p \varphi : \operatorname{gr}^{F^{(1)}}_p A \to \operatorname{gr}^{F^{(1)}}_p B$ is an $(n-1)$-strict monomorphism.*

Then φ is an n-strict monomorphism.

Proof. We consider the exact sequence $0 \to R_F A \xrightarrow{R_F\varphi} R_F B \to M \to 0$ and we wish to show the R-flatness of M, where $R = \mathbb{C}[z_1, \dots, z_n]$. We will apply the criterion of Proposition 15.2.6 to M.

It is clear that multiplication by z_1 is injective on $R_F A$ and $R_F B$. On the other hand, (c) means that the sequence

$$0 \longrightarrow R_F A/z_1 R_F A \longrightarrow R_F B/z_1 R_F B \longrightarrow M/z_1 M \longrightarrow 0,$$

is exact and that M/z_1M is R/z_1R-flat. The snake lemma implies that $z_1 : M \to M$ is injective, so, because of (a), the flatness criterion of Proposition 15.2.6 applies to M. □

***15.3.5. Definition* (Strictness of a multi-filtered complex).** If we have a complex of objects with n *distributive* filtrations and differentials that preserve the filtrations, we consider the associated complex of flat graded R-modules; if all of its cohomology modules are again flat over R, we say that the original filtered complex is *strict*.

The interpretation of distributivity in term of flatness yields the following criterion (see Definition 15.3.1).

15.3.6. Proposition. *A complex of objects with n* distributive *filtrations and differentials that preserve the filtrations, and which is bounded from above, is strict if and only if each individual differential is an n-strict morphism.* □

15.4. Appendix. Compatible filtrations

The definition of compatibility of three subobjects of A given before Lemma 15.1.3 has a natural extension for n subobjects. We will see that it is equivalent to the notion of distributivity, but sheds a new light on other properties.

More precisely, the condition is the following: there should exist an n-dimensional commutative diagram $C(A_1, \dots, A_n; A)$, consisting of 3^n objects placed at the points $\{-1, 0, 1\}^n$ and $2n \cdot 3^{n-1}$ morphisms corresponding to the line segments connecting those points, such that A sits at the point $(0, \dots, 0)$, each A_i sits at the point $(0, \dots, -1, \dots, 0)$ on the i-th coordinate axis, and all lines parallel to the coordinate axes form short exact sequences in the abelian category. It is easy to see that the objects at points in $\{-1, 0\}^n$ are just intersections: if the i-th coordinate of such a point is -1 for $i \in I \subset \{1, \dots, n\}$ and 0 for $i \notin I$, then the exactness of the diagram forces the corresponding object to be

$$\bigcap_{i \in I} A_i,$$

with the convention that the intersection equals A when I is empty. In particular, the object $A_1 \cap \cdots \cap A_n$ always sits at the point with coordinates $(-1, \dots, -1)$.

On the other hand, given a subset $I \subset \{1, \dots, n\}$, fixing the coordinate $\varepsilon_i^o \in \{-1, 0, 1\}$ for every $i \in I$ produces a sub-diagram of size $n - \#I$, hence $n - \#I$ compatible sub-objects of the term placed at $(\varepsilon^o_{i \in I}, 0_{i \notin I})$, that we denote by $A(\varepsilon^o_{i \in I}, 0_{i \notin I})$. For example, fixing $\varepsilon_i^o = 0$ for each $i \in I$ shows that the sub-family $(A_i)_{i \notin I}$ is a compatible family.

As another example, fix $\varepsilon_n^o = -1$. Then the induced family $(A_i \cap A_n)_{i \in \{0, \dots, n-1\}}$ of sub-objects of A_n is also compatible. In the definition of compatibility, the object A does not play a relevant role and one can replace it by a sub-object provided that all A_i are contained in it. Similarly one can replace it by a sup-object. This is shown in Exercise 15.6. So the induced family $(A_i \cap A_n)_{i \in \{0, \dots, n-1\}}$ is also compatible in A.

As still another example, let us fix $\varepsilon_n^o = 1$. We have an exact sequence

$$A_n = A(0,\dots,0,-1) \longrightarrow A = A(0,\dots,0) \longrightarrow A/A_n = A(0,\dots,0,1).$$

Our new diagram has central term A/A_n and the term placed at $(0,\dots,(-1)_i,\dots,0,1)$ is $A_i/A_i\cap A_n = (A_i+A_n)/A_n$: the induced family $\big((A_i+A_n)/A_n\big)_{i\in\{0,\dots,n-1\}}$ is also compatible.

The definition of a compatible family of filtrations is similar to Definition 15.1.7 by replacing the word "distributive" with the word "compatible". Then any sub-family of filtrations of a compatible family remains compatible. Moreover, any finite family of sub-objects consisting of terms of the filtrations $F^{(1)}_\bullet A,\dots,F^{(n)}_\bullet A$ is compatible, and Lemma 15.4.1 below, whose proof is postponed at the end of this section, implies that the induced filtrations $F^{(1)}_\bullet,\dots,F^{n-1}_\bullet$ on each $\mathrm{gr}^{F^{(n)}}_\ell A$ are compatible.

15.4.1. Lemma. *Let $A_1,\dots,A_n\subset A$ be a family of sub-objects of A. Assume the following properties:*

(1) $A_1\subset A_2$.
(2) *Both sub-families $A_1,A_3,\dots,A_n$ and $A_2,A_3,\dots,A_n$ are compatible.*

Then the family $A_1,\dots,A_n$ is compatible. Moreover, the family $(A_i\cap A_2)/(A_i\cap A_1)$ $(i=3,\dots,n)$ of sub-objects of A_2/A_1 is also compatible.

Lemma 15.1.3 extends to any $n\geqslant 4$:

15.4.2. Proposition. *A family of n subobjects of A is distributive if and only if it is compatible.*

Proof. We show that Theorem 15.2.2 holds when compatibility replaces distributivity. The proof that compatibility implies flatness is similar to that of Theorem 15.2.2 in the case of distributive filtrations, in view of the remark above.

The proof of the converse is simpler than in the case of distributive filtrations. Fix $k,\ell\in\mathbb{Z}^n$. Observe that because R is graded, the graded submodules $z_1^{\ell_1}R,\dots,z_n^{\ell_n}R$ are trivially distributive; in fact, the required n-dimensional commutative diagram exists in the category of graded R-modules. If we tensor this diagram by M, it remains exact everywhere, due to the fact that M is flat. Take the graded piece of degree $k+\ell$ everywhere; for $n=2$, for example, the result looks like this:

$$\begin{array}{ccccc}
* & \longrightarrow & * & \longrightarrow & * \\
\uparrow & & \uparrow & & \uparrow \\
M_{k_1+\ell_1,k_2} & \longrightarrow & M_{k_1+\ell_1,k_2+\ell_2} & \longrightarrow & * \\
\uparrow & & \uparrow & & \uparrow \\
M_{k_1,k_2} & \longrightarrow & M_{k_1,k_2+\ell_2} & \longrightarrow & *
\end{array}$$

Apply the direct limit over $\ell \in \mathbb{Z}^n$; this operation preserves exactness. For $n = 2$, for example, the resulting n-dimensional commutative diagram looks like this:

$$\begin{array}{ccccc} * & \longrightarrow & * & \longrightarrow & * \\ \uparrow & & \uparrow & & \uparrow \\ F^2_{k_2}A & \longrightarrow & A & \longrightarrow & * \\ \uparrow & & \uparrow & & \uparrow \\ M_{k_1,k_2} & \longrightarrow & F^{(1)}_{k_1}A & \longrightarrow & * \end{array}$$

The existence of such a diagram proves that $F^{(1)}_{k_1}A, \dots, F^{(n)}_{k_n}A$ are compatible subobjects of A, and also that (15.2.8) holds. $\square$

***Proof of Lemma* 15.4.1.** We wish to define a diagram with vertices $A(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ ($\varepsilon_i \in \{-1, 0, 1\}$) satisfying the properties above. The second assumption means that we have the diagrams with vertices $A(\varepsilon_1, 0, \varepsilon_3, \dots, \varepsilon_n)$ and $A(0, \varepsilon_2, \dots, \varepsilon_n)$. On the other hand, if the diagram we search for exists, the inclusion $A_1 \cap A_2 = A_1 \subset A_2$ is satisfied for all terms of the diagram, namely

$$A(-1,-1,\varepsilon_{\geqslant 3}) = A(-1,0,\varepsilon_{\geqslant 3}) \subset A(0,-1,\varepsilon_{\geqslant 3}). \tag{15.4.3}$$

We are thus forced to set

$$\begin{aligned} A(1,-1,\varepsilon_{\geqslant 3}) &:= A(0,-1,\varepsilon_{\geqslant 3})/A(-1,-1,\varepsilon_{\geqslant 3}) \\ A(1,1,\varepsilon_{\geqslant 3}) &:= A(0,1,\varepsilon_{\geqslant 3}). \end{aligned} \tag{15.4.4}$$

In such a way, we obtain a commutative diagram where the columns are exact sequences (by assumption for the middle one, by our setting for the left and right ones), as well as the middle horizontal line

$$\begin{array}{ccccc} A(1,-1,\varepsilon_{\geqslant 3}) & \longrightarrow & A(1,0,\varepsilon_{\geqslant 3}) & \longrightarrow & A(1,1,\varepsilon_{\geqslant 3}) \\ \uparrow\!\!\uparrow & & \uparrow\!\!\uparrow & & \| \\ A(0,-1,\varepsilon_{\geqslant 3}) & \hookrightarrow & A(0,0,\varepsilon_{\geqslant 3}) & \twoheadrightarrow & A(0,1,\varepsilon_{\geqslant 3}) \\ \cup\!\!\uparrow & & \cup\!\!\uparrow & & \uparrow \\ A(-1,-1,\varepsilon_{\geqslant 3}) & = & A(-1,0,\varepsilon_{\geqslant 3}) & \longrightarrow & A(-1,1,\varepsilon_{\geqslant 3}) = 0 \end{array} \tag{15.4.5}$$

It is then easy to check that the upper horizontal line is exact. This shows that, in the diagram of size n, the lines where ε_1 varies in $\{-1, 0, 1\}$ and all other ε_i fixed, as well as the lines where ε_2 varies and all other ε_i are fixed, are exact. Let us now vary ε_3, say, by fixing all other ε_i and let us omit ε_i for $i \geqslant 4$ in the notation. From the diagram above, we see that the only possibly non-obvious exact sequence has terms $A(1,-1,\varepsilon_3)_{\varepsilon_3=-1,0,1}$. We now consider the commutative diagram where the columns are exact and only the upper horizontal line is possibly non-exact:

$$\begin{array}{ccccc} A(1,-1,-1) & \longrightarrow & A(1,-1,0) & \longrightarrow & A(1,-1,1) \\ \uparrow & & \uparrow & & \uparrow \\ A(0,-1,-1) & \hookrightarrow & A(0,-1,0) & \twoheadrightarrow & A(0,-1,1) \\ \uparrow & & \uparrow & & \uparrow \\ A(-1,-1,-1) & \hookrightarrow & A(-1,-1,0) & \twoheadrightarrow & A(-1,-1,1) \end{array} \tag{15.4.6}$$

But the snake lemma shows its exactness. We conclude that the family $A_1, A_2, \dots, A_n$ is compatible. We now remark that

$$A_2/A_1 = A_2/(A_1 \cap A_2) = A(1,-1,0,\dots,0).$$

The compatibility of the family $(A_i \cap A_2/A_i \cap A_1)_{i=3,\dots,n}$ will be proved if we prove $(A_3 \cap A_2)/(A_3 \cap A_1) = A(1,-1,-1,0,\dots,0)$, and similarly for $i \geqslant 4$. Let us consider the previous diagram when fixing $\varepsilon_i = 0$ for $i \geqslant 4$. The left vertical inclusion reads $A_1 \cap A_2 \cap A_3 \hookrightarrow A_2 \cap A_3$, hence the desired equality. □

15.5. Exercises

***Exercise 15.1* (Basics on Rees modules).** We take up the notation of Section 15.2.a. Set $\mathbb{C}[z] = \mathbb{C}[z_1,\dots,z_n]$. Let $M = \bigoplus_{\boldsymbol{k}\in\mathbb{Z}^n} M_{\boldsymbol{k}}$ be a $\mathbb{Z}^n$-graded $\mathbb{C}[z]$-module.

(1) Show that the subset $T_{\mathrm{m}}M \subset M$ consisting of elements $m \in M$ annihilated by a monomial in $z_1,\dots,z_n$ is a graded $\mathbb{C}[z]$-submodule of M. Conclude that M/T_{m} is a graded $\mathbb{C}[z]$-module.

(2) Let $T \subset M$ be the $\mathbb{C}[z]$-torsion submodule of M. Show that $T = T_{\mathrm{m}}$. [*Hint*: Assume that $T_{\mathrm{m}} = 0$ by working in M/T_{m}; if $pm = 0$ with $p = \sum p_{\boldsymbol{j}} z^{\boldsymbol{j}} \in \mathbb{C}[z]$ and $m = \sum_{\boldsymbol{k}} m_{\boldsymbol{k}} \in M$, choose a linear form L with non-negative coefficients such that $\max\{L(\boldsymbol{j}) \mid p_{\boldsymbol{j}} \neq 0\}$ is achieved for a unique index $\boldsymbol{j} = \boldsymbol{j}_o$ and similarly for $\boldsymbol{k}$ and $\boldsymbol{k}_o$; show that $z^{\boldsymbol{j}_o} m_{\boldsymbol{k}_o} = 0$ and conclude that $m = 0$.]

(3) Show that M is $\mathbb{C}[z]$-torsion free if and only if the natural morphism $M \to M[z^{-1}] := M \otimes_{\mathbb{C}[z]} \mathbb{C}[z^{-1}]$ is injective.

(4) Set $A = M/\sum_i (z_i - 1)M$. Show that M is $\mathbb{C}[z]$-torsion free if and only if there exists an exhaustive $\mathbb{Z}^n$-filtration $F_\bullet A$ such that $M = R_F A := \bigoplus_{\boldsymbol{k}\in\mathbb{Z}^n} (F_{\boldsymbol{k}} A) z^{\boldsymbol{k}}$. [*Hint*: Show first that $A = M[z^{-1}]/\sum_i (z_i - 1)M[z^{-1}]$ and $M[z^{-1}] = A \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}]$; consider then the graded inclusion $M \hookrightarrow A \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}]$.]

(5) *Omitting indices.* Let $(A, F_\bullet A)$ be a multi-filtered vector space, let $I \subset \{1,\dots,n\}$ be a subset and denote by I^{c} its complement. Let $F_\bullet^{(I)} A$ be the $\mathbb{Z}^I$-filtration defined by $F_{\boldsymbol{k}_I}^{(I)} A := \bigcup_{\boldsymbol{k}_{I^{\mathrm{c}}} \in \mathbb{Z}^{I^{\mathrm{c}}}} F_{(\boldsymbol{k}_I, \boldsymbol{k}_{I^{\mathrm{c}}})}^{(I)} A$. Show that

$$R_{F^{(I)}} A = \big(R_F A / \textstyle\sum_{i\in I^{\mathrm{c}}} (z_i - 1) R_F A\big) \,/\, \mathbb{C}[z_I]\text{-torsion}.$$

Conclude that if $R_F A$ is $\mathbb{C}[z]$-flat, then $R_{F^{(I)}} A$ is $\mathbb{C}[z_I]$-flat. [*Hint*: Use that flatness is preserved by base change.]

(6) *Grading.* Set

$$F^{(I)}_{\gneqq \boldsymbol{k}_I}A=\sum_{i\in I}F^{(I)}_{\boldsymbol{k}_I-\boldsymbol{1}_i}A,\quad \mathrm{gr}^{F^{(I)}}_{\boldsymbol{k}_I}A=F^{(I)}_{\boldsymbol{k}_I}A/F^{(I)}_{\gneqq\boldsymbol{k}_I}A,\quad F_{(\gneqq\boldsymbol{k}_I,\boldsymbol{k}_{I^c})}A=\sum_{i\in I}F_{(\boldsymbol{k}_I-\boldsymbol{1}_i,\boldsymbol{k}_{I^c})}A,$$

$$F^{(I^c)}_{\boldsymbol{k}_{I^c}}\mathrm{gr}^{F^{(I)}}_{\boldsymbol{k}_I}A=F_{(\boldsymbol{k}_I,\boldsymbol{k}_{I^c})}A\bigm/\bigl[F_{(\boldsymbol{k}_I,\boldsymbol{k}_{I^c})}A\cap F^{(I)}_{\gneqq\boldsymbol{k}_I}A\bigr].$$

Show that there exist isomorphisms as $\mathbb{Z}^n$-graded modules:

$$R_FA/\textstyle\sum_{i\in I}z_iR_FA\simeq\displaystyle\bigoplus_{(\boldsymbol{k}_I,\boldsymbol{k}_{I^c})\in\mathbb{Z}^n}F_{(\boldsymbol{k}_I,\boldsymbol{k}_{I^c})}A/F_{(<\boldsymbol{k}_I,\boldsymbol{k}_{I^c})}A\cdot z^{\boldsymbol{k}_{I^c}}$$

and

$$\bigoplus_{\boldsymbol{k}_I\in\mathbb{Z}^I}R_{F^{(I^c)}}\mathrm{gr}^{F^{(I)}}_{\boldsymbol{k}_I}A\simeq\bigl(R_FA/\textstyle\sum_{i\in I}z_iR_FA\bigr)\bigm/\mathbb{C}[z_{I^c}]\text{-torsion}.$$

Identify $R_{F^{(I^c)}}\mathrm{gr}^{F^{(I)}}_{\boldsymbol{k}_I}A$ with the term of I-degree $\boldsymbol{k}_I$ in the right-hand side. Conclude that if R_FA is $\mathbb{C}[z]$-flat, then $R_{F^{(I^c)}}\mathrm{gr}^{F^{(I)}}_{\boldsymbol{k}_I}A$ is $\mathbb{C}[z_{I^c}]$-flat. [*Hint*: Use that flatness is preserved by base change.] Conclude that, if R_FA is $\mathbb{C}[z]$-flat, the inclusion $F_{(\gneqq\boldsymbol{k}_I,\boldsymbol{k}_{I^c})}A\subset F_{(\boldsymbol{k}_I,\boldsymbol{k}_{I^c})}A\cap F^{(I)}_{\gneqq\boldsymbol{k}_I}A$ is an equality.

(7) Show that if R_FA is $\mathbb{C}[z]$-flat, then $F_{\boldsymbol{k}}A=\bigcap_{i=1}^nF^{(i)}_{k_i}A$. [*Hint*: Argue by induction on n and prove $F_{\boldsymbol{k}}A=F^{(1)}_{k_1}A\cap F^{\{1\}^c}_{\boldsymbol{k}_{\{1\}^c}}A$ by using the last result of (6).]

Exercise 15.2 **(Regular sequences and the Koszul complex).** We keep the notation as in Proposition 15.2.3.

(1) Show that the sequence $z_1,\dots,z_n$ is a regular sequence on M if and only if for every $k=1,\dots,n$, the Koszul complex $K(M;z_1,\dots,z_k)$ is a graded resolution of $M/(z_1,\dots,z_k)M$.

(2) Deduce that the following properties are equivalent:

(a) any permutation of $z_1,\dots,z_n$ is a regular sequence on M,

(b) any subsequence of $z_1,\dots,z_n$ is a regular sequence on M,

(c) for every subset $J\subset\{1,\dots,n\}$ the Koszul complex $K(M;(z_j)_{j\in J})$ is a graded resolution of $M/(z_j)_{j\in J}M$.

Exercise 15.3 **(Applications of the flatness criterion).**

(1) Let A be an object with n distributive filtrations $F^{(1)}_\bullet A,\dots,F^{(n)}_\bullet A$ and let $F^{(n+1)}A$ be a filtration which jumps at one index at most, for example $F^{(n+1)}_{-1}A=0$ and $F^{(n+1)}_0A=A$. Show that the family $F^{(1)}_\bullet A,\dots,F^{(n+1)}_\bullet A$ is still distributive. [*Hint*: Show that the new Rees module is obtained from the old one by tensoring over $\mathbb{C}$ with $\mathbb{C}[z_{n+1}]$.]

(2) Let A be an object with n distributive filtrations $F^{(1)}_\bullet A,\dots,F^{(n)}_\bullet A$. Show that any family of filtrations $G^{(1)}_\bullet A,\dots,G^{(m)}_\bullet A$ where each $G^{(i)}_\bullet A$ is obtained by *convolution* of some of the filtrations $F^j_\bullet A$, i.e.,

$$G^{(i)}_pA=\sum_{q_1+\cdots+q_k=p}F^{(j_1)}_{q_1}A+\cdots+F^{(j_k)}_{q_k}A,$$

(also denoted by $G^i_\bullet A=F^{(j_1)}_\bullet A\star\cdots\star F^{(j_k)}_\bullet A$) is also a distributive family. [*Hint*: Express the Rees module R^i_GA as obtained by base change from $R_{F^{(j_1)},\dots,F^{(j_k)}}A$ and, more generally express $R_{G^{(1)},\dots,G^{(m)}}A$ as obtained by base change from $R_{F^{(1)},\dots,F^{(n)}}A$; conclude by using that flatness is preserved by base change.]

(3) Let $F^{(1)}_\bullet A,\dots,F^{(n)}_\bullet A$ be filtrations on A. Let B be a sub-object of A and let $F^{(i)}_\bullet B$ and $F^{(i)}_\bullet(A/B)$ be the induced filtrations. Assume that

(a) the families $(F^{(i)}_\bullet B)_i$ and $(F^{(i)}_\bullet(A/B))_i$ are distributive,

(b) for all $k_1,\dots,k_n$, the following sequence is exact:

$$0\longrightarrow\bigcap_{i=1}^n F^{(i)}_{k_i}B\longrightarrow\bigcap_{i=1}^n F^{(i)}_{k_i}A\longrightarrow\bigcap_{i=1}^n F^{(i)}_{k_i}(A/B)\longrightarrow 0.$$

Then the family $(F^{(i)}_\bullet A)_i$ is distributive. [*Hint*: Show that there is an exact sequence of the associated Rees modules, and use that flatness of the extreme terms implies flatness of the middle term.]

Exercise 15.4 **(External products and flatness).**

(1) Let $R=\mathbb{C}[z_1,\dots,z_n]$ and $R'=\mathbb{C}[z'_1,\dots,z'_m]$ be polynomial rings set $R''=R\otimes_{\mathbb{C}}R'=\mathbb{C}[z_1,\dots,z'_m]$. Let M resp. M' be a graded flat R- resp. R'- module. Show that $M'':=M\otimes_{\mathbb{C}}M'$ is R''-flat as a graded R''-module. [*Hint*: Use the criterion of Exercise 15.2.]

(2) Assume now that R and R' are polynomial rings (with variables as above) over a polynomial ring $\mathbb{C}[z''_1,\dots,z''_p]$. Let M,M' be as above. Show that $M'':=M\otimes_{\mathbb{C}[z'']}M'$ is R''-flat as a graded R''-module. [*Hint*: Define M'' in terms of $M\otimes_{\mathbb{C}}M'$.]

(3) Reprove Lemma 8.6.10 by using the argument of (2) and that flatness commutes with base change (in a way similar to that of Remark 15.2.9). [*Hint*: Set $\widetilde{\mathcal{M}}=R_F\mathcal{M}$ and consider $\widetilde{\mathcal{M}}_X\boxtimes_{\mathbb{C}}\widetilde{\mathcal{M}}_Y$; show that this is a flat bi-graded $\mathbb{C}[z_1,z_2]$-module; deduce that restricting first to $z_1=z_2$ and then to $z=0$, or restricting to $z_1=0$ and then to $z_2=0$ give the same result.]

(4) Let $\widetilde{\mathcal{M}}_X,\widetilde{\mathcal{M}}_Y$ be *strict* $\widetilde{\mathcal{D}}$-modules equipped with coherent $F_\bullet\widetilde{\mathcal{D}}$-filtrations $F_\bullet\widetilde{\mathcal{M}}_X,F_\bullet\widetilde{\mathcal{M}}_Y$. Assume that $\operatorname{gr}^F\widetilde{\mathcal{M}}_X,\operatorname{gr}^F\widetilde{\mathcal{M}}_Y$ are strict. Show that

$$\operatorname{gr}^F(\widetilde{\mathcal{M}}_X\boxtimes_{\widetilde{\mathcal{D}}}\widetilde{\mathcal{M}}_Y)\simeq\operatorname{gr}^F\widetilde{\mathcal{M}}_X\boxtimes_{\operatorname{gr}^F\widetilde{\mathcal{D}}}\operatorname{gr}^F\widetilde{\mathcal{M}}_Y.$$

[*Hint*: Show with Exercise 15.2 that $R_F\widetilde{\mathcal{M}}:=\bigoplus_k F_k\widetilde{\mathcal{M}}z_1^k$ is $\mathbb{C}[z,z_1]$-flat and use (2).]

Exercise 15.5. Show as in the beginning of Section 15.4 that the object $A(1_{i\in I},0_{i\notin I})$ is equal to $A/\sum_{i\in I}A_i$.

Exercise 15.6 **(Some properties of compatible families).**

(1) Let $A_1,\dots,A_n\subset A$ be a compatible family of sub-objects of A and let $B\supset A$. Show that $A_1,\dots,A_n,A$ is a compatible family in B (in particular, $A_1,\dots,A_n$ is a compatible family in B). [*Hint*: Note first that, for $\varepsilon=(\varepsilon_1,\dots,\varepsilon_n)$ with $\varepsilon_i\geqslant 0$ for all i, A surjects to $A(\varepsilon)$ and set $A(\varepsilon)=A/I(\varepsilon)$, with $I(0)=0$; define then $B(\varepsilon,\varepsilon_{n+1})$

by

$$\begin{aligned} B(\varepsilon,-1) &= A(\varepsilon) \quad \forall\,\varepsilon,\\ B(\varepsilon,0) &= \begin{cases} A(\varepsilon) & \text{if } \exists\, i,\ \varepsilon_i=-1,\\ B/I(\varepsilon) & \text{if } \forall\, i,\ \varepsilon_i \geqslant 0,\end{cases}\\ B(\varepsilon,1) &= \begin{cases} 0 & \text{if } \exists\, i,\ \varepsilon_i=-1,\\ B/A & \text{if } \forall\, i,\ \varepsilon_i \geqslant 0;\end{cases} \end{aligned}$$

check the exactness of sequences like $B(-1,\varepsilon',0)\to B(0,\varepsilon',0)\to B(1,\varepsilon',0)$.]

(2) Let $A_1,\dots,A_n\subset A$ be a family of sub-objects of A which is compatible in B, for some $B\supset A$. Then this family is compatible in A. [*Hint*: Set $A(\varepsilon)=B(\varepsilon)$ if $\varepsilon_i=-1$ for some i, and if $\varepsilon_i\geqslant 0$ for all i, set $A(\varepsilon)=A/I(\varepsilon)$, where $B(\varepsilon)=B/I(\varepsilon)$ and show first that $I(\varepsilon)\subset\sum_i A_i$ by using Exercise 15.5.]

(3) Let $A_0,A_1,\dots,A_n \subset A$ be a family of sub-objects of A. Assume that $A_1,\dots,A_{n-1}\subset A_n$. Show that the family $A_0,A_1,\dots,A_n$ is compatible if and only if the family $A_0\cap A_n,A_1,\dots,A_n$ of sub-objects of A_n is compatible. [*Hint*: If the diagram $C(A_0,\dots,A_n;A)$ exists, there should be an exact sequence

$$0\to C(A_0\cap A_n,\dots,A_n;A_n)\to C(A_0,\dots,A_n;A)\to C\Big(\frac{A_0}{A_0\cap A_n},0,\dots,0;\frac{A}{A_n}\Big)\to 0,$$

corresponding to exact sequences

$$0\longrightarrow A(\varepsilon_0,\varepsilon',-1)\longrightarrow A(\varepsilon_0,\varepsilon',0)\longrightarrow A(\varepsilon_0,\varepsilon',1);$$

show that $A(\varepsilon_0,\varepsilon',1)=0$ if $\varepsilon'_i=-1$ for some $i=1,\dots,n-1$; set thus $A(\varepsilon_0,\varepsilon',0):=A(\varepsilon_0,\varepsilon',-1)$ for such an ε'; to determine $A(\varepsilon_0,\varepsilon',0)$ for $\varepsilon'_i\geqslant 0$ for all i, use Exercise 15.5 if $\varepsilon_0\geqslant 0$ and deduce the case $\varepsilon_0=-1$; end by checking that all possibly exact sequences are indeed exact.]

CHAPTER 15

$\widetilde{\mathcal{D}}$-MODULES OF NORMAL CROSSING TYPE
PART 2: FUNDAMENTAL PROPERTIES

Summary. Starting from the simple model of a monodromic $\mathcal{D}_X$-module, we first introduce the notion of $\mathcal{D}_X$-module of normal crossing type, obtained by analytifying a monodromic one. The notion of a filtered $\mathcal{D}_X$-module, or a $\widetilde{\mathcal{D}}_X$-module, of normal crossing type needs a different approach, as in general such an object does not come by analytification from a monodromic filtered $\mathcal{D}_X$-module. The notion of distributivity or compatibility of filtrations, introduced in Part 1, is essential in the definition. On the other hand, as sesquilinear pairings do not involve the F-filtration, they can be analyzed from the simple monodromic setting, and the results are higher dimensional analogues of those of Section 7.3 in dimension one.

15.6. Introduction

15.6.1. Notation. In this chapter, the setting is as follows. The space $X = \Delta^n$ is a polydisc in $\mathbb{C}^n$ with analytic coordinates $x_1, \dots, x_n$, we fix $\ell \leqslant n$ and we denote by D the divisor $\{x_1 \cdots x_\ell = 0\}$. We also denote by D_i $(i = 1, \dots, \ell)$ the smooth components of D and by $D_{(\ell)}$ their intersection $D_1 \cap \cdots \cap D_\ell$. We will shorten the notation $\mathbb{C}[x_1, \dots, x_\ell]$ into $\mathbb{C}[x]$ and $\mathbb{C}[x_1, \dots, x_\ell]\langle\partial_{x_1}, \dots, \partial_{x_\ell}\rangle$ into $\mathbb{C}[x]\langle\partial_x\rangle$. We will set $I = \{1, \dots, \ell\}$.

We will mainly consider *right* $\mathcal{D}$-modules.

15.6.2. Simplifying assumptions. All over this part, we will consider the simpler case where $\ell = n$, that is, $D_{(\ell)}$ is reduced to the origin in $X = \Delta^n$, in order to make the computations clearer. We then have $I = \{1, \dots, n\}$. The general case $\ell \neq n$ brings up objects which are $\mathcal{O}_{D_{(\ell)}}$-locally free and the adaptation is straightforward.

In higher dimensions, similarly to what was done in Section 7.2, the theory of vector bundles on X with meromorphic integrable connections with poles along D starts with the simplest objects, namely those with regular singularities [**Del70**]. One first extends naturally these objects as locally free $\mathcal{O}_X(*D)$-modules with integrable connection and the regularity property amounts to the existence of locally free $\mathcal{O}_X$-module of maximal rank on which the connection has logarithmic poles. The category of such objects is equivalent to that of locally constant sheaves on $X \smallsetminus D$, that is, of finite dimensional representations of $\pi_1(X \smallsetminus D) \simeq \mathbb{Z}^n$. These objects behave like products of meromorphic bundles with connection having a regular singularity in dimension 1. We say that these objects are *of normal crossing type*.

Our first aim is to extend this notion to other holonomic $\mathcal{D}_X$-modules. We mainly have in mind the middle extension of such meromorphic connections. In terms of general $\mathcal{D}$-module theory—that we will not use—we could characterize such $\mathcal{D}$-modules as the regular holonomic $\mathcal{D}$-modules whose characteristic variety is adapted to the natural stratification of the divisor D. In other words, these are the simplest objects in higher dimension.

We can settle the problem as follows. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. Assume that $\mathcal{M}$ is $\mathbb{R}$-specializable along each component D_i of D. How do the various V-filtrations interact? The notion of *normal crossing type* aims at reflecting that these V-filtrations behave independently, i.e., without any interaction. In other words, the transversality property of the components of D is extended to the transversality property of the V-filtrations. This is first explained in Section 15.7.a for the simpler "algebraic case" and then in Section 15.7.b for the general holomorphic case.

Sesquilinear pairings between coherent $\mathcal{D}$-modules of normal crossing type have then a simple expression in terms of *basic distributions or currents* (Section 15.8).

When thinking in terms of characteristic varieties, one can expect that the notion of "normal crossing type" is stable with respect to taking nearby or vanishing cycles along a monomial function in the given coordinates. However, obtaining an explicit expression of the various monodromies in terms of the original ones leads to a delicate combinatorial computation, which is achieved in Section 15.12 both for the simpler "monodromic case" and the general holomorphic case.

We are mainly interested in the previous results in the presence of an F-filtration and, for a coherently F-filtered $\mathcal{D}$-module $(\mathcal{M}, F_\bullet\mathcal{M})$, we will express the independence of the V-filtrations in the presence of $F_\bullet\mathcal{M}$. By looking in dimension 1, one first realizes that $(\mathcal{M}, F_\bullet\mathcal{M})$ should be $\mathbb{R}$-specializable along any component D_i of D. But adding an F-filtration to the picture also leads us to take much care of the behaviour of this filtration with respect to the various V-filtrations along the components D_i of the divisor D. The compatibility property (Definition 15.1.7) is essential in order to have a reasonable control on various operations on these filtered $\mathcal{D}$-modules.

An important question, given a filtered $\mathcal{D}$-module $(\mathcal{M}, F_\bullet\mathcal{M})$ such that $\mathcal{M}$ is of normal crossing type along D, is to have an effective criterion on the F-filtration for $(\mathcal{M}, F_\bullet\mathcal{M})$ to be of normal crossing type. We give such a criterion in terms of *parabolic bundles* (Section 15.9.c) by applying the criterion of Section 15.2.b.

15.6.3. Notation for logarithmic modules. The V-filtration of $\mathcal{D}_X$ along D_i, or that of a $\mathcal{D}_X$-module $\mathcal{M}$which is $\mathbb{R}$-specializable along D_i, will be denote by $V_\bullet^{(i)}$, where $\bullet$ runs in $\mathbb{Z}$ or $\mathbb{R}$. We will then set (when the simplifying assumption 15.6.2 holds)

$$V_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{D}_X := \bigcap_{i=1}^n V_{a_i}^{(i)}\mathcal{D}_X, \quad V_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M} := \bigcap_{i=1}^n V_{a_i}^{(i)}\mathcal{M}, \qquad \boldsymbol{a} := (a_1, \dots, a_n),$$

which are modules over the sheaf $V_{\boldsymbol{0}}^{(\boldsymbol{n})}\mathcal{D}_X$ of *logarithmic differential operators with respect to the divisor D*. We use the notation

$$\mathcal{D}_X(\log D) := V_{\boldsymbol{0}}^{(\boldsymbol{n})}\mathcal{D}_X = \mathcal{O}_X\langle x_1\partial_{x_1}, \dots, x_n\partial_{x_n}\rangle.$$

For the $\mathcal{D}_X$-modules of normal crossing type that we will consider in this chapter, the $V_{\boldsymbol{0}}^{(\boldsymbol{n})}\mathcal{D}_X$-modules $V_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}$ are $\mathcal{O}_X$-coherent and $V_{\boldsymbol{0}}^{(\boldsymbol{n})}\mathcal{M}$ contains most of the information on $\mathcal{M}$, and more importantly, the same property applies to filtered objects. For multi-indices, we use the following notation:

$$\begin{cases} \boldsymbol{a} \leqslant \boldsymbol{b} & \text{if } a_i \leqslant b_i \ \forall\, i \in \{1, \dots, n\}, \\ \boldsymbol{a} \lneqq \boldsymbol{b} & \text{if } \boldsymbol{a} \leqslant \boldsymbol{b} \text{ and } \boldsymbol{a} \neq \boldsymbol{b} \\ & \text{i.e., } a_i \leqslant b_i \ \forall\, i \in \{1, \dots, n\} \text{ and } \exists\, i,\ a_i < b_i, \\ \boldsymbol{a} < \boldsymbol{b} & \text{if } a_i < b_i \ \forall\, i \in \{1, \dots, n\}. \end{cases} \tag{15.6.3 $*$}$$

It is thus natural to introduce the notation

$$\mathcal{M}_{\leqslant \boldsymbol{0}} = V_{\boldsymbol{0}}^{(\boldsymbol{n})}\mathcal{M}.$$

For the $\mathcal{D}_X$-modules which are middle extension along each component D_i of D, that will be of most importance for us, we will consider instead

$$\mathcal{M}_{<\boldsymbol{0}} = V_{<\boldsymbol{0}}^{(\boldsymbol{n})}\mathcal{M} := \sum_{\boldsymbol{a}<\boldsymbol{0}} V_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}.$$

In the algebraic setting, we consider the Weyl algebra $A_n := \mathbb{C}[x]\langle\partial_x\rangle$ of differential operators in n variables with polynomial coefficients, and correspondingly (right) $\mathcal{D}_X$-modules with (right) A_n-modules, that we denote by a capital letter like M. Similarly, we set $A_n(\log D) = \mathbb{C}[x]\langle x\partial_x\rangle$.

15.7. Normal crossing type

15.7.a. Monodromic A_n-modules. In this section, we consider the algebraic setting. Let M be an A_n-module and let us consider, for every $\boldsymbol{a} \in \mathbb{R}^n$, the subspace $M_{\boldsymbol{a}}$ of M defined by

$$M_{\boldsymbol{a}} = \bigcap_{i\in I} \bigcup_k \operatorname{Ker}(x_i\partial_{x_i} - a_i)^k.$$

This is a $\mathbb{C}$-vector subspace of M. The endomorphism $x_i\partial_{x_i}$ acting on $M_{\boldsymbol{a}}$ will be denoted by E_i and $(x_i\partial_{x_i} - a_i)$ by N_i. The family $(\mathrm{N}_1, \dots, \mathrm{N}_n)$ forms a commuting family of endomorphisms of $M_{\boldsymbol{a}}$, giving $M_{\boldsymbol{a}}$ a natural $\mathbb{C}[\mathrm{N}_1, \dots, \mathrm{N}_n]$-module structure, and every element of $M_{\boldsymbol{a}}$ is annihilated by some power of each N_i. Moreover, for $i \in I$, the morphism $x_i : M \to M$ (resp. $\partial_{x_i} : M \to M$) induces a $\mathbb{C}$-linear morphism $x_i : M_{\boldsymbol{a}} \to M_{\boldsymbol{a}-\boldsymbol{1}_i}$ (resp. $\partial_{x_i} : M_{\boldsymbol{a}} \to M_{\boldsymbol{a}+\boldsymbol{1}_i}$). For each fixed $\boldsymbol{a} \in \mathbb{R}^n$, we have

$$M_{\boldsymbol{a}} \cap \Big(\sum_{\boldsymbol{a}'\neq\boldsymbol{a}} M_{\boldsymbol{a}'}\Big) = 0 \quad \text{in } M.$$

Indeed, for $m = \sum_{\boldsymbol{a}'\neq\boldsymbol{a}} m_{\boldsymbol{a}'}$, if $m \in M_{\boldsymbol{a}}$, then $m - \sum_{a'_1=a_1} m_{\boldsymbol{a}'}$ is annihilated by some power of $x_1\partial_{x_1} - a_1$ and by a polynomial $\prod_{a'_1\neq a_1}(x_1\partial_{x_1} - a'_1)^{k_{a'_1}}$, hence is zero, so we can restrict the sum above to $a'_1 = a_1$. Arguing similarly for $i = 2, \dots, n$ yields finally $m = 0$. It follows that

$$M' := \bigoplus_{\boldsymbol{a}\in\mathbb{R}^n} M_{\boldsymbol{a}} \subset M \tag{15.7.1}$$

is a A_n-submodule of M. The actions of x_i and ∂_{x_i} satisfy, for each $i \in \{1, \dots, n\}$:

- $x_i : M_{\boldsymbol{a}} \to M_{\boldsymbol{a}-\mathbf{1}_i}$ is an isomorphism if $a_i < 0$,
- $\partial_{x_i} : M_{\boldsymbol{a}} \to M_{\boldsymbol{a}+\mathbf{1}_i}$ is an isomorphism if $a_i > -1$.

(See Exercise 15.7.)

***15.7.2. Definition* (Monodromicity).** Let M be an A_n-module. We say that M is *monodromic* if the following properties are satisfied.

(a) There exists a finite subset $\boldsymbol{A} \subset [-1, 0)^n$, called the set of *exponents of* M, such that $M_{\boldsymbol{a}} = 0$ for $\boldsymbol{a} \notin \boldsymbol{A} + \mathbb{Z}^n$.

(b) Each $M_{\boldsymbol{a}}$ ($\boldsymbol{a} \in \mathbb{R}^n$) is finite-dimensional.

(c) The natural inclusion (15.7.1) is an *equality.*

15.7.3. Proposition. *Let M be a monodromic A_n-module. Then*

(1) *M is $\mathbb{R}$-specializable along each D_i and*

$$V^{(i)}_{b_i} M = \bigoplus_{\substack{\boldsymbol{a} \in \mathbb{R}^n \\ a_i \leqslant b_i}} M_{\boldsymbol{a}}, \quad \operatorname{gr}^{V^{(i)}}_{b_i} M = \bigoplus_{\substack{\boldsymbol{a} \in \mathbb{R}^n \\ a_i = b_i}} M_{\boldsymbol{a}}, \quad V^{(\boldsymbol{n})}_{\boldsymbol{b}} M = \bigoplus_{\substack{\boldsymbol{a} \in \mathbb{R}^n \\ \boldsymbol{a} \leqslant \boldsymbol{b}}} M_{\boldsymbol{a}};$$

(2) *The A_n-module M is uniquely determined, up to isomorphism, from the $A_n(\log D)$-module $M_{\leqslant \mathbf{0}}$;*

(3) *Each $V^{(\boldsymbol{n})}_{\boldsymbol{a}} M$ is an $A_n(\log D)$-module of finite type and, if $a_i < 0$ for all i, it is a free $\mathbb{C}[x]$-module of finite rank;*

(4) *Decomposing the set of variables as $(x', x'') = (x_1, \dots, x_{n'}, x_{n'+1}, \dots, x_{n'+n''})$ with $n' + n'' = n$, then for any $\boldsymbol{a}''_o \in \mathbb{R}^{\boldsymbol{n}''}$, the $\mathbb{C}[x']\langle \partial_{x'} \rangle$-module $M_{(\bullet, \boldsymbol{a}''_o)} = \bigoplus_{\boldsymbol{a}' \in \mathbb{R}^{n'}} M_{(\boldsymbol{a}', \boldsymbol{a}''_o)}$ is monodromic.*

(5) *With the decomposition as in (4), for any $\boldsymbol{a}' \in (\mathbb{R}_{<0})^{n'}$, the $\mathbb{C}[x']$-module $V^{(\boldsymbol{n}')}_{\boldsymbol{a}'} M$ is flat.*

Proof.

(1) The first equality follows from the characterization of the V-filtration, and the other ones are immediate consequences.

(2) For $a_i > 0$ set $k = \lceil a_i \rceil$ and $a'_i = a_i - k \in (-1, 0]$. Set also $a'_j = a_j$ if $j \neq i$. Then $\partial^k_{x_i} : M_{\boldsymbol{a}'} \to M_{\boldsymbol{a}}$ is an isomorphism which, composed with x_i^k, yields the endomorphism $\prod_{\ell=0}^{k-1} (a_i - \ell + \mathrm{N}_i)$ of $M_{\boldsymbol{a}'}$. By replacing each such $M_{\boldsymbol{a}}$ with $M_{\boldsymbol{a}'}$ in M and defining the action of ∂_{x_i} as the identity and that of x_i as $(a_i - (k-1) + \mathrm{N}_i)$ leads to a monodromic A_n-module isomorphic to M. This argument applied for each $i \in \{1, \dots, n\}$ yields the conclusion.

(3) For every $\boldsymbol{\alpha} \in \boldsymbol{A}$, let us set

$$M_{\boldsymbol{\alpha}+\mathbb{Z}^n} = \bigoplus_{\boldsymbol{k} \in \mathbb{Z}^n} M_{\boldsymbol{\alpha}+\boldsymbol{k}},$$

so that $M = \bigoplus_{\boldsymbol{\alpha} \in \boldsymbol{A}} M_{\boldsymbol{\alpha}+\mathbb{Z}^n}$. Then $M_{\boldsymbol{\alpha}+\mathbb{Z}^n}$ is an A_n-module. In such a way, M is the direct sum of monodromic A_n-modules having a single exponent, and it is enough to prove the statement in this case. Then one checks that, for each $\boldsymbol{a} \in \mathbb{R}^n$, $V^{(\boldsymbol{n})}_{\boldsymbol{a}} M = V^{(\boldsymbol{n})}_{\boldsymbol{\alpha}+\boldsymbol{k}} M$ where $\boldsymbol{k} \in \mathbb{Z}^n$ is such that, for each i, $a_i \in (\alpha_i + k_i - 1, \alpha_i + k_i]$, i.e., $k_i = \lceil a_i - \alpha_i \rceil$. The condition $a_i < 0$ for all i is equivalent to $k_i \leqslant 0$ for all i and we find in such a case

$$V_{\boldsymbol{a}}^{(\boldsymbol{n})} M = M_{\boldsymbol{\alpha}+\boldsymbol{k}} \otimes_{\mathbb{C}} \mathbb{C}[x],$$

hence the assertion.

(4) We can assume that $\boldsymbol{a}''_o$ belongs to the projection of $\boldsymbol{A} + \mathbb{Z}^n$ to $\mathbb{R}^{n''}$, otherwise $M_{(\bullet,\boldsymbol{a}''_o)}$ is zero. Then the conditions for being monodromic are clearly satisfied for $M_{(\bullet,\boldsymbol{a}''_o)}$, whose set $\boldsymbol{A}' + \mathbb{Z}^{n'}$ is the pullback of $\boldsymbol{a}''_o$ by the projection $\boldsymbol{A} + \mathbb{Z}^n \to \mathbb{R}^{n''}$.

(5) We can assume, as in (3), that $\boldsymbol{A}$ has only one element $\boldsymbol{\alpha}$. We argue by induction on n'' and we only treat the case where $n'' = 1$ and $y_1 = x_n$. By (3), $V^{(\boldsymbol{n})}_{(\boldsymbol{a}',\alpha_n)} M$ is $\mathbb{C}[x]$-free of finite rank, hence $\mathbb{C}[x']$-flat. We show by increasing induction on $k_n \in \mathbb{N}$ that $V^{(\boldsymbol{n})}_{(\boldsymbol{a}',\alpha_n+k_n)} M$ is $\mathbb{C}[x']$-flat, and the desired assertion is obtained at the limit $k_n = \infty$. We have

$$V^{(\boldsymbol{n})}_{(\boldsymbol{a}',\alpha_n+k_n)} M / V^{(\boldsymbol{n})}_{(\boldsymbol{a}',\alpha_n+k_n-1)} M = V^{(\boldsymbol{n}')}_{\boldsymbol{a}'} (\mathrm{gr}^{(V^{(n)}}_{\alpha_n+k_n} M),$$

which is $\mathbb{C}[x']$-free, hence $\mathbb{C}[x']$-flat, according to (4). By induction on k_n, $V^{(\boldsymbol{n})}_{(\boldsymbol{a}',\alpha_n+k_n)} M$ is thus $\mathbb{C}[x']$-flat. □

The category of monodromic A_n-modules is, by definition, the full subcategory of that of A_n-modules whose objects are monodromic.

15.7.4. Proposition. *Every morphism between monodromic A_n-modules is graded with respect to the decomposition* (15.7.1)*, and the category of monodromic A_n-modules is abelian.*

Proof. By A_n-linearity and using Bézout's theorem, one checks that any morphism $\varphi : M_1 \to M_2$ sends $M_{1,\boldsymbol{a}}$ to $M_{2,\boldsymbol{a}}$, and has a zero component from $M_{1,\boldsymbol{a}}$ to $M_{2,\boldsymbol{b}}$ if $\boldsymbol{b} \neq \boldsymbol{a}$. □

15.7.5. Proposition **(Description by quivers).** *Let us fix $\boldsymbol{\alpha} \in [-1,0)^n$ and let us set $I(\boldsymbol{\alpha}) = \{i \in I \mid \alpha_i = -1\}$. Then the category of monodromic A_n-modules with exponent $\boldsymbol{\alpha}$, that is, of the form $M_{\boldsymbol{\alpha}+\mathbb{Z}^n}$, is equivalent to the category of $I(\boldsymbol{\alpha})$-quivers having the vertex $M_{\boldsymbol{\alpha}+\boldsymbol{k}}$ equipped with its $\mathbb{C}[\mathrm{N}_1,\dots,\mathrm{N}_n]$-module structure at the place $\boldsymbol{\varepsilon} \in \{0,1\}^{I(\boldsymbol{\alpha})}$ and arrows*

$$\begin{matrix} \mathrm{can}_i : M_{\boldsymbol{\alpha}+\boldsymbol{\varepsilon}} \longrightarrow M_{\boldsymbol{\alpha}+\boldsymbol{\varepsilon}+\mathbf{1}_i}, \\ \mathrm{var}_i : M_{\boldsymbol{\alpha}+\boldsymbol{\varepsilon}+\mathbf{1}_i} \longrightarrow M_{\boldsymbol{\alpha}+\boldsymbol{\varepsilon}}, \end{matrix} \quad \text{if } \varepsilon_i = 0,$$

subject to the conditions

$$\begin{cases} \mathrm{var}_i \circ \mathrm{can}_i = \mathrm{N}_i : M_{\boldsymbol{\alpha}+\boldsymbol{\varepsilon}} \longrightarrow M_{\boldsymbol{\alpha}+\boldsymbol{\varepsilon}}, \\ \mathrm{can}_i \circ \mathrm{var}_i = \mathrm{N}_i : M_{\boldsymbol{\alpha}+\boldsymbol{\varepsilon}+\mathbf{1}_i} \longrightarrow M_{\boldsymbol{\alpha}+\boldsymbol{\varepsilon}+\mathbf{1}_i}, \end{cases} \quad \text{if } \varepsilon_i = 0.$$

(It is understood that if $I(\boldsymbol{\alpha}) = \varnothing$, then the quiver has only one vertex and no arrows.)

Proof. It is straightforward, by using that, for $\boldsymbol{k} \in \mathbb{Z}^n$, $\partial_{x_i} : M_{\boldsymbol{\alpha}+\boldsymbol{k}} \to M_{\boldsymbol{\alpha}+\boldsymbol{k}+\mathbf{1}_i}$ is an isomorphism if $i \notin I(\boldsymbol{\alpha})$ or $i \in I(\boldsymbol{\alpha})$ and $\varepsilon_i \geqslant 0$, while $x_i : M_{\boldsymbol{\alpha}+\boldsymbol{k}} \to M_{\boldsymbol{\alpha}+\boldsymbol{k}-\mathbf{1}_i}$ is an isomorphism if $i \notin I(\boldsymbol{\alpha})$ or $i \in I(\boldsymbol{\alpha})$ and $\varepsilon_i \leqslant -1$. □

15.7.6. Remark. In order not to specify a given exponent of a monodromic A_n-module, it is convenient to define the quiver with vertices indexed by $\{0,1\}^I$ instead of $\{0,1\}^{I(\boldsymbol{\alpha})}$. We use the convention that, for a fixed $\boldsymbol{\alpha}\in[-1,0)^n$ and for $i\notin I(\boldsymbol{\alpha})$, $\mathrm{var}_i=\mathrm{Id}$ and $\mathrm{can}_i=\alpha_i\,\mathrm{Id}+\mathrm{N}_i=\mathrm{E}_i$ (hence both are isomorphisms). Then the category of monodromic A_n-modules is equivalent to the category of such quivers.

For $i\in\{1,\dots,n\}$, the definition of the localization, dual localization and middle extension of M along D_i of Chapter 11 can be adapted in a straightforward way in the present algebraic setting.

***15.7.7. Proposition* (Localization, dual localization and middle extension along one component of D)**

Let M be a monodromic A_n-module. Then, for each $i\in I$, the A_n-modules $M(*D_i)$, $M(!D_i)$ and $M(!*D_i):=\mathrm{image}[M(!D_i)\to M(*D_i)]$ are monodromic. Furthermore, M is localized, resp. dual localized, resp. a middle extension*) along D_i, that we denote by $M=M(*D_i)$, resp. $M=M(!D_i)$, resp. $M=M(!*D_i)$, if and only if* var_i *is bijective, resp.* can_i *is bijective, resp.* can_i *is onto and* var_i *is injective.*

Proof. The case of $M=M(*D_i)$ is treated in Exercise 15.9. The other cases are done similarly. □

15.7.8. Definition. We say that M is a *middle extension along $D_{i\in I}$* if for each $i\in I$, every can_i is onto and every var_i is injective.

See Exercises 15.10–15.11.

***15.7.9. Example* (The case of a simple A_n-module).** Let M be a monodromic A_n-module which is *simple* (i.e., has no non-trivial such sub or quotient module). By Exercise 15.11, it must be a middle extension along $D_{i\in I}$ with support in D. Moreover, every nonzero vertex of its quiver has dimension 1, so that E_i acts as a_i on $M_{\boldsymbol{a}}$ and N_i acts by zero.

***15.7.10. Remark* (Suppressing the simplifying assumptions 15.6.2)**

If $\ell<n$, every $M_{\boldsymbol{a}}$ ($\boldsymbol{a}\in\mathbb{R}^\ell$) has to be assumed $\mathcal{O}_{D_{(\ell)}}$-coherent in Definition 15.7.2(b) (or $\mathbb{C}[x_{\ell+1},\dots,x_n]$-Noetherian if we remain in the algebraic setting). Since it is a $\mathcal{D}_{D_{(\ell)}}$-module, it must be $\mathcal{O}_{D_{(\ell)}}$-locally free of finite rank. All the previous results extend in a straightforward way to this setting by replacing $\mathbb{C}[x]$ with $\mathcal{O}_{D_{(\ell)}}[x]$ (where $x:=(x_1,\dots,x_\ell)$) and $\mathbb{C}[x]\langle\partial_x\rangle$ with $\mathcal{D}_{D_{(\ell)}}[x]\langle\partial_x\rangle$. The subset $\boldsymbol{A}$ is contained in $[-1,0)^\ell$ and M decomposes as $M=\bigoplus_{\boldsymbol{\alpha}\in\boldsymbol{A}}M_{\boldsymbol{\alpha}+\mathbb{Z}^\ell}$.

In such a way, the notion of monodromic module is stable by restriction to strata of D. Indeed, let J be a subset of $\{1,\dots,\ell\}$, let J^{c} denote its supplementary subset, and let us consider the stratum

$$D_J^\circ=\bigcap_{i\in J}D_i\smallsetminus\bigcup_{i\in J^{\mathrm{c}}}D_i.$$

Algebraically, restricting M to the complement of $\bigcup_{i\in J^c} D_i$ means tensoring with $\mathbb{C}[x,(x_i^{-1})_{i\in J^c}]$. Denoting by $M(J)$ the restriction of M, by π_J the projection $\boldsymbol{A}\to\mathbb{R}^J$ to the J-components and setting $\boldsymbol{A}_J=\pi_J(\boldsymbol{A})$, we find the decomposition of $M(J)$ as

$$M(J)=\bigoplus_{\boldsymbol{\alpha}_J\in\boldsymbol{A}_J}\bigoplus_{\boldsymbol{k}_J\in\mathbb{Z}^J} M(J)_{\boldsymbol{\alpha}_J+\boldsymbol{k}_J},$$

with
$$M(J)_{\boldsymbol{\alpha}_J+\boldsymbol{k}_J}=\bigoplus_{\boldsymbol{\alpha}\in\pi_J^{-1}(\boldsymbol{\alpha}_J)}\Big(\bigoplus_{\boldsymbol{k}_{J^c}\in\mathbb{Z}^{J^c}} M_{\boldsymbol{\alpha}+(\boldsymbol{k}_J,0)}x^{\boldsymbol{k}_{J^c}}\Big)$$

being $\mathcal{O}_{D_{(\ell)}}[(x_i^{\pm1})_{i\in J^c}]$-coherent, and after analytification, $\mathcal{O}_{D_J^\circ}$-coherent.

15.7.b. Coherent $\mathcal{D}_X$-modules of normal crossing type. Given a monodromic A_n-module M, its analytification M^{an} is the $\mathcal{O}_X$-module defined so that, for each open set $U\subset X$,

$$M^{\mathrm{an}}(U)=M\otimes_{\mathbb{C}[x]}\mathcal{O}_X(U)=M\otimes_{A_n}\mathcal{D}_X(U).$$

For each $x\in X$, due to $\mathbb{C}[x]$-flatness of the ring $\mathcal{O}_{X,x}$ of germs at x of holomorphic functions, the correspondence $M\mapsto M_x^{\mathrm{an}}$ is an exact functor.

This is the prototype of a $\mathcal{D}_X$-module of normal crossing type. More precisely:

15.7.11. Definition. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. We say that $\mathcal{M}$ is *of normal crossing type along D* if there exists a monodromic A_n-module M such that

$$\mathcal{M}\simeq M^{\mathrm{an}}=\Big(\bigoplus_{\boldsymbol{a}\in\mathbb{R}^I} M_{\boldsymbol{a}}\Big)^{\mathrm{an}}. \tag{15.7.11 $*$}$$

The monodromic A_n-module M can be recovered from $\mathcal{M}$. Let $\mathcal{M}_0$ denote the germ of $\mathcal{M}$ at the origin, and for every $\boldsymbol{a}\in\mathbb{R}^n$ let us consider the sub-space $\mathcal{M}_{0,\boldsymbol{a}}$ of $\mathcal{M}_0$ defined by

$$\mathcal{M}_{0,\boldsymbol{a}}=\bigcap_{i\in I}\bigcup_k \mathrm{Ker}(x_i\partial_{x_i}-a_i)^k.$$

This is a $\mathbb{C}$-vector subspace of $\mathcal{M}_0$. We have $\mathbb{C}$-linear morphisms $x_i:\mathcal{M}_{0,\boldsymbol{a}}\to\mathcal{M}_{0,\boldsymbol{a}-\mathbf{1}_i}$ (resp. $\partial_{x_i}:\mathcal{M}_{0,\boldsymbol{a}}\to\mathcal{M}_{0,\boldsymbol{a}+\mathbf{1}_i}$) as in the algebraic setting, so that $\bigoplus_{\boldsymbol{a}}\mathcal{M}_{0,\boldsymbol{a}}$ is an A_n-module. If $\mathcal{M}=M^{\mathrm{an}}$ for some monodromic A_n-module M, then it is easily checked that $(M^{\mathrm{an}})_{0,\boldsymbol{a}}=M_{\boldsymbol{a}}$. In conclusion, if $\mathcal{M}$ is of normal crossing type, the A_n-module $\bigoplus_{\boldsymbol{a}}\mathcal{M}_{0,\boldsymbol{a}}$ is monodromic and the natural morphism

$$(\bigoplus_{\boldsymbol{a}}\mathcal{M}_{0,\boldsymbol{a}})\otimes_{\mathbb{C}[x]}\mathcal{O}_X=(\bigoplus_{\boldsymbol{a}}\mathcal{M}_{0,\boldsymbol{a}})\otimes_{A_n}\mathcal{D}_X\longrightarrow\mathcal{M} \tag{15.7.12}$$

is an isomorphism.

In the next proposition, we use Notation 15.6.3.

15.7.13. Proposition. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module which is of normal crossing type along D. Then the following properties are satisfied.*

(1) *$\mathcal{M}$ is $\mathbb{R}$-specializable along each D_i ($i\in I$), giving rise to V-filtrations $V_\bullet^{(i)}\mathcal{M}$. In particular, all properties of Definition* 9.3.18 *hold for each filtration $V_\bullet^{(i)}\mathcal{M}$.*

(2) *The V-filtrations $V_\bullet^{(i)}\mathcal{M}$ ($i\in I$) are distributive, in the sense of Definition* 15.1.7 *(see also Theorem* 15.2.2*);*

(3) *The* $\mathcal{D}_X$*-module* $\mathcal{M}$ *is uniquely determined, up to isomorphism, from the* $\mathcal{D}_X(\log D)$*-module* $\mathcal{M}_{\leqslant \mathbf{0}}$*;*

(4) *For any* $i \in I$ *and any* $a_i \in \mathbb{R}$, $\operatorname{gr}^{V^{(i)}}_{a_i}\mathcal{M}$ *is of normal crossing type on* $(D_i, \bigcup_{j\neq i} D_j)$ *and* $V^{(j)}_{\bullet}\operatorname{gr}^{V^{(i)}}_{a_i}\mathcal{M}$ *is the filtration naturally induced by* $V^{(j)}_{\bullet}\mathcal{M}$ *on* $\operatorname{gr}^{V^{(i)}}_{a_i}\mathcal{M}$*, that is,*

$$V^{(j)}_{\bullet}\operatorname{gr}^{V^{(i)}}_{a_i}\mathcal{M} = \frac{V^{(j)}_{\bullet}\mathcal{M} \cap V^{(i)}_{a_i}\mathcal{M}}{V^{(j)}_{\bullet}\mathcal{M} \cap V^{(i)}_{<a_i}\mathcal{M}}.$$

(5) *For* $\boldsymbol{a} = (a_1, \dots, a_n) \in \mathbb{R}^n$, $V^{(\boldsymbol{n})}_{\boldsymbol{a}}\mathcal{M}$ *is a* $V^{(\boldsymbol{n})}_{\mathbf{0}}\mathcal{D}_X$*-module which is* $\mathcal{O}_X$*-coherent, and* $\mathcal{O}_X$*-locally free if* $a_i < 0$ *for all* $i \in I$*.*

(6) *For any decomposition* $X = X' \times X''$ *with projection* p', p'' *to* X', X''*, and* $n = n' + n''$ *as in Proposition* 15.7.3(4)*, if* $\boldsymbol{a}'$ *belongs to* $(\mathbb{R}_{<0})^{n'}$*, then* $V^{(\boldsymbol{n}')}_{\boldsymbol{a}'}\mathcal{M}$ *is* $p'^{-1}\mathcal{O}_{X'}$*-flat.*

(7) *For any multi-index* $\boldsymbol{a} \in \mathbb{R}^n$*, the natural morphism of* $\mathbb{C}[\mathrm{N}_1, \dots, \mathrm{N}_n]$*-modules*

$$M_{\boldsymbol{a}} \longrightarrow \operatorname{gr}^{V^{(\boldsymbol{n})}}_{\boldsymbol{a}}\mathcal{M} := \operatorname{gr}^{V^{(1)}}_{a_1} \cdots \operatorname{gr}^{V^{(n)}}_{a_n}\mathcal{M}$$

is an isomorphism (see Remark 15.2.9 *for the multi-grading).*

(8) *In the setting of* (6)*, for any* $\boldsymbol{a}'$ *in* $\mathbb{R}^{n'}$, $\operatorname{gr}^{V^{(\boldsymbol{n}')}}_{\boldsymbol{a}'}\mathcal{M}$ *is of normal crossing type on* $(X'', \bigcup_{j=1}^{n''} D''_j)$*, and the natural morphism* $(\bigoplus_{\boldsymbol{a}''} M_{(\boldsymbol{a}',\boldsymbol{a}'')}) \otimes_{\mathbb{C}[x'']} \mathcal{O}_{X''} \to \operatorname{gr}^{V^{(\boldsymbol{n}')}}_{\boldsymbol{a}'}\mathcal{M}$ *is an isomorphism.*

15.7.14. Caveat. In order to apply Definition 15.1.7, one should regard $V^{(i)}_{\bullet}\mathcal{M}$ as a filtration indexed by $\mathbb{Z}$, by numbering the sequence of real numbers a_i such that $\operatorname{gr}^{V^{(i)}}_{a_i}\mathcal{M} \neq 0$. See also Section 5.1.d and the setup in Section 10.6.a. Setting (see Notation (15.6.3 $*$))

$$V^{(\boldsymbol{n})}_{\lneqq \boldsymbol{a}}\mathcal{M} := \sum_{\boldsymbol{b} \lneqq \boldsymbol{a}} V^{(\boldsymbol{n})}_{\boldsymbol{b}}\mathcal{M},$$

the distributivity implies $\operatorname{gr}^{V^{(\boldsymbol{n})}}_{\boldsymbol{a}}\mathcal{M} = V^{(\boldsymbol{n})}_{\boldsymbol{a}}\mathcal{M} / V^{(\boldsymbol{n})}_{\lneqq \boldsymbol{a}}\mathcal{M}$.

***Proof of Proposition* 15.7.13.**

(1) For each i and $a_i \in \mathbb{R}$, we define

$$V^{(i)}_{a_i}\mathcal{M} = V^{(i)}_{a_i}M \otimes_{V^{(i)}_0\mathbb{C}[x]\langle\partial_x\rangle} V^{(i)}_0\mathcal{D}_X. \tag{15.7.15}$$

Then this filtration satisfies the characteristic properties of the Kashiwara-Malgrange filtration along D_i of a $\mathcal{D}_X$-module, since $V^{(i)}_{\bullet}M$ is the Kashiwara-Malgrange filtration along D_i of M as a $\mathbb{C}[x]\langle\partial_x\rangle$-module. In such a way, we get the $\mathbb{R}$-specializability of $\mathcal{M}$ along D_i.

(2) Let us set $M_{\leqslant \boldsymbol{a}} = V^{(\boldsymbol{n})}_{\boldsymbol{a}}M$ and $\boldsymbol{a} = (\boldsymbol{a}_I, \boldsymbol{a}_J, \boldsymbol{a}_K)$ and let us choose $\boldsymbol{a}'_I \leqslant \boldsymbol{a}_I$ and $\boldsymbol{a}'_J \leqslant \boldsymbol{a}_J$. We will check the compatibility property, which is equivalent to distributivity, and amounts to complete the star in any diagram as below in order to

produce exact sequences:

$$\begin{array}{ccccc}
\dfrac{M_{\leqslant(\boldsymbol{a}'_I,\boldsymbol{a}_J,\boldsymbol{a}_K)}}{M_{\leqslant(\boldsymbol{a}'_I,\boldsymbol{a}'_J,\boldsymbol{a}_K)}} & \longrightarrow & \dfrac{M_{\leqslant(\boldsymbol{a}_I,\boldsymbol{a}_J,\boldsymbol{a}_K)}}{M_{\leqslant(\boldsymbol{a}_I,\boldsymbol{a}'_J,\boldsymbol{a}_K)}} & \longrightarrow & \star \\
\uparrow & & \uparrow & & \uparrow \\
M_{\leqslant(\boldsymbol{a}'_I,\boldsymbol{a}_J,\boldsymbol{a}_K)} & \longrightarrow & M_{\leqslant(\boldsymbol{a}_I,\boldsymbol{a}_J,\boldsymbol{a}_K)} & \longrightarrow & \dfrac{M_{\leqslant(\boldsymbol{a}_I,\boldsymbol{a}_J,\boldsymbol{a}_K)}}{M_{\leqslant(\boldsymbol{a}'_I,\boldsymbol{a}_J,\boldsymbol{a}_K)}} \\
\uparrow & & \uparrow & & \uparrow \\
M_{\leqslant(\boldsymbol{a}'_I,\boldsymbol{a}'_J,\boldsymbol{a}_K)} & \longrightarrow & M_{\leqslant(\boldsymbol{a}_I,\boldsymbol{a}'_J,\boldsymbol{a}_K)} & \longrightarrow & \dfrac{M_{\leqslant(\boldsymbol{a}_I,\boldsymbol{a}'_J,\boldsymbol{a}_K)}}{M_{\leqslant(\boldsymbol{a}'_I,\boldsymbol{a}'_J,\boldsymbol{a}_K)}}
\end{array}$$

The order $\leqslant$ is the partial natural order on $\mathbb{R}^n$: $\boldsymbol{a}' \leqslant \boldsymbol{a} \iff a'_i \leqslant a_i$, $\forall\, i$. Let us set

$$\mathbb{R}^n(\boldsymbol{a}_I,\boldsymbol{a}'_I,\boldsymbol{a}_J,\boldsymbol{a}'_J,\boldsymbol{a}_K) = \{\boldsymbol{a}'' \in \mathbb{R}^n \mid \boldsymbol{a}'_I \not\leqslant \boldsymbol{a}''_I \leqslant \boldsymbol{a}_I,\ \boldsymbol{a}'_J \not\leqslant \boldsymbol{a}''_J \leqslant \boldsymbol{a}_J,\ \boldsymbol{a}''_K \leqslant \boldsymbol{a}_K\}.$$

Then a natural choice in order to complete the diagram is

$$\star = \bigoplus_{\boldsymbol{a}'' \in \mathbb{R}^n(\boldsymbol{a}_I,\boldsymbol{a}'_I,\boldsymbol{a}_J,\boldsymbol{a}'_J,\boldsymbol{a}_K)} M_{\boldsymbol{a}''}.$$

By flatness of $V_{\boldsymbol{0}}^{(\boldsymbol{n})}\mathcal{D}_X$ over $V_{\boldsymbol{0}}^{(\boldsymbol{n})}\mathbb{C}[x]\langle\partial_x\rangle = \mathbb{C}[x]\langle x\partial_x\rangle$, the similar diagram for $\mathcal{M}$ is obtained by tensoring by $V_{\boldsymbol{0}}^{(\boldsymbol{n})}\mathcal{D}_X$, and is thus also exact, leading to the compatibility property of $V_\bullet^{(i)}\mathcal{M}$ $(i \in I)$.

(3) The assertion follows from Proposition 15.7.3(2) and the fact that the isomorphism (15.7.12) identifies $\mathcal{M}_{\leqslant\boldsymbol{0}}$ with $M_{\leqslant\boldsymbol{0}} \otimes_{\mathcal{O}_X} V_{\boldsymbol{0}}^{(\boldsymbol{n})}\mathcal{D}_X$, as seen in the proof of (1).

(4) Due to the isomorphism (15.7.12), it is enough to prove the result for the multi-graded module $M := \mathrm{gr}^{V^{(\boldsymbol{n})}}\mathcal{M}$, for which all assertions are clear.

(5) The relation in (1) reduces the proof of (5) to the case of a monodromic M, which has been obtained in Proposition 15.7.3(3).

(6) We argue by a double induction exactly as in Proposition 15.7.3(5), by making use of (4) for the induction.

(7) This is now obvious from the previous description, since $\mathrm{gr}_{\boldsymbol{a}}^{V^{(\boldsymbol{n})}}\mathcal{M} = \mathrm{gr}_{\boldsymbol{a}}^{V^{(\boldsymbol{n})}}M$.

(8) The proof of (8) is also straightforward and left as an exercise. □

Morphisms between $\mathcal{D}_X$-modules of normal crossing type can also be regarded as being of normal crossing type, as follows from the next proposition.

Let $\varphi : \mathcal{M}_1 \to \mathcal{M}_2$ be a morphism between coherent $\mathcal{D}_X$-modules of normal crossing type. Then φ is compatible with the V-filtrations $V_\bullet^{(i)}$ and, for every $\boldsymbol{a} \in \mathbb{R}^n$, its multi-graded components $\mathrm{gr}_{\boldsymbol{a}}^{V^{(\boldsymbol{n})}}\mathcal{M}_1 \to \mathrm{gr}_{\boldsymbol{a}}^{V^{(\boldsymbol{n})}}\mathcal{M}_2$ do not depend on the order of grading (according to the compatibility of the V-filtrations and Remark 15.2.12). We denote this morphism by $\mathrm{gr}_{\boldsymbol{a}}^{V^{(\boldsymbol{n})}}\varphi$. On the other hand, regarding $M_{\boldsymbol{a}}$ as a $\mathbb{C}$-submodule of $\mathcal{M}$, we note that φ sends $M_{1,\boldsymbol{a}}$ to $M_{2,\boldsymbol{a}}$, due to $\mathcal{D}$-linearity, and has no component from $M_{1,\boldsymbol{a}}$ to $M_{2,\boldsymbol{b}}$ if $\boldsymbol{b} \neq \boldsymbol{a}$. We denote by $\varphi_{\boldsymbol{a}}$ the induced morphism $M_{1,\boldsymbol{a}} \to M_{2,\boldsymbol{a}}$. The following is now obvious.

15.7.16. Proposition. *The morphism φ is the morphism induced by $\bigoplus_{\boldsymbol{a}} \varphi_{\boldsymbol{a}}$ by means of the isomorphism* (15.7.12) *and, with respect to the isomorphism $M_{\boldsymbol{a}} \xrightarrow{\sim} \mathrm{gr}_{\boldsymbol{a}}^{V^{(\boldsymbol{n})}}\mathcal{M}$ of Proposition* 15.7.13(7)*, $\varphi_{\boldsymbol{a}}$ coincides with $\mathrm{gr}_{\boldsymbol{a}}^{V^{(\boldsymbol{n})}}\varphi$. In particular, φ is uniquely determined from $\varphi_{\leqslant\boldsymbol{0}}$.* □

15.7.17. Corollary. *The category of $\mathcal{D}_X$-modules of normal crossing type along D is abelian and each morphism is n-strict with respect to $V_\bullet^{(1)}\mathcal{M},\dots,V_\bullet^{(n)}\mathcal{M}$.*

(See Definition 15.3.1 for the notion of n-strictness.)

Proof. It is quite obvious that the morphism $\bigoplus_{\boldsymbol{a}}\varphi_{\boldsymbol{a}}:M\to N$ is n-strict with respect to the V-filtrations of M and N. Due to formula (15.7.15) for the V-filtrations of $\mathcal{M}$ and $\mathcal{N}$, and to flatness of $V_0^{(i)}\mathcal{D}_X$ over $V_0^{(i)}\mathbb{C}[x]\langle\partial_x\rangle$, we deduce the n-strictness of φ. $\square$

15.7.18. Remarks.

(1) Let us fix $i\in I$ and set $\boldsymbol{a}=(\boldsymbol{a}',a_i)$. By $\mathbb{R}$-specializability along D_i we have isomorphisms

$$x_i:V_{a_i}^{(i)}\mathcal{M}\xrightarrow{\sim}V_{a_i-1}^{(i)}\mathcal{M},\ (a_i<0)\quad\text{and}\quad\partial_{x_i}:\mathrm{gr}_{a_i}^{V^{(i)}}\mathcal{M}\xrightarrow{\sim}\mathrm{gr}_{a_i+1}^{V^{(i)}}\mathcal{M},\ (a_i>-1).$$

One checks on M, and then on $\mathcal{M}$ due to (15.7.12) and 15.7.13(8), that they induce isomorphisms

$$(15.7.18\,*)\qquad\begin{aligned}x_i:V_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}&\xrightarrow{\sim}V_{\boldsymbol{a}-\boldsymbol{1}_i}^{(\boldsymbol{n})}\mathcal{M},\quad(a_i<0)\\ \partial_{x_i}:V_{\boldsymbol{a}''}^{(\boldsymbol{n}'')}\mathrm{gr}_{a_i}^{V^{(i)}}\mathcal{M}&\xrightarrow{\sim}V_{\boldsymbol{a}''}^{(\boldsymbol{n}'')}\mathrm{gr}_{a_i+1}^{V^{(i)}}\mathcal{M},\quad(a_i>-1),\end{aligned}$$

where we have set $\boldsymbol{a}''=(a_j)_{j\neq i}$.

(2) For any $\boldsymbol{a}\in(\mathbb{R}_{<0})^n$, we can thus regard $(V_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M})^{\mathrm{left}}$ as an $\mathcal{O}_X$-locally free module of finite rank equipped with a flat D-logarithmic connection. Moreover, for any $\boldsymbol{a}\in\mathbb{R}^n$, $V_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}_{X\smallsetminus D}$ is $\mathcal{O}_{X\smallsetminus D}$ locally free, and more precisely $V_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}(*D)$ is $\mathcal{O}_X(*D)$-locally free.

15.7.c. Behaviour with respect to localization, dual localization and middle extension along one component of D

15.7.19. Proposition. *Let $\mathcal{M}$ be a $\mathcal{D}_X$-module of normal crossing type and let $i_o\in I$. Then*

- *$\mathcal{M}(*D_{i_o})$ and $\mathcal{M}(!D_{i_o})$ are of normal crossing type;*
- *$\mathcal{M}=\mathcal{M}(*D_{i_o})$ (resp. $\mathcal{M}=\mathcal{M}(!D_{i_o})$) if and only if var_{i_o} is bijective (resp. can_{i_o} is bijective);*
- *$\mathcal{M}(!*D_{i_o}):=\mathrm{image}[\mathcal{M}(*D_{i_o})\to\mathcal{M}(!D_{i_o})]$ is of normal crossing type.*
- *$\mathcal{M}=\mathcal{M}(!*D_{i_o})$ if and only if can_{i_o} is onto and var_{i_o} is injective.*

Proof. This is obtained from Proposition 15.7.7 by flat tensorization with $\mathcal{O}_X$. $\square$

15.7.20. Definition. We say that $\mathcal{M}$ is a *middle extension along $D_{i\in I}$* if for each $i\in I$, every can_i is onto and every var_i is injective.

15.7.21. Remark **(Suppressing the simplifying assumptions 15.6.2)**

If $\ell<n$, we apply the same changes as in Remark 15.7.10. All the previous results extend in a straightforward way to this setting.

15.8. Sesquilinear pairings of normal crossing type

In this section, we take up the setting of Section 15.7.b. In the setting of this chapter (see Section 15.6), we consider the category $\mathcal{D}$-Triples(X).

15.8.1. Definition **(Triples of $\widetilde{\mathcal{D}}_X$-modules of normal crossing type)**
We say that an object $\widetilde{\mathcal{T}} = (\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}'', \mathfrak{s})$ of $\widetilde{\mathcal{D}}$-Triples(X) is of normal crossing type along D if its components $\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}''$ are strict and the corresponding filtered $\mathcal{D}_X$-modules $(\mathcal{M}', F_\bullet\mathcal{M}'), (\mathcal{M}'', F_\bullet\mathcal{M}'')$ are of normal crossing type along D.

We will perform, in higher dimension, an analysis similar to that of Section 7.3.

15.8.a. Basic distributions. The results of §7.3.a in dimension 1 extend in a straightforward way to $X = \Delta^n$. We will present them in the same context of left $\mathcal{D}$-modules. We continue using the simplifying assumptions 15.6.2.

15.8.2. Proposition. *Fix $\boldsymbol{b}', \boldsymbol{b}'' \in [-1, \infty)^n$ and $k \in \mathbb{N}$, and suppose a distribution $u \in \mathfrak{Db}(\Delta^n)$ solves the system of equations*

$$(15.8.2\,*)\qquad (x_i\partial_{x_i} - b'_i)^k u = (\overline{x}_i\partial_{\overline{x}_i} - b''_i)^k u = \partial_{x_j} u = \partial_{\overline{x}_j} u = 0 \quad (i \in I,\ j \notin I).$$

for an integer $k \geqslant 0$.

(a) *If $\boldsymbol{b}', \boldsymbol{b}'' \in (-1, \infty)^n$, we have $u = 0$ unless $\boldsymbol{b}' - \boldsymbol{b}'' \in \mathbb{Z}^n$.*

(b) *If $\boldsymbol{b}' = \boldsymbol{b}'' = \boldsymbol{b}$, then, up to shrinking Δ^n, u is a $\mathbb{C}$-linear combination of the basic distributions*

$$(15.8.2\,**)\qquad u_{\boldsymbol{b},\boldsymbol{p}} = \prod_{\substack{i\in I\\ b_i > -1}} |x_i|^{2b_i} \frac{\mathrm{L}(x_i)^{p_i}}{p_i!} \prod_{\substack{i\in I\\ b_i = -1}} \partial_{x_i}\partial_{\overline{x}_i} \frac{\mathrm{L}(x_i)^{p_i+1}}{(p_i+1)!},$$

where $0 \leqslant p_1, \dots, p_n \leqslant k-1$. These distributions are $\mathbb{C}$-linearly independent.

Proof. Assume first $\boldsymbol{b}', \boldsymbol{b}'' \in (-1, \infty)^n$. If $\operatorname{Supp} u \subset D$, then $x^{\boldsymbol{m}} u = 0$ for some $\boldsymbol{m} \in \mathbb{N}^n$ and, arguing as in the proof of Proposition 7.3.2, we find $u = 0$.

Otherwise, set $x_i = e^{\xi_i}$ and pullback u as $\widetilde{u}$ on the product of half-planes $\operatorname{Re}\xi_i > 0$. Set $v = e^{-\boldsymbol{b}'\boldsymbol{\xi}} e^{-\boldsymbol{b}''\overline{\boldsymbol{\xi}}}\widetilde{u}$. Then v is annihilated by $(\partial_{\xi_i}\partial_{\overline{\xi}_i})^k$ for every $i = 1, \dots, n$ —therefore by a suitable power of the n-Laplacian $\sum_i \partial_{\xi_i}\partial_{\overline{\xi}_i}$ — and a suitable $k \geqslant 1$, and by ∂_{x_j} and $\partial_{\overline{x}_j}$, that we will now omit. By the regularity of the Laplacian, v is C^∞ and, arguing with respect to each variable as in Proposition 7.3.2, we find that v is a polynomial $P(\boldsymbol{\xi}, \overline{\boldsymbol{\xi}})$ and thus $\widetilde{u} = e^{\boldsymbol{b}'\boldsymbol{\xi}} e^{\boldsymbol{b}''\overline{\boldsymbol{\xi}}} P(\boldsymbol{\xi}, \overline{\boldsymbol{\xi}})$. We now conclude (a), as well as (b) for $\boldsymbol{b}', \boldsymbol{b}'' \in (-1, \infty)^n$, as in dimension 1.

In the general case for (b), we will argue by induction on $\#\{i \in I \mid b_i = -1\}$, assumed to be $\geqslant 1$. Up to renaming the indices, we write $\boldsymbol{b} = (-1, \widehat{\boldsymbol{b}})$ and we decompose correspondingly $\boldsymbol{p} \in \mathbb{N}^n$ as $\boldsymbol{p} = (p_1, \widehat{\boldsymbol{p}})$.

By induction we find

$$|x_1|^2 u = \sum_{\boldsymbol{p}} c_{p_1+2,\boldsymbol{p}'} \cdot u_{(0,\widehat{\boldsymbol{b}}),\boldsymbol{p}}, \quad c_{\boldsymbol{p}} \in \mathbb{C},$$

for $p_i = 0, \dots k-1$ $(i = 1, \dots, n)$, and this is also written as

$$|x_1|^2 \partial_{x_1} \partial_{\overline{x}_1} \sum_{\boldsymbol{q}} c_{\boldsymbol{q}} u_{(0,\widehat{\boldsymbol{b}}),\boldsymbol{q}},$$

with $q_i = 0, \dots k-1$ for $i \neq 1$ and $q_1 = 2, \dots, k+1$. Let us set

$$v = u - \partial_{x_1} \partial_{\overline{x}_1} \sum_{\boldsymbol{q}} c_{\boldsymbol{q}} u_{(0,\widehat{\boldsymbol{b}}),\boldsymbol{q}},$$

so that $|x_1|^2 v = 0$. A computation similar to that in §7.3.a shows that the basic distributions $u_{(0,\widehat{\boldsymbol{b}}),\boldsymbol{q}}$ satisfy the equations (15.8.2 $*$) (with respect to the parameter $\boldsymbol{b}$) except when $q_1 = k+1$, in which case we find

$$(\partial_{x_1} x_1)^k \partial_{x_1} \partial_{\overline{x}_1} u_{(0,\widehat{\boldsymbol{b}}),(k+1,\widehat{\boldsymbol{q}})} = (-1)^{k+1} u_{\widehat{\boldsymbol{b}},\widehat{\boldsymbol{q}}}\, \delta(x_1),$$

and similarly when applying $(\partial_{\overline{x}_1} \overline{x}_1)^k$. Here, $\delta(x_1)$ is the distribution δ in the variable x_1 (see Exercise 7.19): for a distribution w depending on the variables $\neq x_1$, and for a test form η of maximal degree, written as $\eta = \eta_o^{(1)} \wedge \frac{\mathrm{i}}{2\pi}(\mathrm{d}x_1 \wedge \mathrm{d}\overline{x}_1)$, we set

$$\big\langle \eta, w \cdot \delta(x_1) \big\rangle := \big\langle \eta^{(1)}_{o|D_1}, w \big\rangle.$$

On the other hand, according to Exercise 12.2 and as in Proposition 7.3.3, the equation $|x_1|^2 v = 0$ implies

$$v = v_0 \delta(x_1) + \sum_{j \geqslant 0} \big(\partial^j_{x_1}(v'_j \delta(x_1)) + (\partial^j_{\overline{x}_1}(v''_j \delta(x_1))\big),$$

where v_0, v'_j, v''_j are sections of $\mathfrak{Db}_{D_1}$ on a possibly smaller Δ^{n-1}. Applying $(\partial_{x_1} x_1)^k$ and its conjugate to

$$u = \partial_{x_1} \partial_{\overline{x}_1} \sum_{\boldsymbol{q}} c_{\boldsymbol{q}} u_{(0,\widehat{\boldsymbol{b}}),\boldsymbol{q}} + v_0 \delta(x_1) + \sum_{j \geqslant 0} \big(\partial^j_{x_1}(v'_j \delta(x_1)) + (\partial^j_{\overline{x}_1}(v''_j \delta(x_1))\big)$$

yields

$$0 = (-1)^{k+1} c_{k+1,\widehat{\boldsymbol{q}}} \cdot u_{\widehat{\boldsymbol{b}},\widehat{\boldsymbol{q}}}\, \delta(x_1) + \sum_{j \geqslant 1} (-j)^k \partial^j_{x_1}(v'_j \delta(x_1)),$$
$$0 = (-1)^{k+1} c_{k+1,\widehat{\boldsymbol{q}}} \cdot u_{\widehat{\boldsymbol{b}},\widehat{\boldsymbol{q}}}\, \delta(x_1) + \sum_{j \geqslant 1} (-j)^k \partial^j_{\overline{x}_1}(v''_j \delta(x_1)),$$

By the uniqueness of the decomposition in $\mathfrak{Db}_{D_1}[\partial_{x_1}, \partial_{\overline{x}_1}]$, we conclude that

$$c_{k+1,\widehat{\boldsymbol{q}}} = 0, \qquad v'_j = v''_j = 0 \quad (j \geqslant 1),$$

and finally $u = \sum_{\boldsymbol{q}} c_{\boldsymbol{q}} u_{\boldsymbol{b},\boldsymbol{q}} + v_0 \delta(x_1)$, up to changing the notation for $c_{\boldsymbol{q}}$ in order that q_i varies in $0, \dots, k-1$ for all i. Now, v_0 has to satisfy Equations (15.8.2 $*$) on D_1, so has a decomposition on the basic distributions (15.8.2 $**$) on D_1 by the induction hypothesis, and we express $v_0 \delta(x_1)$ as a basic distribution by using the formula proved in Exercise 7.19 with respect to the variable x_1. $\square$

15.8.b. Sesquilinear pairings between holonomic $\mathcal{D}_X$-modules of normal crossing type

We make explicit the expression of a sesquilinear pairing between holonomic $\mathcal{D}_X$-modules of normal crossing type, by extending to higher dimensions Proposition 7.3.6. Due to the simplifying assumptions 15.6.2, the modules $M^{\boldsymbol{b}}$ considered below are finite dimensional $\mathbb{C}$-vector spaces.

15.8.3. Proposition. *Let $\mathfrak{s}$ be a sesquilinear pairing between $\mathcal{M}', \mathcal{M}''$ of normal crossing type.*

(1) *The induced pairing $\mathfrak{s} : M'^{\boldsymbol{b}'} \otimes \overline{M''^{\boldsymbol{b}''}} \to \mathfrak{Db}_{\Delta^n}$ vanishes if $\boldsymbol{b}' - \boldsymbol{b}'' \notin \mathbb{Z}^n$.*

(2) *If $m' \in M'^{\boldsymbol{b}}$ and $m'' \in M''^{\boldsymbol{b}}$ with $\boldsymbol{b} \geqslant -1$, then the induced pairing $\mathfrak{s}(\boldsymbol{b})(m', \overline{m''})$ is a $\mathbb{C}$-linear combination of the basic distributions $u_{\boldsymbol{b},\boldsymbol{p}}$ ($\boldsymbol{p} \in \mathbb{N}^n$).* □

As in dimension 1 (see Section 7.3.b), we find a decomposition

$$\mathfrak{s}(\boldsymbol{b}) = \sum_{\boldsymbol{p} \in \mathbb{N}^n} \mathfrak{s}^{(b)} g_{\boldsymbol{p}} \cdot u_{\boldsymbol{b},\boldsymbol{p}},$$

where $\mathfrak{s}^{(b)} g_{\boldsymbol{p}} : M'^{\boldsymbol{b}} \otimes_{\mathbb{C}} M''^{\boldsymbol{b}} \to \mathbb{C}$ is a sesquilinear pairing (between finite-dimensional $\mathbb{C}$-vector spaces) and, setting $\mathfrak{s}^{\boldsymbol{b}} = \mathfrak{s}^{(b)} g_0$, we can write in a symbolic way (recall (7.3.8))

$$\mathfrak{s}(\boldsymbol{b})(m', \overline{m''}) = \prod_{i | b_i = -1} \partial_{x_i} \partial_{\overline{x}_i} \, \mathfrak{s}^{(b)} g \Big(\prod_{i | b_i > -1} |x_i|^{2(b_i \operatorname{Id} - \mathrm{N}_i)} \prod_{i | b_i = -1} \frac{|x_i|^{-2\mathrm{N}_i} - 1}{\mathrm{N}_i} \, m', \overline{m''} \Big),$$

where $\mathrm{N}_i = -(x_i \partial_i - b_i)$. As a corollary we obtain:

15.8.4. Corollary. *With the assumptions of the proposition, we have*

$$x_i \partial_{x_i} \mathfrak{s}(m', \overline{m''}) = \overline{x_i} \partial_{\overline{x_i}} \mathfrak{s}(m', \overline{m''}).$$ □

Notice also that the same property holds for $-(x_i \partial_{x_i} - b_i)$ since b_i is real. Therefore, with respect to the nilpotent operator N_i, $\mathfrak{s} : M'^{\boldsymbol{b}} \otimes \overline{M''^{\boldsymbol{b}}} \to \mathfrak{Db}_X$ satisfies

$$\mathfrak{s}(\mathrm{N}_i m', \overline{m''}) = \mathfrak{s}(m', \overline{\mathrm{N}_i m''}).$$

15.8.5. Remark. In the context of right $\mathcal{D}$-modules, we consider currents instead of distributions. We denote by Ω_n the (n, n)-form $\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n \wedge \mathrm{d}\overline{x}_1 \wedge \cdots \wedge \mathrm{d}\overline{x}_n$, that we also abbreviate by $\mathrm{d}x \wedge \mathrm{d}\overline{x}$. In order to state similar results, we set $a = -b - 1$ and we consider the basic currents $\Omega_n u_{\boldsymbol{b},\boldsymbol{p}}$. Given a sesquilinear pairing $\mathfrak{s} : \mathcal{M}' \otimes_{\mathbb{C}} \overline{\mathcal{M}''} \to \mathfrak{C}_{\Delta^n}$, the induced pairing $\mathfrak{s} : M'_{\boldsymbol{a}'} \otimes \overline{M''_{\boldsymbol{a}''}} \to \mathfrak{C}_{\Delta^n}$ vanishes if $\boldsymbol{a}' - \boldsymbol{a}'' \notin \mathbb{Z}^n$, and for $m' \in M'_{\boldsymbol{a}}$ and $m'' \in M''_{\boldsymbol{a}}$ with $\boldsymbol{a} \leqslant 0$, the induced pairing $\mathfrak{s}(\boldsymbol{a})(m', \overline{m''})$ can be written as

$$\mathfrak{s}(\boldsymbol{a})(m', \overline{m''}) = \Omega_n \mathfrak{s}_{\boldsymbol{a}} \Big(m' \prod_{i | a_i < 0} |x_i|^{-2(1 + a_i + \mathrm{N}_i)} \prod_{i | a_i = 0} \frac{|x_i|^{-2\mathrm{N}_i} - 1}{\mathrm{N}_i}, \overline{m''} \Big) \cdot \prod_{i | a_i = 0} \partial_{x_i} \partial_{\overline{x}_i},$$

where $\mathrm{N}_i = (x_i \partial_i - a_i)$. Similarly, N_i is self-adjoint with respect to $\mathfrak{s}$.

15.9. Filtered normal crossing type

15.9.a. Coherent filtrations of normal crossing type. We now extend the notion of "normal crossing type" to filtered coherent $\mathcal{D}$-modules. Of course the underlying $\mathcal{D}$-module should be of normal crossing type, but the isomorphism (15.7.12), together with the decomposition (15.7.11 $*$), is not expected to hold at the filtered level. This would be a too strong condition.[1] On the other hand, the properties in Proposition 15.7.13 can be naturally extended to the filtered case. We keep the simplifying assumptions 15.6.2.

15.9.1. Definition. Let $(\mathcal{M}, F_\bullet\mathcal{M})$ be a coherently F-filtered $\mathcal{D}_X$-module. We say that $(\mathcal{M}, F_\bullet\mathcal{M})$ is of *normal crossing type along D* if

(1) $\mathcal{M}$ is of normal crossing type along D (see Definition 15.7.11),

(2) $(\mathcal{M}, F_\bullet\mathcal{M})$ is $\mathbb{R}$-specializable along D_i for every component D_i of D (see Section 10.5),

(3) the filtrations $(F_\bullet\mathcal{M}, V_\bullet^{(1)}\mathcal{M}, \dots, V_\bullet^{(n)}\mathcal{M})$ are distributive (or compatible) (see Definition 15.1.7 or Section 15.4).

15.9.2. Remarks.

(a) Condition (3) implies that $\mathrm{gr}_p^F \mathrm{gr}_{\boldsymbol{a}}^{V^{(\boldsymbol{n})}}\mathcal{M}$ does not depend on the way $\mathrm{gr}_{\boldsymbol{a}}^{V^{(\boldsymbol{n})}}\mathcal{M}$ is computed.

(b) Note that (2) implies 15.7.13(1) for $\mathcal{M}$, and similarly (3) implies 15.7.13(2). So the condition that $\mathcal{M}$ is of normal crossing type along D only adds the existence of the isomorphism (15.7.12).

(c) Let us recall that $V_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}$ is $\mathcal{O}_X$-coherent for every $\boldsymbol{a} \in \mathbb{R}^n$ (see Proposition 15.7.13(5)). Since $F_p\mathcal{M}$ is $\mathcal{O}_X$-coherent, it follows that $F_pV_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M} := F_p\mathcal{M} \cap V_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}$ (see §10.5) and $\mathrm{gr}_p^F V_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}$ are also $\mathcal{O}_X$-coherent and therefore the filtration $F_\bullet V_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}$ is locally finite, hence is a coherent $F_\bullet V_{\boldsymbol{0}}^{(\boldsymbol{n})}\mathcal{D}_X$-filtration.

(d) Since each $\mathrm{gr}_{\boldsymbol{a}}^{V^{(\boldsymbol{n})}}\mathcal{M}$ is finite dimensional, the induced filtration $F_\bullet\mathrm{gr}_{\boldsymbol{a}}^{V^{(\boldsymbol{n})}}\mathcal{M}$ is finite, and there exists a (non-canonical) splitting compatible with $F_\bullet$:

$$F_p\mathrm{gr}_{\boldsymbol{a}}^{V^{(\boldsymbol{n})}}\mathcal{M} \simeq \bigoplus_{q\leqslant p} \mathrm{gr}_q^F \mathrm{gr}_{\boldsymbol{a}}^{V^{(\boldsymbol{n})}}\mathcal{M}.$$

(e) There are a priori two ways for defining the filtration $F_\bullet M_{\boldsymbol{a}}$, namely, either by inducing it on $M_{\boldsymbol{a}} \subset \mathcal{M}$, or by inducing it on $\mathrm{gr}_{\boldsymbol{a}}^{V^{(\boldsymbol{n})}}\mathcal{M}$ and transport it by means of the isomorphism $M_{\boldsymbol{a}} \xrightarrow{\sim} \mathrm{gr}_{\boldsymbol{a}}^{V^{(\boldsymbol{n})}}\mathcal{M}$. We always consider the latter one. The filtration $F_\bullet\mathcal{M}$ is a priori not isomorphic to $\bigoplus_a F_\bullet\mathrm{gr}_{\boldsymbol{a}}^{V^{(\boldsymbol{n})}}\mathcal{M}$ by means of the isomorphism $\mathcal{M} \simeq \bigoplus_a \mathrm{gr}_{\boldsymbol{a}}^{V^{(\boldsymbol{n})}}\mathcal{M}$ induced by 15.7.13(7) and (15.7.12). Using the compatibility of the filtrations, we have

$$F_pM_{\boldsymbol{a}} = M_{\boldsymbol{a}} \cap (F_pV_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M} + V_{<\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}) \subset \mathcal{M}.$$

[1] Such a filtered decomposition holds however for *monodromic* mixed Hodge modules, see **[Sai22]** and **[CD23]**.

The graded filtered module $(\bigoplus_{\boldsymbol{a}} M_{\boldsymbol{a}}, \bigoplus_{\boldsymbol{a}} F_\bullet M_{\boldsymbol{a}})$ is obviously of normal crossing type if $(\mathcal{M}, F_\bullet\mathcal{M})$ is so.

As the category of coherently filtered $\mathcal{D}_X$-modules is not abelian, one cannot expect, in contrast with Corollary 15.7.17, that the category of filtered $\mathcal{D}_X$-modules of normal crossing type is abelian. However, some morphisms have kernel and cokernel in this category.

15.9.3. Proposition. *Let $\varphi : (\mathcal{M}_1, F_\bullet\mathcal{M}_1) \to (\mathcal{M}_2, F_\bullet\mathcal{M}_2)$ be a morphism between filtered $\mathcal{D}_X$-modules of normal crossing type. Assume that φ is $(n+1)$-strict (see Definition* 15.3.1*), i.e.,* $\operatorname{Coker} R_{FV}\varphi$ *is $\mathbb{C}[z, z_1, \dots, z_n]$-flat. Then* $\operatorname{Ker}\varphi$*,* $\operatorname{Im}\varphi$ *and* $\operatorname{Coker}\varphi$*, equipped with the induced F- and V-filtrations, are filtered $\mathcal{D}_X$-modules of normal crossing type.*

Proof. That Property 15.9.1(1) holds for $\operatorname{Ker}\varphi$, $\operatorname{Im}\varphi$ and $\operatorname{Coker}\varphi$ follows from Corollary 15.7.17, and 15.9.1(3) holds by assumption. On the other hand, $(n+1)$-strictness of φ implies its 2-strictness for each i, that is, $\operatorname{Coker} R_{FV^{(i)}}\varphi$ is $\mathbb{C}[z, z_i]$-flat: indeed, $\operatorname{Coker} R_{FV^{(i)}}\varphi$ is obtained by base change $z_j = 1$ for all $j \neq i$, and flatness is preserved by base change. By a similar argument (restricting to $z_i = 0$), we obtain that for each a_i, $\operatorname{gr}^{V^{(i)}}_{a_i}(R_F\varphi)$ is strict, which means that $R_F\varphi$ is strictly $\mathbb{R}$-specializable along D_i, and this implies 15.9.1(2) for $\operatorname{Ker}\varphi$, $\operatorname{Im}\varphi$ and $\operatorname{Coker}\varphi$, according to Proposition 9.3.31. □

15.9.b. Behaviour with respect to specialization, localization, dual localization and middle extension along one component of D

The properties (1) and (2) of Proposition 15.7.13 have been taken as a model for defining the notion of a filtered $\mathcal{D}_X$-module of normal crossing type. We now deduce the analogues of the stability and flatness properties (4)–(6) of Proposition 15.7.13.

***15.9.4. Proposition* (Stability by specialization and flatness).** *Let $(\mathcal{M}, F_\bullet\mathcal{M})$ be a coherently F-filtered $\mathcal{D}_X$-module of normal crossing type along D.*

(1) *For any $i \in I$ and any $a_i \in \mathbb{R}$, $(\operatorname{gr}^{V^{(i)}}_{a_i}\mathcal{M}, F_\bullet\operatorname{gr}^{V^{(i)}}_{a_i}\mathcal{M})$ is of normal crossing type on $(D_i, \bigcup_{j\neq i} D_j)$, where $F_\bullet\operatorname{gr}^{V^{(i)}}_{a_i}\mathcal{M}$ is the filtration naturally induced by $F_\bullet\mathcal{M}$ on $\operatorname{gr}^{V^{(i)}}_{a_i}\mathcal{M}$.*

(2) *For $\boldsymbol{a} = (a_1, \dots, a_n) \in \mathbb{R}^n$ and each $p \in \mathbb{Z}$, $\operatorname{gr}^F_p V^{(\boldsymbol{n})}_{\boldsymbol{a}}\mathcal{M}$ is a coherent $\mathcal{O}_X$-module which is $\mathcal{O}_X$-locally free in the neighborhood of D if $a_i < 0$ for all $i \in I$.*

(3) *For any decomposition $X = X' \times X''$ with projection $p' : X \to X'$ and $n = n' + n''$ as in Proposition* 15.7.3(4)*, if $\boldsymbol{a}'$ belongs to $(\mathbb{R}_{<0})^{n'}$, then for each $p \in \mathbb{Z}$, $\operatorname{gr}^F_p V^{(\boldsymbol{n}')}_{\boldsymbol{a}'}\mathcal{M}$ is $p'^{-1}\mathcal{O}_{X'}$-flat in the neighborhood of D.*

Proof.

(1) We know by Proposition 15.7.13(4) that $\operatorname{gr}^{V^{(i)}}_{a_i}\mathcal{M}$ is of normal crossing type on $(D_i, \bigcup_{j\neq i} D_j)$, and that the filtrations $V^{(j)}_\bullet$ on $\operatorname{gr}^{V^{(i)}}_{a_i}\mathcal{M}$ are naturally induced by $V^{(j)}_\bullet\mathcal{M}$. It follows that the family $(F_\bullet\operatorname{gr}^{V^{(i)}}_{a_i}\mathcal{M}, (V^{(j)}_\bullet\operatorname{gr}^{V^{(i)}}_{a_i}\mathcal{M})_{j\neq i})$ is distributive

(see Remark 15.1.8). We know, by Proposition 10.7.3, that $(\mathrm{gr}_{a_i}^{V^{(i)}}\mathcal{M}, F_\bullet\mathrm{gr}_{a_i}^{V^{(i)}}\mathcal{M})$ is coherent as a filtered $\mathcal{D}_{D_i}$-module. Note also that, setting $\boldsymbol{a}' = (a_j)_{j\neq i}$ and $\boldsymbol{n}' = (n_j)_{j\neq i}$, we have

$$\mathrm{gr}_p^F\mathrm{gr}_{\boldsymbol{a}'}^{V^{(\boldsymbol{n}')}}\mathrm{gr}_{a_i}^{V^{(i)}}\mathcal{M} = \mathrm{gr}_p^F\mathrm{gr}_{\boldsymbol{a}}^{V^{(\boldsymbol{n})}}\mathcal{M}$$

(since, by the distributivity property, we can take graded objects in any order).

The strict $\mathbb{R}$-specializability property along each D_j ($j\neq i$) remains to be shown, namely,

$$\begin{aligned}x_j : F_pV_{a_j}^{(j)}\mathrm{gr}_{a_i}^{V^{(i)}}\mathcal{M} &\xrightarrow{\ \sim\ } F_pV_{a_j-1}^{(j)}\mathrm{gr}_{a_i}^{V^{(i)}}\mathcal{M}, \quad \forall\,p,\,\forall\,j\neq i,\,\forall\,a_j<0,\\ \partial_{x_j} : F_p\mathrm{gr}_{a_j}^{V^{(j)}}\mathrm{gr}_{a_i}^{V^{(i)}}\mathcal{M} &\xrightarrow{\ \sim\ } F_{p+1}\mathrm{gr}_{a_j+1}^{V^{(j)}}\mathrm{gr}_{a_i}^{V^{(i)}}\mathcal{M}, \quad \forall\,p,\,\forall\,j\neq i,\,\forall\,a_j>-1.\end{aligned}$$

Let us first show that, by applying $\mathrm{gr}_{a_i}^{V^{(i)}}$, we get isomorphisms

$$x_j : \mathrm{gr}_{a_i}^{V^{(i)}}F_pV_{a_j}^{(j)}\mathcal{M} \xrightarrow{\ \sim\ } \mathrm{gr}_{a_i}^{V^{(i)}}F_pV_{a_j-1}^{(j)}\mathcal{M}, \quad \forall\,p,\,\forall\,j\neq i,\,\forall\,a_j<0, \tag{15.9.5}$$

$$\partial_{x_j} : \mathrm{gr}_{a_i}^{V^{(i)}}F_p\mathrm{gr}_{a_j}^{V^{(j)}}\mathcal{M} \xrightarrow{\ \sim\ } \mathrm{gr}_{a_i}^{V^{(i)}}F_{p+1}\mathrm{gr}_{a_j+1}^{V^{(j)}}\mathcal{M}, \quad \forall\,p,\,\forall\,j\neq i,\,\forall\,a_j>-1. \tag{15.9.6}$$

By the strict $\mathbb{R}$-specializability of $(\mathcal{M}, F_\bullet\mathcal{M})$ along D_j and since $\mathcal{M}$ is of normal crossing type, so that (15.7.18 $*$) holds, we have isomorphisms under the conditions of (15.9.5):

$$F_pV_{a_j}^{(j)}\mathcal{M} \xrightarrow[\sim]{x_j} F_pV_{a_j-1}^{(j)}\mathcal{M}, \qquad \begin{cases}V_{a_i}^{(i)}V_{a_j}^{(j)}\mathcal{M}\\ V_{<a_i}^{(i)}V_{a_j}^{(j)}\mathcal{M}\end{cases} \xrightarrow[\sim]{x_j} \begin{cases}V_{a_i}^{(i)}V_{a_j-1}^{(j)}\mathcal{M}\\ V_{<a_i}^{(i)}V_{a_j-1}^{(j)}\mathcal{M},\end{cases}$$

hence isomorphisms

$$\begin{cases}V_{a_i}^{(i)}F_pV_{a_j}^{(j)}\mathcal{M}\\ V_{<a_i}^{(i)}F_pV_{a_j}^{(j)}\mathcal{M}\end{cases} \xrightarrow[\sim]{x_j} \begin{cases}V_{a_i}^{(i)}F_pV_{a_j-1}^{(j)}\mathcal{M}\\ V_{<a_i}^{(i)}F_pV_{a_j-1}^{(j)}\mathcal{M},\end{cases}$$

and thus the isomorphisms (15.9.5). We argue similarly for the isomorphisms (15.9.6). Now, the desired assertion follows from the compatibility property 15.9.1(3) which enables us to switch $F_pV_{a_j}^{(j)}$ or $F_p\mathrm{gr}_{a_j}^{V^{(j)}}$ with $\mathrm{gr}_{a_i}^{V^{(i)}}$.

By the same argument as above, the filtered analogue of (15.7.18 $*$) holds (any $\boldsymbol{a}'\in\mathbb{R}^{n-1}$, $p\in\mathbb{Z}$):

$$\begin{aligned}F_pV_{\boldsymbol{a}'}^{(\boldsymbol{n}')}V_{a_i}^{(i)}\mathcal{M} &\xrightarrow[\sim]{x_i} F_pV_{\boldsymbol{a}'}^{(\boldsymbol{n}')}V_{a_i-1}^{(j)}\mathcal{M} \quad \text{if } a_i<0,\\ F_pV_{\boldsymbol{a}'}^{(\boldsymbol{n}')}\mathrm{gr}_{a_i}^{V^{(i)}}\mathcal{M} &\xrightarrow[\sim]{\partial_{x_i}} F_{p+1}V_{\boldsymbol{a}'}^{(\boldsymbol{n}')}\mathrm{gr}_{a_i+1}^{V^{(i)}}\mathcal{M} \quad \text{if } a_i>-1.\end{aligned} \tag{15.9.7}$$

(2) Coherence has already been noticed (Remark 15.9.2(c)), and we will show local freeness at the origin, the case of other points of D being similar (and needs to avoid the simplifying assumption 15.6.2). Therefore, X will denote a small enough neighbourhood of the origin. Let $i_0 : 0 \hookrightarrow X$ denote the inclusion. From the first line of (15.9.7) one deduces that $i_0^*\mathrm{gr}_p^FV_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M} = \mathrm{gr}_p^F(V_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}/V_{\boldsymbol{a}-\boldsymbol{1}}^{(\boldsymbol{n})}\mathcal{M})$. Let us denote by rk the generic rank on X of a coherent $\mathcal{O}_X$-module. By local $\mathcal{O}_X$-freeness of $V_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}$, we have $\mathrm{rk}\,V_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M} = \dim i_0^*V_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}$, and on the other hand, by $\mathcal{O}_X$-coherence, for each p, $\mathrm{rk}\,\mathrm{gr}_p^FV_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M} \leqslant \dim i_0^*\mathrm{gr}_p^FV_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}$ with equality if and only if $\mathrm{gr}_p^FV_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}$ is

$\mathcal{O}_X$-locally free. It follows that both sums over p of the latter terms are equal, and therefore these terms are equal for each p.

(3) Iterating the argument in (1) shows that, for any $\boldsymbol{a}'' \in \mathbb{R}^{n''}$, the coherently filtered $\mathcal{D}_{X'}$-module $(\mathrm{gr}^{V^{(\boldsymbol{n}'')}}_{\boldsymbol{a}''}\mathcal{M}, F_\bullet\mathrm{gr}^{V^{(\boldsymbol{n}'')}}_{\boldsymbol{a}''}\mathcal{M})$ is of normal crossing type on (X', D'), with $D' = \{\prod_{i=1}^{n'} x_i = 0\}$. Therefore, by (2), if $a_i < 0$ for $i = 1, \dots, n'$, the $\mathcal{O}'_X$-module $\mathrm{gr}^F_p V^{(\boldsymbol{n}')}_{\boldsymbol{a}'}\mathrm{gr}^{V^{(\boldsymbol{n}'')}}_{\boldsymbol{a}''}\mathcal{M}$ is locally $\mathcal{O}_{X'}$-free, hence $\mathcal{O}_{X'}$-flat. Since $\mathrm{gr}^F_p V^{(\boldsymbol{n}')}_{\boldsymbol{a}'}V^{(\boldsymbol{n}'')}_{\boldsymbol{a}''}\mathcal{M}$ is also $\mathcal{O}_{X'}$-flat (being $\mathcal{O}_X$-locally free) if moreover $a_j < 0$ for all $j = n'+1, \dots, n'+n'' = n$, it follows by an easy induction that it is also $\mathcal{O}_{X'}$-flat for any $\boldsymbol{a}''$. Passing to the limit with respect to $\boldsymbol{a}''$ yields the $\mathcal{O}_{X'}$-flatness of $\mathrm{gr}^F_p V^{V^{(\boldsymbol{n}')}}_{\boldsymbol{a}'}\mathcal{M}$. □

The following lemma is similar to Exercise 15.14, but weaker when considering surjectivity for can_{i_o}.

15.9.8. Lemma. *Assume that* $(\mathcal{M}, F_\bullet\mathcal{M})$ *is of normal crossing type along* D. *Let us fix* $i \in I$ *and let* $\widehat{\boldsymbol{n}}$ *be* $\boldsymbol{n}$ *with* i *omitted. Then, for every* $\widehat{\boldsymbol{a}} \in \mathbb{R}^{n-1}$, *each of the following properties*

$$(15.9.8\,*)\qquad \begin{aligned} &\mathrm{can}_i : F_p V^{(\widehat{\boldsymbol{n}})}_{\widehat{\boldsymbol{a}}}\mathrm{gr}^{V^{(i)}}_{-1}\mathcal{M} \longrightarrow F_{p+1}V^{(\widehat{\boldsymbol{n}})}_{\widehat{\boldsymbol{a}}}\mathrm{gr}^{V^{(i)}}_{0}\mathcal{M} \quad \textit{is bijective,}\\ &\mathrm{var}_i : F_p V^{(\widehat{\boldsymbol{n}})}_{\widehat{\boldsymbol{a}}'}\mathrm{gr}^{V^{(i)}}_{0}\mathcal{M} \longrightarrow F_{p}V^{(\widehat{\boldsymbol{n}})}_{\widehat{\boldsymbol{a}}}\mathrm{gr}^{V^{(i)}}_{-1}\mathcal{M} \quad \textit{is} \begin{cases}\textit{injective,}\\ \textit{resp. bijective,}\end{cases} \end{aligned}$$

holds for all p *as soon as it holds when omitting* $V^{(\widehat{\boldsymbol{n}})}_{\widehat{\boldsymbol{a}}}$. □

15.9.9. Remark. As a consequence, if var_i is injective, then the first line of (15.9.7) with $j = i$ also holds for $a_j = 0$. That the lemma does not a priori hold when can_i is only onto leads to the definition below.

15.9.10. Definition **(Middle extension along $D_{i\in I}$).** Let $(\mathcal{M}, F_\bullet\mathcal{M})$ be a coherently F-filtered $\mathcal{D}_X$-module of normal crossing type along D. We say that $(\mathcal{M}, F_\bullet\mathcal{M})$ is a *middle extension along* $D_{i\in I}$ if $\mathcal{M}$ is a middle extension independently along each D_i ($i \in I$) and moreover, for each $i \in I$, and every $\widehat{\boldsymbol{a}} \in \mathbb{R}^{n-1}$ (equivalently, every $\widehat{\boldsymbol{a}} \in [-1, 0]^{n-1}$),

$$\mathrm{can}_i : F_p V^{(\widehat{\boldsymbol{n}})}_{\widehat{\boldsymbol{a}}}\mathrm{gr}^{V^{(i)}}_{-1}\mathcal{M} \longrightarrow F_{p+1}V^{(\widehat{\boldsymbol{n}})}_{\widehat{\boldsymbol{a}}}\mathrm{gr}^{V^{(i)}}_{0}\mathcal{M} \quad \text{is onto, } \forall\, p.$$

If $n = 1$ this notion is equivalent to that of Definition 9.7.3, but if $n \geqslant 2$ it is a priori stronger than the condition of filtered middle extension along each D_i independently (see Definition 10.5.1).

15.9.c. Logarithmic filtered normal crossing type. It is easier to deal with coherent $\mathcal{O}$-modules instead of coherent $\mathcal{D}$-modules. We will focus on the coherent $\mathcal{O}_X$-modules $\mathcal{M}_{\leqslant\boldsymbol{0}} := V^{(\boldsymbol{n})}_{\boldsymbol{0}}\mathcal{M}$ and $\mathcal{M}_{<\boldsymbol{0}} := V^{\boldsymbol{n}}_{<\boldsymbol{0}}\mathcal{M} = \bigcap_{i\in I} V^{(i)}_{<0}\mathcal{M}$, the latter being locally free (Proposition 15.7.13(5)).

Our aim is to deduce properties on $F_\bullet\mathcal{M}$ from properties on $F_\bullet\mathcal{M}_{\leqslant\boldsymbol{0}}$ and, in the case of a middle extension along $D_{i\in I}$, from $F_\bullet\mathcal{M}_{<\boldsymbol{0}}$. Both are modules over the sheaf $V^{(\boldsymbol{n})}_{\boldsymbol{0}}\mathcal{D}_X$ of logarithmic differential operators. We first explain which properties should be expected on the latter $V_{\mathbf{0}}^{(\boldsymbol{n})}\mathcal{D}_X$-module, in order to recover the normal crossing property of $(\mathcal{M}, F_\bullet\mathcal{M})$ from them. We will then give a criterion to check whether they are satisfied.

***15.9.11. Proposition* (Properties of $F_pV_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}$).** *Let $(\mathcal{M}, F_\bullet\mathcal{M})$ be a coherently F-filtered $\mathcal{D}_X$-module of normal crossing type along D. Set $\mathcal{M}_{\leqslant\mathbf{0}} := V_{\mathbf{0}}^{(\boldsymbol{n})}\mathcal{M}$. For $\boldsymbol{a} \in \mathbb{R}^n$, let us set $F_pV_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M} := F_p\mathcal{M} \cap V_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}$. Then*

(1) *for any $\boldsymbol{a} \in \mathbb{R}^n$, $F_\bullet V_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}$ is a coherent $F_\bullet V_{\mathbf{0}}^{(\boldsymbol{n})}\mathcal{D}_X$-filtration;*

(2) *we have $F_p\mathcal{M}_{<\mathbf{0}} = j_*(F_p\mathcal{M}|_{X\smallsetminus D}) \cap \mathcal{M}_{<\mathbf{0}}$ for any p, where $j : X \smallsetminus D$ is the open inclusion;*

(3) *the filtrations $(F_\bullet\mathcal{M}_{\leqslant\mathbf{0}}, V_\bullet^{(1)}\mathcal{M}_{\leqslant\mathbf{0}}, \dots, V_\bullet^{(n)}\mathcal{M}_{\leqslant\mathbf{0}})$ are distributive and*

$$F_p\mathcal{M} = \sum_{q\geqslant 0}(F_{p-q}\mathcal{M}_{\leqslant\mathbf{0}}) \cdot F_q\mathcal{D}_X.$$

Proof. The first point has been seen in Remark 15.9.2(c). For the second point, the inclusion $\subset$ is clear; on the other hand, let m be a local section of $j_*(F_p\mathcal{M}|_{X\smallsetminus D}) \cap \mathcal{M}_{<\mathbf{0}}$; it is also a local section of $F_q\mathcal{M}_{<\mathbf{0}}$ for q large enough; if $q > p$, then the class of m in the locally free $\mathcal{O}_X$-module $\mathrm{gr}_q^F\mathcal{M}_{<\mathbf{0}}$ (Proposition 15.9.4(2)) is supported on D, hence is zero.

The distributivity property of the filtrations on $\mathcal{M}_{\leqslant\mathbf{0}}$ clearly follows from that on $\mathcal{M}$, as noted in Remark 15.1.8(2). By the same argument we have distributivity for the family of filtrations on each $V_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}$ ($\boldsymbol{a} \in \mathbb{R}^n$).

It remains to justify the expression for $F_p\mathcal{M}$. We have seen in the proof of Proposition 15.9.4 that, for $\boldsymbol{k} \geqslant 0$ and any $i \in I$, setting $\boldsymbol{k} = (\boldsymbol{k}', k_i)$, we have an isomorphism

$$\partial_{x_i} : F_{p-1}V_{\boldsymbol{k}'}^{(\boldsymbol{n}')}\mathrm{gr}_{k_i}^{V^{(i)}}\mathcal{M} \xrightarrow{\sim} F_pV_{\boldsymbol{k}'}^{(\boldsymbol{n}')}\mathrm{gr}_{k_i+1}^{V^{(i)}}\mathcal{M},$$

and thus

$$F_pV_{\boldsymbol{k}+\mathbf{1}_i}^{(\boldsymbol{n})}\mathcal{M} = F_{p-1}V_{\boldsymbol{k}}^{(\boldsymbol{n})}\mathcal{M} \cdot \partial_{x_i} + F_pV_{\boldsymbol{k}}^{(\boldsymbol{n})}\mathcal{M},$$

which proves (3) by an easy induction. ☐

The property 15.9.11(3) can be made more precise. For $\boldsymbol{\alpha} \in [-1, 0]^n$ and $p \in \mathbb{Z}$, let us choose a finite $\mathbb{C}$-vector space $E_{\boldsymbol{\alpha},p}$ of sections of $F_pV_{\boldsymbol{\alpha}}^{(\boldsymbol{n})}\mathcal{M}$ which maps bijectively to $\mathrm{gr}_p^F\mathrm{gr}_{\boldsymbol{\alpha}}^{V^{(\boldsymbol{n})}}\mathcal{M}$. Given any $\boldsymbol{a} \in \mathbb{R}^n$, we decompose it as $(\boldsymbol{a}', \mathbf{0}, \boldsymbol{a}'')$, where each component a_i of $\boldsymbol{a}'$ (resp. $\boldsymbol{a}''$) satisfies $a_i < 0$ (resp. $a_i > 0$). When $\boldsymbol{a}$ is fixed, any $\boldsymbol{\alpha} \in [-1, 0]^n$ decomposes correspondingly as $(\boldsymbol{\alpha}', \boldsymbol{\alpha}^o, \boldsymbol{\alpha}'')$, of respective sizes n', n^o, n''.

15.9.12. Proposition. *With these assumptions and notation, for every $\boldsymbol{a} \in \mathbb{R}^n$ and $p \in \mathbb{Z}$,*

- *if $\boldsymbol{a} < 0$, i.e., $a_i < 0$ for all i (i.e., $n' = n$), then $F_pV_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}$ is locally $\mathcal{O}_X$-free and decomposes as*

$$F_pV_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M} \simeq \bigoplus_{q\leqslant p}\ \bigoplus_{\boldsymbol{\alpha}\in[-1,0)^n} E_{\boldsymbol{\alpha},q} \otimes_{\mathbb{C}} x^{\boldsymbol{e}(\boldsymbol{\alpha},\boldsymbol{a})}\mathcal{O}_X,$$

where $\boldsymbol{e}(\boldsymbol{\alpha}, \boldsymbol{a}) \in \mathbb{N}^n$ is defined by $e_i(\boldsymbol{\alpha}, \boldsymbol{a}) = \max(0, \lceil\alpha_i - a_i\rceil)$;

- *if* $\boldsymbol{a} = (\boldsymbol{a}', \boldsymbol{a}^o)$ *with* $\boldsymbol{a}^o = 0$ *(i.e.,* $n'' = 0$*), then taking the sum in* $\mathcal{M}$*, we have*

$$F_p V^{(\boldsymbol{n})}_{\boldsymbol{a}} \mathcal{M} \simeq \sum_{\widetilde{\boldsymbol{\alpha}}' \in [-1,0)^{n^o}} F_p V^{(\boldsymbol{n})}_{(\boldsymbol{a}', \widetilde{\boldsymbol{\alpha}}')} \mathcal{M} + \sum_{q \leqslant p} \sum_{\boldsymbol{\alpha}' \in [-1,0)^{n'}} E_{(\boldsymbol{\alpha}',0),q} \otimes x^{\boldsymbol{e}(\boldsymbol{\alpha}', \boldsymbol{a}')} \mathcal{O}_X,$$

where each $F_p V^{(\boldsymbol{n})}_{(\boldsymbol{a}', \widetilde{\alpha}')} \mathcal{M}$ *is described in the first point;*

- *In general, we have*

$$F_p V^{(\boldsymbol{n})}_{\boldsymbol{a}} \mathcal{M} = \sum_{\boldsymbol{\alpha}'' \in (-1,0]^{n''}} \sum_{\substack{\boldsymbol{b}'' \in \mathbb{N}^{n''} \\ \forall\, i,\, b_i + \alpha_i \leqslant a_i}} F_{p-|\boldsymbol{b}''|} V^{(\boldsymbol{n})}_{(\boldsymbol{a}',0,\boldsymbol{\alpha}'')} \mathcal{M} \cdot \partial_x^{\boldsymbol{b}''},$$

where all terms are described in the previous points.

Proof. The last point is obtained by induction from the second line of (15.9.7), and the first point comes from the first line of (15.9.7) together with the local $\mathcal{O}_X$-freeness of $F_p V^{(\boldsymbol{n})}_{\boldsymbol{a}} \mathcal{M}$ if $\boldsymbol{a} < 0$. The second point is then straightforward. □

15.9.13. Remark **(The case of a middle extension along $D_{i \in I}$)**

In that case (Definition 15.9.10), Proposition 15.9.11 holds with the replacement of $\mathcal{M}_{\leqslant \mathbf{0}}$ with $\mathcal{M}_{<\mathbf{0}}$, and Proposition 15.9.12 reads as follows. We now decompose $\boldsymbol{a}$ as $(\boldsymbol{a}', \boldsymbol{a}'')$, where each component a_i of $\boldsymbol{a}'$ (resp. $\boldsymbol{a}''$) satisfies $a_i < 0$ (resp. $a_i \geqslant 0$), and correspondingly $n = n' + n''$. Then

$$F_p V^{(\boldsymbol{n})}_{\boldsymbol{a}} \mathcal{M} = \sum_{\boldsymbol{\alpha} \in [-1,0)^n} \sum_{\substack{\boldsymbol{b}'' \in \mathbb{N}^{n''} \\ \forall\, i,\, b_i + \alpha_i \leqslant a_i}} E_{\boldsymbol{\alpha}, p-|\boldsymbol{b}''|} \cdot x^{\boldsymbol{e}(\boldsymbol{\alpha}, \boldsymbol{a}-\boldsymbol{b}'')} \partial_x^{\boldsymbol{b}''} \mathcal{O}_X,$$

where we have set $\boldsymbol{a} - \boldsymbol{b}'' = (\boldsymbol{a}', \boldsymbol{a}'' - \boldsymbol{b}'')$ and $\boldsymbol{e}$ is as in Proposition 15.9.12.

As the proposition below shows, it is much easier to check $\mathbb{R}$-specializability of $(\mathcal{M}, F_\bullet \mathcal{M})$ and distributivity of the filtrations $(F_\bullet \mathcal{M}, V^{(1)}_\bullet \mathcal{M}, \dots, V^{(n)}_\bullet \mathcal{M})$ on $V^{(\boldsymbol{n})}_{\mathbf{0}} \mathcal{M}$, since one does not need to check strictness of the derivations ∂_{x_i}.

15.9.14. Proposition **(From $\mathcal{M}_{\leqslant \mathbf{0}}$ to $\mathcal{M}$).** *Let* $\mathcal{M}$ *be a coherent* $\mathcal{D}_X$*-module of normal crossing type along* D*. Set* $\mathcal{M}_{\leqslant \mathbf{0}} := V^{(\boldsymbol{n})}_{\mathbf{0}} \mathcal{M}$*. Denote by* $V^{(i)}_\bullet \mathcal{M}_{\leqslant \mathbf{0}}$ *the filtration naturally induced by* $V^{(i)}_\bullet \mathcal{M}$ *and let* $F_\bullet \mathcal{M}_{\leqslant \mathbf{0}}$ *be any coherent* $F_\bullet V^{(\boldsymbol{n})}_{\mathbf{0}} \mathcal{D}_X$*-filtration such that* $(F_\bullet \mathcal{M}_{\leqslant \mathbf{0}}, V^{(1)}_\bullet \mathcal{M}_{\leqslant \mathbf{0}}, \dots, V^{(n)}_\bullet \mathcal{M}_{\leqslant \mathbf{0}})$ *are compatible filtrations and that* $(\mathcal{M}_{\leqslant \mathbf{0}}, F_\bullet \mathcal{M}_{\leqslant \mathbf{0}})$ *is* $\mathbb{R}$*-specializable along each* D_i*, in the sense that* $F_p V^{(i)}_{a_i} \mathcal{M}_{\leqslant \mathbf{0}} \cdot x_i = F_p V^{(i)}_{a_i - 1} \mathcal{M}_{\leqslant \mathbf{0}}$ *for every* i *and* $a_i < 0$*, and* ∂_{x_i} *sends* $F_p V^{(i)}_{-1} \mathcal{M}_{\leqslant \mathbf{0}}$ *to* $F_{p+1} V^{(i)}_0 \mathcal{M}_{\leqslant \mathbf{0}}$*. Set*

$$F_p \mathcal{M} := \sum_{q \geqslant 0} (F_{p-q} \mathcal{M}_{\leqslant \mathbf{0}}) \cdot F_q \mathcal{D}_X.$$

Then

(1) $(\mathcal{M}, F_\bullet \mathcal{M})$ *is* $\mathbb{R}$*-specializable along each* D_i*, and for* $\boldsymbol{\alpha} \in [-1,0]^n$*,*

$$F_p V^{(\boldsymbol{n})}_{\boldsymbol{\alpha}} \mathcal{M}_{\leqslant \mathbf{0}} := F_p \mathcal{M}_{\leqslant \mathbf{0}} \cap V^{(\boldsymbol{n})}_{\boldsymbol{\alpha}} \mathcal{M}_{\leqslant \mathbf{0}} = F_p \mathcal{M} \cap V^{(\boldsymbol{n})}_{\boldsymbol{\alpha}} \mathcal{M}_{\leqslant \mathbf{0}},$$

(2) *and* $(F_\bullet \mathcal{M}, V^{(1)}_\bullet \mathcal{M}, \dots, V^{(n)}_\bullet \mathcal{M})$ *are compatible filtrations.*

Before entering the proof of Proposition 15.9.14, let us emphasize a useful criterion for $\mathbb{R}$-specializability.

15.9.15. Corollary. *Let $(\mathcal{M}, F_\bullet\mathcal{M})$ be a coherently F-filtered $\mathcal{D}_X$-module. Assume that*

- *$\mathcal{M}$ is of normal crossing type along D,*
- *$F_p\mathcal{M} := \sum_{q\geqslant 0}(F_{p-q}\mathcal{M}_{\leqslant \mathbf{0}})\cdot F_q\mathcal{D}_X$,*
- *$(F_\bullet\mathcal{M}, V_\bullet^{(1)}\mathcal{M}, \dots, V_\bullet^{(n)}\mathcal{M})$ are compatible filtrations.*

Then $(\mathcal{M}, F_\bullet\mathcal{M})$ is of normal crossing type along D if and only if $x_iF_pV_{a_i}^{(i)}\mathcal{M}_{\leqslant\mathbf{0}} = F_pV_{a_i-1}^{(i)}\mathcal{M}_{\leqslant\mathbf{0}}$ for every i and $a_i<0$.

Proof. The condition is necessary by definition. Let us show it is sufficient. The coherent F-filtration $F_\bullet\mathcal{M}$ induces a coherent F-filtration $F_\bullet\mathcal{M}_{\leqslant\mathbf{0}}$ and the family of induced filtrations $(F_\bullet\mathcal{M}_{\leqslant\mathbf{0}}, V_\bullet^{(1)}\mathcal{M}_{\leqslant\mathbf{0}}, \dots, V_\bullet^{(n)}\mathcal{M}_{\leqslant\mathbf{0}})$ on $\mathcal{M}_{\leqslant\mathbf{0}}$ remains distributive. The assumptions of Proposition 15.9.14 are thus satisfied and the conclusion follows. □

15.9.16. Remark. We can replace the above condition with the condition that $x_iF_pV_{a_i}^{(i)}\mathcal{M} = F_pV_{a_i-1}^{(i)}\mathcal{M}$ for every i and $a_i<0$. Indeed, the main point in Proposition 15.9.14 concerns the behaviour of ∂_{x_i}, and the latter property is obtained as a consequence of the condition in the corollary, which is not used otherwise, so we may as well assume this property.

***Proof of Proposition* 15.9.14.** For every $\boldsymbol{a}\in\mathbb{R}^n$, there is a natural way to define a filtration on $V_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}$ from that on $\mathcal{M}_{\leqslant\mathbf{0}}$ by refining the formula for $F_p\mathcal{M}$ and setting

$$G_p(V_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}) := \sum_{\substack{\boldsymbol{c}\leqslant 0,\,\boldsymbol{j}\geqslant 0\\ \boldsymbol{c}+\boldsymbol{j}\leqslant\boldsymbol{a}}} F_{p-|\boldsymbol{j}|}V_{\boldsymbol{c}}^{(\boldsymbol{n})}\mathcal{M}\cdot\partial_x^{\boldsymbol{j}}. \tag{15.9.17}$$

For example, this formula yields $G_p(V_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}) = F_pV_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}$ if $\boldsymbol{a}\leqslant 0$, i.e., $a_i\leqslant 0$ for all i. Similarly, if $\boldsymbol{a}'' = (a_i)_{i|a_i>0}$ denotes the "positive part" of $\boldsymbol{a}$ and $\boldsymbol{a}'$ the non-positive part, we have, with obvious notation,

$$G_p(V_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}) = \sum_{\substack{\boldsymbol{c}''\leqslant 0,\,\boldsymbol{j}''\geqslant 0\\ \boldsymbol{c}''+\boldsymbol{j}''\leqslant\boldsymbol{a}''}} F_{p-|\boldsymbol{j}''|}V_{(\boldsymbol{a}',\boldsymbol{c}'')}^{(\boldsymbol{n})}\mathcal{M}\cdot\partial_{x''}^{\boldsymbol{j}''}. \tag{15.9.18}$$

As a consequence, if $a_i\leqslant 0$, we find the relation

$$G_p(V_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M})\cdot x_i = G_p(V_{\boldsymbol{a}-\mathbf{1}_i}^{(\boldsymbol{n})}\mathcal{M}) \tag{15.9.19}$$

and, if $a_i>-1$,

$$G_{p+1}(V_{\boldsymbol{a}+\mathbf{1}_i}^{(\boldsymbol{n})}\mathcal{M}) = G_p(V_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M})\cdot\partial_{x_i} + G_p(V_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}). \tag{15.9.20}$$

We also note that

$$\varinjlim_{\boldsymbol{a}} G_p(V_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}) = \sum_{\boldsymbol{c}\leqslant 0,\,\boldsymbol{j}\geqslant 0} F_{p-|\boldsymbol{j}|}V_{\boldsymbol{c}}^{(\boldsymbol{n})}\mathcal{M}\cdot\partial_x^{\boldsymbol{j}} = \sum_{\boldsymbol{j}\geqslant 0} F_{p-|\boldsymbol{j}|}\mathcal{M}_{\leqslant\mathbf{0}}\cdot\partial_x^{\boldsymbol{j}} =: F_p\mathcal{M}.$$

We set $V_\bullet^{(i)}V_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M} = V_\bullet^{(i)}\mathcal{M}\cap V_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}$. We will prove the following properties under the assumptions in the proposition.

(a) Let $\boldsymbol{b} < \boldsymbol{a}$ (i.e., $b_i \leqslant a_i$ for all i and $\boldsymbol{b} \neq \boldsymbol{a}$). Then $G_p(V_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}) \cap V_{\boldsymbol{b}}^{(\boldsymbol{n})}\mathcal{M} = G_p(V_{\boldsymbol{b}}^{(\boldsymbol{n})}\mathcal{M})$.

(b) $(G_\bullet(V_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}), V_\bullet^{(1)}V_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}, \dots, V_\bullet^{(n)}V_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M})$ are compatible filtrations,

(c) the following inclusion is $(n+1)$-strict for $\boldsymbol{b} \leqslant \boldsymbol{a}$:

$$(V_{\boldsymbol{b}}^{(\boldsymbol{n})}\mathcal{M}, G_\bullet(V_{\boldsymbol{b}}^{(\boldsymbol{n})}\mathcal{M}), (V_\bullet^{(i)}V_{\boldsymbol{b}}^{(\boldsymbol{n})}\mathcal{M})_{i\in I}) \longrightarrow (V_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}, G_\bullet(V_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}), (V_\bullet^{(i)}V_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M})_{i\in I}).$$

Let us indicate how to obtain the proposition from (a)–(c). $\mathbb{R}$-specializability of $(\mathcal{M}, F_\bullet\mathcal{M})$ along D_i amounts to

$$\begin{cases}(F_p\mathcal{M} \cap V_{b_i}^{(i)}\mathcal{M}) \cdot x_i = F_p\mathcal{M} \cap V_{b_i-1}^{(i)}\mathcal{M}) & \text{if } b_i \leqslant 0,\\ F_{p+1}\mathcal{M} \cap V_{b_i+1}^{(i)}\mathcal{M} \subset (F_p\mathcal{M} \cap V_{b_i}^{(i)}\mathcal{M}) \cdot \partial_{x_i} + V_{<b_i+1}^{(i)}\mathcal{M}) & \text{if } b_i > -1.\end{cases}$$

By taking inductive limit on $\boldsymbol{a} > 0$ in (a), we obtain

$$F_p\mathcal{M} \cap V_{\boldsymbol{b}}^{(\boldsymbol{n})}\mathcal{M} = G_p(V_{\boldsymbol{b}}^{(\boldsymbol{n})}\mathcal{M})$$

for every $\boldsymbol{b}$. From (15.9.19) and (15.9.20), and by taking inductive limit $b_k \to \infty$ for any $k \neq i$, we obtain that the both properties are fulfilled. The other assertions in 15.9.14 are also obtained by taking the inductive limit on $\boldsymbol{a}$. We also note that (a) and (b) for $\boldsymbol{a}$ imply (c) for $\boldsymbol{a}$, according to Example 15.3.3. Conversely, (c) for $\boldsymbol{a}$ implies (a) for $\boldsymbol{a}$.

We will prove (a) and (b) by induction on the lexicographically ordered pair $(n, m, \boldsymbol{a})$ with $m = |\boldsymbol{a}''|$. Let us first exemplify the proof of (a) and (b) in the case $n = 1$. Condition (b) is empty. For (a), we can assume $a > 0$, and it is enough, by an easy induction on $a - b$, to prove $G_p(V_a^{(1)}\mathcal{M}) \cap V_{<a}^{(1)}\mathcal{M} = G_p(V_{<a}^{(1)}\mathcal{M})$. For that purpose, we notice that (15.9.20) yields

$$G_p(V_a^{(1)}\mathcal{M}) = G_p(V_{<a}^{(1)}\mathcal{M}) + F_{p-k}V_\alpha^{(1)}\mathcal{M} \cdot \partial_{x_1}^k,$$

where $k \in \mathbb{N}$ is such that $\alpha := a - k \in (-1, 0]$. Hence

$$G_p(V_a^{(1)}\mathcal{M}) \cap V_{<a}^{(1)}\mathcal{M} = G_p(V_{<a}^{(1)}\mathcal{M}) + \big(F_{p-k}V_\alpha^{(1)}\mathcal{M} \cdot \partial_{x_1}^k \cap V_{<a}^{(1)}\mathcal{M}\big).$$

Since $\partial_x^k : \mathrm{gr}_\alpha^V\mathcal{M} \to \mathrm{gr}_a^V\mathcal{M}$ is injective (in fact, an isomorphism), we have the equality

$$(F_{p-j}V_\alpha^{(1)}\mathcal{M}) \cdot \partial_{x_1}^k \cap V_{<a}^{(1)}\mathcal{M} = (F_{p-j}V_{<\alpha}^{(1)}\mathcal{M}) \cdot \partial_{x_1}^k,$$

so we obtain (a) in this case.

We now assume $n \geqslant 2$. Moreover, if $|\boldsymbol{a}''| = 0$, i.e., if $\boldsymbol{a} \leqslant 0$, there is nothing to prove. For induction purpose (on n), let us make precise how the filtration G_p behaves under taking $\mathrm{gr}_{a_i}^{V^{(i)}}$. Let us fix $i \in I$ and let us set

$$\widehat{\boldsymbol{a}} = (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n), \quad \widehat{\boldsymbol{n}} = (1, \dots, i-1, i+1, \dots, n), \quad \mathcal{M}^{(a_i)} = \mathrm{gr}_{a_i}^{V^{(i)}}\mathcal{M},$$

the latter being a $\mathcal{D}_{D_i}$-module of normal crossing type, with the induced filtrations $(V_\bullet^{(j)})_{j\neq i}$. We set $\mathcal{M}_0^{(a_i)} = V_{\boldsymbol{0}}^{(\widehat{\boldsymbol{n}})}\mathcal{M}^{(a_i)}$, that we equip with the naturally induced filtrations $(V_\bullet^{(j)}\mathcal{M}_0^{(a_i)})_{j\neq i}$.

If $a_i \leqslant 0$, we also equip it with the induced filtration $F_\bullet \mathcal{M}_0^{(a_i)}$. In such a case, by Remark 15.1.8(1), the family $(F_\bullet \mathcal{M}_0^{(a_i)}, (V_\bullet^{(j)} \mathcal{M}_0^{(a_i)})_{j\neq i})$ is distributive. We can thus consider the filtration $G_p(V_{\widehat{\boldsymbol{a}}}^{(\widehat{\boldsymbol{n}})}(\mathcal{M}^{(a_i)})$.

Assume now $a_i > 0$. We can produce an F-filtration $F_p\mathcal{M}_0^{(a_i)}$ in two ways: either by inducing $G_p(V_{\widehat{\boldsymbol{0}},a_i}^{(\boldsymbol{n})}\mathcal{M})$ on $\mathcal{M}_0^{(a_i)} = V_{\widehat{\boldsymbol{0}},a_i}^{(\boldsymbol{n})}\mathcal{M}/V_{\widehat{\boldsymbol{0}},<a_i}^{(\boldsymbol{n})}\mathcal{M}$ (by distributivity of the family $(V_\bullet^{(j)}\mathcal{M})_{j\in I})$ or, setting $a_i = \alpha_i + k_i$ with $\alpha_i \in (-1,0]$ and $k_i \in \mathbb{N}$, by considering the image of $F_{p-k_i}\mathcal{M}_0^{(\alpha_i)}$ by the isomorphism $\partial_{x_i}^{k_i} : \mathcal{M}_0^{(\alpha_i)} \xrightarrow{\sim} \mathcal{M}_0^{(a_i)}$ (once more by distributivity). We claim that both filtrations coincide: indeed, we have by definition

$$G_p(V_{\widehat{\boldsymbol{0}},a_i}^{(\boldsymbol{n})}\mathcal{M}) = \sum_{\substack{c_i\leqslant 0,\, j_i\geqslant 0\\ c_i+j_i\leqslant a_i}} F_{p-j_i}(V_{\widehat{\boldsymbol{0}},c_i}^{(\boldsymbol{n})}\mathcal{M})\cdot \partial_{x_i}^{j_i},$$

which implies

$$G_p(V_{\widehat{\boldsymbol{0}},a_i}^{(\boldsymbol{n})}\mathcal{M}) + V_{\widehat{\boldsymbol{0}},<a_i}^{(\boldsymbol{n})}\mathcal{M} = F_{p-k_i}(V_{\widehat{\boldsymbol{0}},\alpha_i}^{(\boldsymbol{n})}\mathcal{M})\cdot \partial_{x_i}^{k_i} + V_{\widehat{\boldsymbol{0}},<a_i}^{(\boldsymbol{n})}\mathcal{M},$$

as desired. By the second definition, the family $(F_\bullet\mathcal{M}_0^{(a_i)}, (V_\bullet^{(j)}\mathcal{M}_0^{(a_i)})_{j\neq i})$, which is the image by the isomorphism $\partial_{x_i}^{k_i}$ of the family $(F_\bullet\mathcal{M}_0^{(\alpha_i)}, (V_\bullet^{(j)}\mathcal{M}_0^{(\alpha_i)})_{j\neq i})$, is distributive.

For any $\widehat{\boldsymbol{a}}$, we can produce the filtration $G_p(V_{\widehat{\boldsymbol{a}}}^{(\widehat{\boldsymbol{n}})}\mathcal{M}^{(a_i)})$ by a formula similar to (15.9.18):

$$G_p(V_{\widehat{\boldsymbol{a}}}^{(\widehat{\boldsymbol{n}})}(\mathcal{M}^{(a_i)}) = \sum_{\substack{\widehat{\boldsymbol{c}}''\leqslant 0,\, \widehat{\boldsymbol{j}}''\geqslant 0\\ \widehat{\boldsymbol{c}}''+\widehat{\boldsymbol{j}}''\leqslant \widehat{\boldsymbol{a}}''}} F_{p-|\widehat{\boldsymbol{j}}''|}V_{(\widehat{\boldsymbol{a}}',\widehat{\boldsymbol{c}}'')}^{(\widehat{\boldsymbol{n}})}\mathcal{M}^{(a_i)}\cdot \partial_{x''}^{\widehat{\boldsymbol{j}}''}.$$

This filtration is the image by the isomorphism $\partial_{x_i}^{k_i}$, of $G_{p-k_i}(V_{\widehat{\boldsymbol{a}}}^{(\widehat{\boldsymbol{n}})}\mathcal{M}^{(\alpha_i)})$.

15.9.21. Lemma. *For any a_i, the filtration $G_p(V_{\widehat{\boldsymbol{a}}}^{(\widehat{\boldsymbol{n}})}(\mathcal{M}^{(a_i)})$ is the image of $G_p(V_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M})$ by the natural morphism $V_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M} \to V_{\widehat{\boldsymbol{a}}}^{(\widehat{\boldsymbol{n}})}\mathcal{M}^{(a_i)} = V_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}/V_{\widehat{\boldsymbol{a}},<a_i}^{(\boldsymbol{n})}\mathcal{M}$.*

Proof. Assume first that $a_i \leqslant 0$. By the distributivity assumption in the proposition, the $\mathcal{O}_X$-module $F_{p-|\boldsymbol{j}''|}V_{(\boldsymbol{a}',\boldsymbol{c}'')}^{(\boldsymbol{n})}\mathcal{M}$ induces $F_{p-|\boldsymbol{j}''|}V_{(\widehat{\boldsymbol{a}}',\boldsymbol{c}'')}^{(\boldsymbol{n}')}\mathcal{M}^{(a_i)}$, which implies that $G_p(V_{\widehat{\boldsymbol{a}}}^{(\widehat{\boldsymbol{n}})}\mathcal{M}^{(a_i)})$ is the filtration induced by $G_p(V_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M})$ on $\mathcal{M}^{(a_i)}$, since ∂_{x_i} does not occur in (15.9.18).

If $a_i > 0$, both filtrations considered in the lemma are the images by the isomorphism $\partial_{x_i}^{k_i}$ of the corresponding filtrations with a_i replaced by α_i: we have noted this property just above for the first one, and the property for the second one follows from (15.9.20). Since the latter coincide, according to the first part of the proof, so do the former. $\square$

We now fix $(n, m, \boldsymbol{a})$ with $m = |\boldsymbol{a}''| \geqslant 1$, and we assume that (a)–(c) holds for strictly smaller triples.

In order to prove (a), we can argue by decreasing induction on $\boldsymbol{b}$ with $\boldsymbol{b} < \boldsymbol{a}$, and we are reduced to the case where $\boldsymbol{b}$ is the predecessor in one direction, say 1, of $\boldsymbol{a}$, that is, $b_i = a_i$ for $i \neq 1$ and b_1 is the predecessor of a_1.

• Assume first that $a_1 > 0$. We will set $a_1 = \alpha_1 + k_1$, with $\alpha_1 \in (-1, 0]$ and $k_1 \in \mathbb{N}$. We then have

$$G_p(V^{(\boldsymbol{n})}_{\boldsymbol{a}}\mathcal{M}) = G_{p-1}(V^{(\boldsymbol{n})}_{\boldsymbol{a}-\mathbf{1}_1}\mathcal{M}) \cdot \partial_{x_1} + G_p(V^{(\boldsymbol{n})}_{\boldsymbol{b}}\mathcal{M}),$$

and we are reduced to proving

$$G_{p-1}(V^{(\boldsymbol{n})}_{\boldsymbol{a}-\mathbf{1}_1}\mathcal{M}) \cdot \partial_{x_1} \cap V^{(\boldsymbol{n})}_{\boldsymbol{b}}\mathcal{M} \subset G_p(V^{(\boldsymbol{n})}_{\boldsymbol{b}}\mathcal{M}).$$

Since $a_1 > 0$ and $\mathcal{M}$ is of normal crossing type, the morphism

$$\partial_{x_1} : V^{(\widehat{\boldsymbol{n}})}_{\widehat{\boldsymbol{a}}-\mathbf{1}_1}\mathcal{M}/V^{(\widehat{\boldsymbol{n}})}_{\widehat{\boldsymbol{b}}-\mathbf{1}_1}\mathcal{M} \xrightarrow{\sim} V^{(\widehat{\boldsymbol{n}})}_{\widehat{\boldsymbol{a}}}\mathcal{M}/V^{(\widehat{\boldsymbol{n}})}_{\widehat{\boldsymbol{b}}}\mathcal{M}$$

is injective, so that

$$G_{p-1}(V^{(\boldsymbol{n})}_{\boldsymbol{a}-\mathbf{1}_i}\mathcal{M}) \cdot \partial_{x_i} \cap V^{(\boldsymbol{n})}_{\boldsymbol{b}}\mathcal{M} = \big[G_{p-1}(V^{(\boldsymbol{n})}_{\boldsymbol{a}-\mathbf{1}_i}\mathcal{M}) \cap V^{(\boldsymbol{n})}_{\boldsymbol{b}-\mathbf{1}_i}\mathcal{M}\big] \cdot \partial_{x_i}.$$

By induction on $(m, \boldsymbol{a})$, the latter term is contained in $G_{p-1}(V^{(\boldsymbol{n})}_{\boldsymbol{b}-\mathbf{1}_i}\mathcal{M}) \cdot \partial_{x_i}$, hence in $G_p(V^{(\boldsymbol{n})}_{\boldsymbol{b}}\mathcal{M})$.

• Let us now assume that $a_1 \leqslant 0$. Since $|\boldsymbol{a}''| \geqslant 1$, there exists an index, say $i \neq 1$, such that $a_i > 0$. To prove $G_p(V^{(\boldsymbol{n})}_{\boldsymbol{a}}\mathcal{M}) \cap V^{(\boldsymbol{n})}_{\boldsymbol{b}}\mathcal{M} = G_p(V^{(\boldsymbol{n})}_{\boldsymbol{b}}\mathcal{M})$ for all p, it is enough to prove $G_p(V^{(\boldsymbol{n})}_{\boldsymbol{a}}\mathcal{M}) \cap G_{p+1}(V^{(\boldsymbol{n})}_{\boldsymbol{b}}\mathcal{M}) = G_p(V^{(\boldsymbol{n})}_{\boldsymbol{b}}\mathcal{M})$ for all p, and (replacing p with $p-1$), this amounts to proving for all p the injectivity of

$$\operatorname{gr}^G_p V^{(\boldsymbol{n})}_{\boldsymbol{b}}\mathcal{M} \longrightarrow \operatorname{gr}^G_p V^{(\boldsymbol{n})}_{\boldsymbol{a}}\mathcal{M}.$$

Set $\boldsymbol{a} = (a_1, \dots, a_n) = (\widehat{\boldsymbol{a}}, a_i)$, and $\boldsymbol{b} = ({<}a_1, a_2, \dots, a_{n-1}, a_n) = (\widehat{\boldsymbol{b}}, a_i)$. We will also consider $(\widehat{\boldsymbol{a}}, {<}a_i)$ and $(\widehat{\boldsymbol{b}}, {<}a_i)$. The induction hypothesis on n implies that(a)–(c) hold for $V^{(\widehat{\boldsymbol{n}})}_{\widehat{\boldsymbol{a}}}\mathcal{M}^{(a_i)}$. Note that $V^{(\widehat{\boldsymbol{n}})}_{\widehat{\boldsymbol{a}}}\mathcal{M}^{(a_i)} = V^{(\boldsymbol{n})}_{(\widehat{\boldsymbol{a}}, a_i)}\mathcal{M}/V^{(\boldsymbol{n})}_{(\widehat{\boldsymbol{a}}, <a_i)}\mathcal{M}$.

Lemma 15.9.21 provides an exact sequence

$$0 \longrightarrow G_p V^{(\boldsymbol{n})}_{(\widehat{\boldsymbol{a}}, <a_i)}\mathcal{M} \longrightarrow G_p V^{(\boldsymbol{n})}_{\boldsymbol{a}}\mathcal{M} \longrightarrow G_p V^{(\widehat{\boldsymbol{n}})}_{\widehat{\boldsymbol{a}}}\mathcal{M}^{(a_i)} \longrightarrow 0, \tag{15.9.22}$$

and a similar one with $\boldsymbol{b}$, thus a commutative diagram with horizontal exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{gr}^G_p V^{(\boldsymbol{n})}_{(\widehat{\boldsymbol{b}}, <a_i)}\mathcal{M} & \longrightarrow & \operatorname{gr}^G_p V^{(\boldsymbol{n})}_{\boldsymbol{b}}\mathcal{M} & \longrightarrow & \operatorname{gr}^G_p V^{(\widehat{\boldsymbol{n}})}_{\widehat{\boldsymbol{b}}}\mathcal{M}^{(a_i)} & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & \operatorname{gr}^G_p V^{(\boldsymbol{n})}_{(\widehat{\boldsymbol{a}}, <a_i)}\mathcal{M} & \longrightarrow & \operatorname{gr}^G_p V^{(\boldsymbol{n})}_{\boldsymbol{a}}\mathcal{M} & \longrightarrow & \operatorname{gr}^G_p V^{(\widehat{\boldsymbol{n}})}_{\widehat{\boldsymbol{a}}}\mathcal{M}^{(a_i)} & \longrightarrow & 0
\end{array}$$

By the induction hypothesis on n and $|\boldsymbol{a}''|$, both extreme vertical arrows are injective (because $|\widehat{\boldsymbol{a}}''| < |\boldsymbol{a}''|$ for the left one, and $|\widehat{\boldsymbol{n}}| < n$ for the right one). We conclude that the middle vertical arrow is injective, which finishes the proof of (a).

Let us now prove (b). We consider the exact sequence (15.9.22). The induction hypothesis implies that (b) holds for $V^{(\boldsymbol{n})}_{(\widehat{\boldsymbol{a}}, <a_i)}\mathcal{M}$ and for $V^{(\widehat{\boldsymbol{n}})}_{\widehat{\boldsymbol{a}}}\mathcal{M}^{(a_i)}$. We can apply Exercise 15.3(3a) to conclude that (b) holds for $V^{(\boldsymbol{n})}_{\boldsymbol{a}}\mathcal{M}$. □

***15.9.23. Remark* (The case of a middle extension along $D_{i \in I}$)**

Assume moreover that, in Proposition 15.9.14, $\mathcal{M}$ is a middle extension along each D_i ($i \in I$). Then we can replace everywhere $\mathcal{M}_{\leqslant \mathbf{0}}$ with $\mathcal{M}_{<\mathbf{0}} := \bigcap_{i \in I} V^{(i)}_{<0}\mathcal{M}$ and we can moreover conclude that $(\mathcal{M}, F_\bullet\mathcal{M})$ is a middle extension along $D_{i\in I}$ (Definition 15.9.10). In the proof, we modify the definition (15.9.17) of $G_p(V_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M})$ as follows: we set

$$G_p(V_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}) := \sum_{\substack{\boldsymbol{c}<0,\,\boldsymbol{j}\geqslant 0\\ \boldsymbol{c}+\boldsymbol{j}\leqslant \boldsymbol{a}}} F_{p-|\boldsymbol{j}|}V_{\boldsymbol{c}}^{(\boldsymbol{n})}\mathcal{M}\cdot\partial_x^{\boldsymbol{j}}.$$

For example, we have $G_p(V_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}) = F_pV_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}$ if $\boldsymbol{a}<0$, i.e., $a_i<0$ for all i. As another example, setting $\boldsymbol{c} = (\boldsymbol{c}', \boldsymbol{0}'')$ if $\boldsymbol{c}\leqslant\boldsymbol{0}$, with $\boldsymbol{c}'<\boldsymbol{0}'$, and correspondingly $n = n'+n''$, we have

$$G_p(V_{\boldsymbol{0}}^{(\boldsymbol{n})}\mathcal{M}) = \sum_{\substack{\boldsymbol{c}=(\boldsymbol{c}',\boldsymbol{0}'')\\ \boldsymbol{c}'<\boldsymbol{0}'}} F_{p-n''}V_{(\boldsymbol{c}',-\boldsymbol{1}'')}^{(\boldsymbol{n})}\mathcal{M}\cdot\partial_{x''}.$$

A useful example. Let $\mathcal{M}$ be a $\mathcal{D}$-module of normal crossing type which is a middle extension along each D_i ($i\in I$) and let us consider the locally free $\mathcal{O}$-module $\mathcal{M}_{<\boldsymbol{0}} = V_{<\boldsymbol{0}}^{(\boldsymbol{n})}\mathcal{M}$, equipped with the induced filtrations $V_\bullet^{(i)}\mathcal{M}_{<\boldsymbol{0}}$ (which are thus compatible). For $\boldsymbol{a}<\boldsymbol{0}$, we have

$$V_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}_{<\boldsymbol{0}} := \bigcap_i V_{a_i}^{(i)}\mathcal{M}_{<\boldsymbol{0}} = V_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}\cap\mathcal{M}_{<\boldsymbol{0}}.$$

Let $F_\bullet\mathcal{M}_{|X\smallsetminus D}$ be a coherent (finite) $\mathcal{D}$-filtration such that each $\mathrm{gr}_p^F\mathcal{M}_{|X\smallsetminus D}$ is $\mathcal{O}$-locally free and let us set

$$F_\bullet\mathcal{M}_{<\boldsymbol{0}} = j_*F_\bullet\mathcal{M}_{|X\smallsetminus D}\cap\mathcal{M}_{<\boldsymbol{0}}$$

and

$$F_p\mathcal{M} = \sum_{q\geqslant 0} F_{p-q}\mathcal{M}_{<\boldsymbol{0}}\cdot F_q\mathcal{D}_X.$$

15.9.24. Proposition. *With these assumptions, let us moreover assume that, for each p and $\boldsymbol{a}$, $\mathrm{gr}_p^F V_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}_{<\boldsymbol{0}}$ is $\mathcal{O}$-locally free and that the natural morphism*

$$F_pV_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}_{<\boldsymbol{0}} = V_{\boldsymbol{a}}^{(\boldsymbol{n})}F_p\mathcal{M}_{<\boldsymbol{0}} \longrightarrow V_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathrm{gr}_p^F\mathcal{M}_{<\boldsymbol{0}}$$

is onto. Then the filtered $\mathcal{D}$-module $(\mathcal{M}, F_\bullet\mathcal{M})$ is of normal crossing type and a middle extension along $D_{i\in I}$.

The morphism in the proposition reads

$$\bigcap_i\big(V_{a_i}^{(i)}\mathcal{M}_{<\boldsymbol{0}}\cap F_p\mathcal{M}_{<\boldsymbol{0}}\big) \longrightarrow \bigcap_i\big((V_{a_i}^{(i)}\mathcal{M}_{<\boldsymbol{0}}\cap F_p\mathcal{M}_{<\boldsymbol{0}}) + F_{p-1}\mathcal{M}_{<\boldsymbol{0}}\big)\big/ F_{p-1}\mathcal{M}_{<\boldsymbol{0}}$$

and the condition amounts to the equality

$$\bigcap_i\big(V_{a_i}^{(i)}\mathcal{M}_{<\boldsymbol{0}}\cap F_p\mathcal{M}_{<\boldsymbol{0}}\big) + F_{p-1}\mathcal{M}_{<\boldsymbol{0}} = \bigcap_i\big((V_{a_i}^{(i)}\mathcal{M}_{<\boldsymbol{0}}\cap F_p\mathcal{M}_{<\boldsymbol{0}}) + F_{p-1}\mathcal{M}_{<\boldsymbol{0}}\big).$$

Proof. We consider the filtrations $F_\bullet, V_\bullet^{(1)}, \dots, V_\bullet^{(n)}$ on $\mathcal{M}_{<\boldsymbol{0}}$. Except possibly compatibility, they satisfy the assumptions of Proposition 15.9.14 in the setting of Remark 15.9.23. We will show that they are compatible. For that purpose, we will use the criterion in term of flatness of Theorem 15.2.2, and more precisely the criterion in terms of regular sequences of Corollary 15.2.5 together with the criteria of Exercise 15.2.

We note (see proof of Proposition 15.2.10) that the second assumption is equivalent to the property that, for each $p, \boldsymbol{a}$, the natural morphism

$$\mathrm{gr}_p^F V_{\boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M}_{<\mathbf{0}} \longrightarrow V_{\boldsymbol{a}}^{(\boldsymbol{n})} \mathrm{gr}_p^F \mathcal{M}_{<\mathbf{0}}$$

is an isomorphism, and the first assertion implies that the latter is $\mathcal{O}$-locally free.

We consider the multi-Rees module $R_{FV}\mathcal{M}_{<\mathbf{0}}$, which is a $\mathbb{C}[z_0, z_1, \dots, z_n]$-module. Exercise 15.2(2b) shows that it is flat if any subsequence of $z_0, z_1, \dots, z_n$ is regular.

• If the subsequence does not contain z_o, then we apply Proposition 15.2.14 with $\mathcal{E} = F_p\mathcal{M}_{<\mathbf{0}}$ for each p. The assumption of freeness of each $\mathrm{gr}_p^F V_{\boldsymbol{a}}^{(\boldsymbol{n})} \mathcal{M}_{<\mathbf{0}}$ implies that of $V_{\boldsymbol{a}}^{(\boldsymbol{n})} F_p\mathcal{M}_{<\mathbf{0}}$, so 15.2.14(1) is satisfied. 15.2.14(2) is also satisfied according to the definition of $F_p\mathcal{M}_{<\mathbf{0}}$.

• If the subsequence contains z_o, we are considering flatness for $R_V\mathrm{gr}^F\mathcal{M}_{<\mathbf{0}}$. We apply Proposition 15.2.14 once more, now with $\mathcal{E} = \mathrm{gr}_p^F\mathcal{M}_{<\mathbf{0}}$ for each p, and freeness of each $V_{\boldsymbol{a}}^{(\boldsymbol{n})}\mathrm{gr}_p^F\mathcal{M}_{<\mathbf{0}}$ implies that 15.2.14(1) is satisfied. Similarly, 15.2.14(2) is also satisfied according to the definition of $F_p\mathcal{M}_{<\mathbf{0}}$. □

15.10. Exercises

Exercise 15.7. Let M be a monodromic A_n-module. Show that $x_i : M_{\boldsymbol{a}} \to M_{\boldsymbol{a}-\mathbf{1}_i}$ is an isomorphism if $a_i < 0$ and $\partial_{x_i} : M_{\boldsymbol{a}} \to M_{\boldsymbol{a}+\mathbf{1}_i}$ is an isomorphism if $a_i > -1$.

Exercise 15.8. Without the simplifying assumption 15.6.2, show that a monodromic A_n-module is of finite type over $\mathbb{C}[x]\langle\partial_x\rangle$. Moreover, show that $V_{\boldsymbol{b}}^{(\boldsymbol{n})}M := \bigoplus_{\boldsymbol{a}\leqslant\boldsymbol{b}} M_{\boldsymbol{a}}$ is a $\mathbb{C}[x]\langle x\partial_x\rangle$-module which is of finite type over $\mathbb{C}[x]$, and $\mathbb{C}[x]$-free if $b_i < 0$ for all $i \in I$. Extend similarly all results of Proposition 15.7.3.

Exercise 15.9. Let $i_o \in I$ and let $M_{\boldsymbol{\alpha}+\mathbb{Z}^n}$ be a monodromic A_n-module with the single exponent $\boldsymbol{\alpha} \in [-1, 0)^n$.

(1) Show that $M_{\boldsymbol{\alpha}+\mathbb{Z}^n}$ is supported on D_{i_o} if and only if $\alpha_{i_o} = -1$ and, for $\boldsymbol{k} \in \mathbb{Z}^n$, $M_{\boldsymbol{\alpha}+\boldsymbol{k}} = 0$ if $k_{i_o} \leqslant 0$, that is, if and only if $i_o \in I(\boldsymbol{\alpha})$ and, setting $\boldsymbol{k} = (\boldsymbol{k}', k_{i_o})$, every vertex $M_{\boldsymbol{\alpha}+(\boldsymbol{k}',0)}$ of the quiver of $M_{\boldsymbol{\alpha}+\mathbb{Z}^n}$ is zero.

(2) Show that $M_{\boldsymbol{\alpha}+\mathbb{Z}^n} = M_{\boldsymbol{\alpha}+\mathbb{Z}^n}(*D_{i_o})$, i.e., x_{i_o} acts in a bijective way on $M_{\boldsymbol{\alpha}+\mathbb{Z}^n}$, if and only if $i_o \notin I(\boldsymbol{\alpha})$ or $i_o \in I(\boldsymbol{\alpha})$ and var_{i_o} is an isomorphism.

(3) Show that the quiver of $M_{\boldsymbol{\alpha}+\mathbb{Z}^n}(*D_{i_o})$ is that of $M_{\boldsymbol{\alpha}+\mathbb{Z}^n}$ if $i_o \notin I(\boldsymbol{\alpha})$ and, otherwise, setting $\boldsymbol{k} = (\boldsymbol{k}', k_{i_o})$, is isomorphic to the quiver is obtained from that of $M_{\boldsymbol{\alpha}+\mathbb{Z}^n}$ by replacing $M_{\boldsymbol{\alpha}+(\boldsymbol{k}',0)}$ with $M_{\boldsymbol{\alpha}+(\boldsymbol{k}',-1)}$, var_{i_o} with Id and can_{i_o} with N_{i_o}.

Let now M be any monodromic A_n-module, and consider its quiver as in Remark 15.7.6.

(4) Show that M is supported on D_{i_o} if and only if, for any exponent $\boldsymbol{\alpha} \in [-1, 0)^n$, we have $\alpha_{i_o} = -1$ and every vertex of the quiver with index $\boldsymbol{k} \in \{0, 1\}^n$ satisfying $k_{i_o} = 0$ vanishes.

(5) Show that $M = M(*D_{i_o})$ if and only if var_{i_o} is bijective.

Exercise 15.10. Define the endofunctors $(!D_{i_o}), (!{*}D_{i_o})$ of the category of monodromic A_n-modules in such a way that the quiver of $M_{\boldsymbol{\alpha}+\mathbb{Z}^n}(!D_{i_o})$, resp. $M_{\boldsymbol{\alpha}+\mathbb{Z}^n}(!{*}D_{i_o})$ is that of $M_{\boldsymbol{\alpha}+\mathbb{Z}^n}$ if $i_o \notin I(\boldsymbol{\alpha})$ and, otherwise, setting $\boldsymbol{k} = (\boldsymbol{k}', k_{i_o})$, the quiver is obtained from that of $M_{\boldsymbol{\alpha}+\mathbb{Z}^n}$ by replacing

- $M_{\boldsymbol{\alpha}+(\boldsymbol{k}',-1)}$ with $M_{\boldsymbol{\alpha}+(\boldsymbol{k}',0)}$, var_{i_o} with N_{i_o} and can_{i_o} with Id,
- resp. $M_{\boldsymbol{\alpha}+(\boldsymbol{k}',0)}$ with $\mathrm{image}[\mathrm{N}_{i_o} : M_{\boldsymbol{\alpha}+(\boldsymbol{k}',-1)} \to M_{\boldsymbol{\alpha}+(\boldsymbol{k}',-1)}]$, var_{i_o} with the natural inclusion and can_{i_o} with N_{i_o}.

Show that there is a natural morphism $M(!D_{i_o}) \to M({*}D_{i_o})$ whose image is $M(!{*}D_{i_o})$.

Exercise 15.11. Say that M is a *middle extension along $D_{i\in I}$ with support in D* if, for each $i \in I$, either the source of can_i is zero, or can_i is onto and var_i is injective. In other words, we accept A_n-modules supported on the intersection of some components of D, which are middle extension along any of the other components.

Show that any monodromic A_n-module M is a successive extension of such A_n-modules which are middle extensions along $D_{i\in I}$ with support in D.

Exercise 15.12 **(Proof of Proposition 15.7.13**(8)**).** Show in detail the statement of this proposition.

Exercise 15.13. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module of normal crossing type along $D_{i\in I}$. Show that $\mathcal{M}$ is a successive extension of $\mathcal{D}_X$-modules of normal crossing type along $D_{i\in I}$, each of which being moreover a middle extension along $D_{i\in I}$ with support in D. [*Hint*: Use Exercise 15.11.]

Exercise 15.14. Assume that $\mathcal{M}$ is of normal crossing type along $D_{i\in I}$. Let us fix $i \in I$ and $\boldsymbol{a} = (\widehat{\boldsymbol{a}}, a_i)$. Show that, for every $\widehat{\boldsymbol{a}} \in \mathbb{R}^{n-1}$, each of the following properties

$$\begin{aligned}
&\mathrm{can}_i : V^{(\widehat{\boldsymbol{n}})}_{\widehat{\boldsymbol{a}}}\mathrm{gr}^{V^{(i)}}_{-1}\mathcal{M} \longrightarrow V^{(\widehat{\boldsymbol{n}})}_{\widehat{\boldsymbol{a}}}\mathrm{gr}^{V^{(i)}}_{0}\mathcal{M} \quad \text{is onto, resp. bijective,}\\
&\mathrm{var}_i : V^{(\widehat{\boldsymbol{n}})}_{\widehat{\boldsymbol{a}}}\mathrm{gr}^{V^{(i)}}_{0}\mathcal{M} \longrightarrow V^{(\widehat{\boldsymbol{n}})}_{\widehat{\boldsymbol{a}}}\mathrm{gr}^{V^{(i)}}_{-1}\mathcal{M} \quad \text{is injective, resp. bijective,}
\end{aligned}$$

holds as soon as it holds when omitting $V^{(\widehat{\boldsymbol{n}})}_{\widehat{\boldsymbol{a}}}$. [*Hint*: Work first with the monodromic M; show that the morphism $x_i : \mathrm{gr}^{V^{(i)}}_0 M \to \mathrm{gr}^{V^{(i)}}_{-1} M$ decomposes as the direct sum of morphisms $x_i : M_{(\widehat{\boldsymbol{a}},0)} \to M_{(\widehat{\boldsymbol{a}},-1)}$, and similarly for ∂_{x_i}; conclude that var_i is injective (resp. bijective) or can_i is surjective (resp. bijective) if and only if each $\widehat{\boldsymbol{a}}$-component is so; conclude for $\mathcal{M}$ by flat tensorization.]

CHAPTER 15

$\widetilde{\mathcal{D}}$-MODULES OF NORMAL CROSSING TYPE
PART 3: NEARBY CYCLES ALONG A MONOMIAL FUNCTION

Summary. In this part, we compute the nearby cycles of a filtered holonomic $\mathcal{D}_X$-module of normal crossing type along a monomial function. As in Part 2, the case of a monodromic $\mathcal{D}_X$-module is simpler, while not straightforward, and we will be able to give an explicit expression of the monodromic decomposition of nearby cycles in this case, together with the behavior of a sesquilinear pairing. The case of $\mathcal{D}_X$-modules of normal crossing type is obtained by analytification, while the case of filtered $\mathcal{D}_X$-modules of normal crossing type needs more care, as the behavior of the compatibility property of filtrations after taking nearby cycles is delicate.

15.11. Introduction

Let $(\mathcal{M}, F_\bullet\mathcal{M})$ be a coherently filtered $\mathcal{D}_X$-module which is of normal crossing type along a normal crossing divisor D. Our main objective in this part is to analyze the nearby cycles of such a filtered $\mathcal{D}$-module along a monomial function $g = x^{\boldsymbol{e}}$ (with respect to coordinates adapted to D) in a way similar to that of Proposition 15.9.4, where the function g is a coordinate. It is stated as follows, where the still undefined notions will be explained with details below.

15.11.1. Theorem **(Strict $\mathbb{R}$-specializability and normal crossing type)**
Let $(\mathcal{M}, F_\bullet\mathcal{M})$ be a coherently F-filtered $\mathcal{D}_X$-module of normal crossing type along D. Assume that $(\mathcal{M}, F_\bullet\mathcal{M})$ is a middle extension along $D_{i\in I}$ (Definition 15.9.10*). Then $(\mathcal{M}, F_\bullet\mathcal{M})$ is $\mathbb{R}$-specializable and a middle extension along (g). Moreover, for every $\lambda \in \mathbb{S}^1$, $(\psi_{g,\lambda}\mathcal{M}, F_\bullet\psi_{g,\lambda}\mathcal{M})$ is of normal crossing type along D.*

A special case of this theorem has already been proved in Section 9.9.c (Proposition 9.9.12) and used in the proof of Theorem 14.6.1 showing that polarizable variations of Hodge structure are polarizable Hodge modules. In turn, Theorem 15.11.1 will be one of the ingredients in the proof of the structure theorem 16.2.1 in Chapter 16.

This theorem also clarifies the relation between the notion of middle extension along $D_{i\in I}$ and middle extension along D in the filtered setting. Indeed, by taking for g a reduced equation of D, we obtain:

***15.11.2. Corollary* (Middle extension and localizability).** *Under the assumptions of Theorem* 15.11.1*,* $(\mathcal{M}, F_\bullet\mathcal{M})$ *is a middle extension along* D*.* □

It can be noticed that Theorem 15.11.1 extends in an obvious way to triples of normal crossing type along D, according to Definition 15.8.1.

15.11.3. Notation. We keep the notation 15.6.1, so that $D = \{x_1 \cdots x_\ell = 0\}$. We also keep the simplifying assumption 15.6.2, so that $\ell = n$. Given $g(x) = x^{\boldsymbol{e}} := \prod_{i\in I} x_i^{e_i}$ ($e_i \in \mathbb{N}$), the indices for which $e_i = 0$ do not play an important role. Let us denote by

$$I_{\boldsymbol{e}} := \{i \mid e_i \neq 0\} \subset \{1, \dots, n\}$$

the subset of relevant indices and $r = \#I_{\boldsymbol{e}}$. Accordingly, we decompose the set of variables $(x_1, \dots, x_n)$ as (x', x''), with $x' = (x_i)_{i\in I_{\boldsymbol{e}}}$. We rename the indices so that

$$I_{\boldsymbol{e}} = \{1, \dots, r\},$$

with $1 \leqslant r \leqslant n$. We decompose correspondingly X as $X = X' \times X''$. We set

$$\delta_j = \frac{x_j\partial_{x_j}}{e_j} - \frac{x_1\partial_{x_1}}{e_1}, \quad j = 2, \dots, r, \text{ i.e., } j \in I_{\boldsymbol{e}} \smallsetminus \{1\}.$$

We denote by ι_g the graph inclusion $x \mapsto (x, t = g(x))$, and we consider the pushforward filtered $\mathcal{D}$-module $(\mathcal{M}_g, F_\bullet\mathcal{M}_g) = {}_{\mathrm{D}}\iota_{g*}(\mathcal{M}, F_\bullet\mathcal{M})$ (see Example 8.7.7). We write $\mathcal{M}_g = \iota_{g*}\mathcal{M}\otimes_{\mathbb{C}}\mathbb{C}[\partial_t]$ with the action of $\mathcal{D}_{X\times\mathbb{C}}$ defined as follows, according to (8.7.7 ∗):

$$(15.11.3\,*)\qquad \begin{aligned}(m\otimes\partial_t^\ell)\cdot\partial_t &= m\otimes\partial_t^{\ell+1}\\ (m\otimes 1)\cdot\partial_{x_i} &= m\partial_{x_i}\otimes 1 - (e_i m x^{\boldsymbol{e}-\boldsymbol{1}_i})\otimes\partial_t\\ (m\otimes 1)\cdot f(x,t) &= mf(x, x^{\boldsymbol{e}})\otimes 1.\end{aligned}$$

As a consequence, for $i \in \{1, \dots, r\}$ we have

$$(15.11.3\,**)\qquad (m\otimes 1)\cdot t\partial_t = (mx^{\boldsymbol{e}}\otimes 1)\cdot\partial_t = \frac{1}{e_i}\big[(mx_i\partial_{x_i}\otimes 1) - (m\otimes 1)x_i\partial_{x_i}\big].$$

Furthermore, the F-filtration is that obtained by convolution:

$$F_p\mathcal{M}_g = \sum_{q+k=p} \iota_{g*}(F_p\mathcal{M})\otimes\partial_t^k.$$

In the following, we omit the functor ι_{g*} in the notation.

15.12. Proof of Theorem 15.11.1 omitting the F-filtration

We forget about the F-filtration in this section. We set

$$\begin{aligned} \mathcal{D}'_X &= \mathcal{O}_X\langle \partial_{x_1}, \dots, \partial_{x_r}\rangle = \mathcal{D}_{X'\times X''/X''},\\ V_0^{(\boldsymbol{r})}\mathcal{D}'_X &= \bigcap_{i=1}^r V_0^{(i)}\mathcal{D}'_X,\\ V_{\alpha\boldsymbol{e}}^{(\boldsymbol{r})}\mathcal{M} &= \bigcap_{i=1}^r V_{\alpha e_i}^{(i)}\mathcal{M} \end{aligned}$$

(see Notation 15.6.3), the latter being a $V_0^{(\boldsymbol{r})}\mathcal{D}'_X$-module.

15.12.a. $\mathbb{R}$-specializability of $\mathcal{M}$ along (g). We show that $\mathcal{M}$ is $\mathbb{R}$-specializable along (g) by making explicit the V-filtration of $\mathcal{M}_g$ along (t). In the proposition below, we regard $V_{\alpha\boldsymbol{e}}^{(\boldsymbol{r})}\mathcal{M}\otimes 1$ and $\bigoplus_k(V_{\alpha\boldsymbol{e}}^{(\boldsymbol{r})}\mathcal{M}\otimes\partial_t^k)$ as $\mathcal{O}_X$-submodules of $\mathcal{M}_g=\bigoplus_k(\mathcal{M}\otimes\partial_t^k)$.

15.12.1. Proposition **($\mathbb{R}$-specializability of $\mathcal{M}_g$ along (t)).** *The $\mathcal{D}_{X\times\mathbb{C}}$-module $\mathcal{M}_g$ is $\mathbb{R}$-specializable along (t). Furthermore, the V-filtration of $\mathcal{M}_g$ is obtained from the V-filtrations $V_\bullet^{(i)}\mathcal{M}$ by the formula*

$$(15.12.1\,*)\qquad V_\alpha\mathcal{M}_g=(V_{\alpha\boldsymbol{e}}^{(\boldsymbol{r})}\mathcal{M}\otimes 1)\cdot\mathcal{D}'_X[t\partial_t]=(V_{\alpha\boldsymbol{e}}^{(\boldsymbol{r})}\mathcal{M}\otimes 1)\cdot\mathcal{D}'_X,\quad \text{if } \alpha<0,$$

and, for $\alpha\in[-1,0)$ and $j\geqslant 1$, by the inductive formula

$$(15.12.1\,**)\qquad V_{\alpha+j}\mathcal{M}_g=V_\alpha\mathcal{M}_g\cdot\partial_t^j+V_{<\alpha+j}\mathcal{M}_g.$$

The second equality in (15.12.1 $*$) follows from the expression of the action of $t\partial_t$ deduced from Formula (15.11.3 $**$).

Proof. Let us denote by $U_\bullet\mathcal{M}_g$ the filtration defined in the proposition. We will show that $U_\bullet\mathcal{M}_g$ satisfies the characteristic properties of the V-filtration along (t).

The inclusions $U_\alpha\mathcal{M}_g\cdot t\subset U_{\alpha-1}\mathcal{M}_g$ and $U_\alpha\mathcal{M}_g\cdot\partial_t\subset U_{\alpha+1}\mathcal{M}_g$ are easily obtained for any α. Furthermore, the stability by $\mathcal{D}'_X$ is by definition, and if $i>r$, ∂_{x_i} acts on $m\otimes 1$ by $m\partial_{x_i}\otimes 1$, according to Formula (15.11.3 $*$). In other words, $U_\alpha\mathcal{M}_g$ is stable by $\mathcal{D}_{X\times\mathbb{C}/\mathbb{C}}$. All this shows in particular that $U_\alpha\mathcal{M}_g$ is a $V_0(\mathcal{D}_{X\times\mathbb{C}})$-module.

For $\alpha<0$, we have $U_\alpha\mathcal{M}_g\cdot t=U_{\alpha-1}\mathcal{M}_g$ since

$$(V_{\alpha\boldsymbol{e}}^{(\boldsymbol{r})}\mathcal{M}\otimes 1)\cdot t=V_{\alpha\boldsymbol{e}}^{(\boldsymbol{r})}\mathcal{M}x^{\boldsymbol{e}}\otimes 1=V_{(\alpha-1)\boldsymbol{e}}^{(\boldsymbol{r})}\mathcal{M}\otimes 1.$$

Furthermore, as $V_{\alpha\boldsymbol{e}}^{(\boldsymbol{r})}\mathcal{M}$ is locally finitely generated over $V_0^{(\boldsymbol{r})}\mathcal{D}_X$, it follows that $U_\alpha\mathcal{M}_g$ is locally finitely generated over $V_0(\mathcal{D}_{X\times\mathbb{C}})$, hence coherent (argue e.g. as in Exercise 8.63(5)). In order to conclude that $U_\bullet\mathcal{M}_g$ is a coherent V-filtration along (t), it remains to be proved that $\mathcal{M}_g=\bigcup_\alpha U_\alpha\mathcal{M}_g$, and so it is enough to prove that any local section of $\mathcal{M}\otimes 1$, equivalently any local section of $V_{\boldsymbol{a}}^{(\boldsymbol{r})}\mathcal{M}$ for any $\boldsymbol{a}$, belongs to some $U_\alpha\mathcal{M}_g$.

If $m\in V_{\boldsymbol{a}}^{(\boldsymbol{r})}\mathcal{M}$ for some $\boldsymbol{a}\in\mathbb{R}^r$, the middle extension property of $\mathcal{M}$ along $D_{i\in I}$ implies that m is a finite sum of terms $m_{\boldsymbol{k}}\cdot\partial_x^{\boldsymbol{k}}$ with $\boldsymbol{k}=(k_1,\dots,k_r)$, $k_i\geqslant 0$, and $m_{\boldsymbol{k}}\in V_{\boldsymbol{a}(\boldsymbol{k})}^{(\boldsymbol{r})}\mathcal{M}$ with $a_i(\boldsymbol{k})<0$ for each $i=1,\dots,r$. Therefore, there exist $\alpha<0$ such that $m_{\boldsymbol{k}} \in V^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}}\mathcal{M}$ for each $\boldsymbol{k}$. We can thus use iteratively (15.11.3 $*$) to write any local section of $V^{(\boldsymbol{r})}_{\boldsymbol{a}}\mathcal{M}\otimes 1$ as a sum of terms $(\mu_{\boldsymbol{k},\ell}\otimes 1)\cdot\partial_x^{\boldsymbol{k}}\partial_t^\ell$, where each $\mu_{\boldsymbol{k},\ell}$ belongs to $V^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}}\mathcal{M}$ for some $\alpha<0$.

It remains to be shown that $(t\partial_t-\alpha)$ is nilpotent on $\mathrm{gr}^U_\alpha\mathcal{M}_g$ if $\alpha<0$.

15.12.2. Notation. In order to distinguish between the action of $x_i\partial_{x_i}$ trivially coming from that on $\mathcal{M}$ and the action $x_i\partial_{x_i}$ on $\mathcal{M}_g$, it will be convenient to denote by D_i the first one, defined by

$$(m\otimes\partial_t^\ell)\cdot\mathrm{D}_i=(mx_i\partial_{x_i})\otimes\partial_t^\ell.$$

Then we can rewrite D_i as

$$(m\otimes\partial_t^\ell)\cdot\mathrm{D}_i=(m\otimes 1)\cdot(x_i\partial_{x_i}+e_it\partial_t)\partial_t^\ell=(m\otimes\partial_t^\ell)\cdot(x_i\partial_{x_i}+e_i(t\partial_t-\ell)),$$

a formula that can also be read

$$(m\otimes\partial_t^\ell)\cdot x_i\partial_{x_i}=(m\otimes\partial_t^\ell)\cdot(\mathrm{D}_i-e_it\partial_t+e_i\ell).\tag{15.12.3}$$

We first notice that there exists $\alpha'<\alpha$ such that, for each $i=1,\dots,n$, some power of $(\mathrm{D}_i-\alpha e_i)$ sends $(V^{(i)}_{\alpha e_i}\mathcal{M}\otimes 1)$ to $(V^{(i)}_{\alpha' e_i}\mathcal{M}\otimes 1)$. Therefore, a power of $\prod_{i\in I_{\boldsymbol{e}}}(\mathrm{D}_i-\alpha e_i)$ sends $(V^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}}\mathcal{M}\otimes 1)$ to $(V^{(\boldsymbol{r})}_{\alpha'\boldsymbol{e}}\mathcal{M}\otimes 1)$. It is thus enough to check that $\prod_{i\in I_{\boldsymbol{e}}}(\mathrm{D}_i-e_it\partial_t)$ sends $(V^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}}\mathcal{M}\otimes 1)$ into $U_{\alpha'}(\mathcal{M}_g)$ for some $\alpha'<\alpha$. We have $\alpha\boldsymbol{e}-\mathbf{1}_{I_{\boldsymbol{e}}}\leqslant\alpha'\boldsymbol{e}$ for some $\alpha'<\alpha$, so $(V^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}}\mathcal{M}\otimes 1)\cdot\prod_{i\in I_{\boldsymbol{e}}}x_i\subset(V^{(\boldsymbol{r})}_{\alpha'\boldsymbol{e}}\mathcal{M}\otimes 1)$, and thus

$$(V^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}}\mathcal{M}\otimes 1)\cdot\prod_{i\in I_{\boldsymbol{e}}}x_i\partial_{x_i}\subset(V^{(\boldsymbol{r})}_{\alpha'\boldsymbol{e}}\mathcal{M}\otimes 1)\cdot\prod_{i\in I_{\boldsymbol{e}}}\partial_{x_i}\subset U_{\alpha'}(\mathcal{M}_g).$$

Therefore, by (15.12.3),

$$(V^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}}\mathcal{M}\otimes 1)\cdot\prod_{i\in I_{\boldsymbol{e}}}(\mathrm{D}_i-e_it\partial_t)\subset U_{\alpha'}(\mathcal{M}_g).\qquad\square$$

***15.12.4. Corollary* (Middle extension property of $\mathcal{M}_g$ along (t))**
The $\mathcal{D}_{X\times\mathbb{C}}$-module $\mathcal{M}_g$ satisfies the equality $\mathcal{M}_g=\mathcal{M}_g[!*t]$.

Proof. We first remark that t acts injectively on $\mathcal{M}_g$: if we consider the filtration $G_\bullet\mathcal{M}_g$ by the degree in ∂_t, then the action of t on $\mathrm{gr}^G\mathcal{M}_g\simeq\mathcal{M}[\tau]$ is equal to the induced action of $x^{\boldsymbol{e}}$ on $\mathcal{M}[\tau]$, hence is injective by the assumption that $\mathcal{M}$ is a middle extension along $D_{i\in I}$; a fortiori, the action of t on $\mathcal{M}_g$ is injective. We thus have $\mathcal{M}_g\subset\mathcal{M}_g[*t]$. By Formula (15.12.1 $**$) and the exhaustivity of $V_\bullet\mathcal{M}_g$, $\mathcal{M}_g$ is the image of $V_{<0}\mathcal{M}_g\otimes_{V_0\mathcal{D}_{X\times\mathbb{C}}}\mathcal{D}_{X\times\mathbb{C}}$ in $\mathcal{M}_g[*t]$. This is nothing but $\mathcal{M}_g[!*t]$ (see Definition 11.5.2 and Definition 11.4.1). $\square$

15.12.b. A resolution of $V_\alpha\mathcal{M}_g$. We continue by providing a suitable presentation of $V_\alpha\mathcal{M}_g$ for $\alpha\in\mathbb{R}$, that we will later enrich with an F-filtration. The tensor product

$$\mathcal{K}^0_\alpha=V^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}}\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{D}'_X$$

has the structure of a right $V_0^{(\boldsymbol{r})}\mathcal{D}'_X$-module with the tensor structure and of a right $\mathcal{D}'_X$-module with the trivial structure. This trivial structure extends as a right

$\mathcal{D}_X$-module structure by setting $(m \otimes 1)\partial_{x_i} = m\partial_{x_i} \otimes 1$ for $i \notin I_{\boldsymbol{e}}$. Both structures commute with each other (see Exercise 8.19).

Since the operators $\cdot_{\mathrm{tens}}\delta_j$ pairwise commute ($j = 2, \dots, r$) and commute with the right $\mathcal{D}_X$-module structure, we can consider the Koszul complex

$$\mathcal{K}^\bullet_\alpha = K\big(V^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}}\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{D}'_X, (\cdot_{\mathrm{tens}}\delta_j)_{j=2,\dots,r}\big)$$

(i.e., the simple complex associated with the $(r-1)$-cube with arrows in the direction j all equal to $\cdot_{\mathrm{tens}}\delta_j$).

***15.12.5. Proposition* (A resolution of $V_\alpha\mathcal{M}_g$).** *For each $\alpha < 0$, the Koszul complex $\mathcal{K}^\bullet_\alpha$ is a resolution of $V_\alpha\mathcal{M}_g$ via the right $\mathcal{D}_X$-linear surjective morphism*

$$\begin{aligned} \mathcal{K}^0_\alpha = V^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}}\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{D}'_X &\longrightarrow V_\alpha\mathcal{M}_g \\ m \otimes P &\longmapsto (m \otimes 1) \cdot P. \end{aligned} \tag{15.12.5 $*$}$$

Beware that the tensor products on both sides of (15.12.5 $*$) do not have the same meaning.

Let J be a subset of $\{1, \dots, r\}$, let J^{c} denote its supplementary subset, and let D°_J be the stratum of D defined as $\bigcap_{i\in J} D_i \smallsetminus \bigcup_{i\in J^{\mathrm{c}}} D_i$. Let $\boldsymbol{A}_J$ denote the projection of $\boldsymbol{A}$ (see Definition 15.7.2) on the J-components and let $\boldsymbol{e}_J$ denote the J-components of $\boldsymbol{e}$.

***15.12.6. Corollary* (Jumping indices for $V_\bullet\mathcal{M}_g$ and resolution of $\mathrm{gr}^V_\alpha\mathcal{M}_g$)**

For $\alpha < 0$, $\mathrm{gr}^V_\alpha\mathcal{M}_g$ vanishes (in some neighborhood of the origin) unless there exists $i \in I_{\boldsymbol{e}} = \{1, \dots, r\}$ such that $\alpha\boldsymbol{e}_i \in \boldsymbol{A}_i + \mathbb{Z}$. Furthermore, setting $\mathcal{K}^\bullet_{<\alpha} = \mathcal{K}^\bullet_{\alpha-\varepsilon}$ for $\varepsilon > 0$ small enough, the Koszul complex

$$K\big((V^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}}\mathcal{M}/V^{(\boldsymbol{r})}_{(\alpha-\varepsilon)\boldsymbol{e}}\mathcal{M}) \otimes_{\mathcal{O}_X} \mathcal{D}'_X, (\cdot_{\mathrm{tens}}\delta_j)_{j=2,\dots,r}\big) = \mathcal{K}^\bullet_\alpha/\mathcal{K}^\bullet_{<\alpha}$$

is a resolution of $\mathrm{gr}^V_\alpha\mathcal{M}_g$ as a right $\mathcal{D}_X$-module.

15.12.7. Example. Assume that $e_i = 1$ for every $i \in I_{\boldsymbol{e}}$, that is, $g = x_1 \cdots x_r$. Then the set of λ's such that $\psi_{g,\lambda}\mathcal{M} \neq 0$ is contained in the union of the sets of λ's such that $\psi_{x_i,\lambda}\mathcal{M} \neq 0$ for some $i \in I_{\boldsymbol{e}}$.

***Proof of Proposition* 15.12.5.** For $\varepsilon > 0$, the surjectivity of (15.12.5 $*$) implies that of the morphism $\big(V^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}}\mathcal{M}/V^{(\boldsymbol{r})}_{(\alpha-\varepsilon)\boldsymbol{e}}\mathcal{M}\big) \otimes_{\mathcal{O}_X} \mathcal{D}'_X \to \mathrm{gr}^V_\alpha\mathcal{M}_g$. If ε is small enough, the source of this morphism reads

$$\Bigg(\bigoplus_{\substack{\boldsymbol{a}\in\boldsymbol{A}+\mathbb{Z}^n \\ \exists\, i\in I_{\boldsymbol{e}},\, a_i=\alpha e_i}} M_{\boldsymbol{a}}\Bigg) \otimes_{\mathbb{C}[x]} \mathcal{D}'_X,$$

hence the first assertion, according to Remark 15.7.10. For the second assertion, since $\mathcal{K}^\bullet_\alpha$, resp. $\mathcal{K}^\bullet_{<\alpha}$, is a resolution of $V_\alpha\mathcal{M}_g$, resp. $V_{<\alpha}\mathcal{M}_g$, and since the morphism $V_{<\alpha}\mathcal{M}_g \to V_\alpha\mathcal{M}_g$ is injective, one deduces that $\mathcal{K}^\bullet_\alpha/\mathcal{K}^\bullet_{<\alpha}$ is a resolution of $\mathrm{gr}^V_\alpha\mathcal{M}_g$. $\square$

We will make use of the next general lemma, whose proof is left as Exercise 15.15.

15.12.8. Lemma. *Let A be a commutative ring and let $(a_1,\dots,a_r)$ be a finite sequence of elements of A. Let M be an A-module. If $(a_2,\dots,a_r)$ is a regular sequence on M, then the sequence $\big((a_2\otimes u_2-a_1\otimes u_1),\dots,(a_r\otimes u_r-a_1\otimes u_1)\big)$ is a regular sequence on $M\otimes_A A[u_1,\dots,u_n]$. Furthermore, let $\widetilde M$ be the quotient module*

$$\big(M\otimes_A A[u_1,\dots,u_n]\big)/\big((a_j\otimes u_j-a_1\otimes u_1)_{j=2,\dots,r}\big)$$

considered as an $A[u_1,u']$-module (with $u'=(u_2,\dots,u_r)$), equipped with the filtrations $F^{(2)}_\bullet,\dots,F^{(r)}_\bullet$ induced by the filtrations by the degree in $u_2,\dots,u_r$ on $A[u_1,u_2,\dots,u_r]$. Then the $(r-1)$-graded module $\mathrm{gr}^{F^{(r)}}\cdots\mathrm{gr}^{F^{(2)}}\widetilde M$ is isomorphic to

$$\bigoplus_{\boldsymbol{k}\in\mathbb{N}^{r-1}}\big(M/(a_2^{k_2},\dots,a_r^{k_r})\big)\otimes_A u'^{\boldsymbol{k}}A[u_1],$$

where the action of $u'^{\boldsymbol{\ell}}$ is via the natural (injective) morphism $M/(a_2^{k_2},\dots,a_r^{k_r})\to M/(a_2^{k_2+\ell_2},\dots,a_r^{k_r+\ell_r})$.

***Proof of Proposition* 15.12.5.** It is enough to consider the algebraic case of a monodromic $\mathbb{C}[x]\langle\partial_x\rangle$-module since, by assumption, $\mathcal{M}=M\otimes_{\mathbb{C}[x]\langle\partial_x\rangle}\mathcal{D}_X$ and a similar property for $\mathcal{M}_g$, and since this is a flat extension. We set $A_n=\mathbb{C}[x]\langle\partial_x\rangle$ and $A'_n=\mathbb{C}[x]\langle\partial_{x_1},\dots,\partial_{x_r}\rangle$. Let M be a monodromic A_n-module. We set $M_g={}_{\mathrm{D}}\iota_{g*}M\simeq M[\partial_t]$, which is an A_{n+1}-module, with $A_{n+1}=\mathbb{C}[x,t]\langle\partial_x,\partial_t\rangle$. Note that M_g is naturally graded: $M_g=\bigoplus_{\boldsymbol{a},\ell}M_{\boldsymbol{a}}\otimes\partial_t^\ell$.

(1) We start with showing that the Koszul complex

$$K^\bullet_\alpha=K\big(V^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}}M\otimes_{\mathbb{C}[x]}A'_n,(\cdot_{\mathrm{tens}}\delta_j)_{j=2,\dots,r}\big)$$

is exact in nonzero degrees. We can simplify this complex by considering the filtration $F_\bullet A'_n$ by the degree of differential operators, so that $\mathrm{gr}^FA'_n\simeq\mathbb{C}[x,\xi']$. The differentials are of F-degree one, so we can filter the complex by setting $(F_pK_\alpha)^k=F_{p+k}(K^k_\alpha)$, with $F_q(V^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}}M\otimes_{\mathbb{C}[x]}A'_n)=V^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}}M\otimes_{\mathbb{C}[x]}F_qA'_n$. The morphism induced by $\cdot_{\mathrm{tens}}\delta_j$ on $\mathrm{gr}^FK^0_\alpha$ is $\mathrm{Id}\otimes(x_j\xi_j/e_j-x_1\xi_1/e_1)$ and the corresponding Koszul complex reads

$$\begin{aligned}\mathrm{gr}^FK^\bullet_\alpha&=\big(V^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}}M\otimes_{\mathbb{C}[x]}\mathbb{C}[x,\xi'],(x_j\xi_j/e_j-x_1\xi_1/e_1)_{j=2,\dots,r}\big)\\&\simeq K\big(V^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}}M\otimes_{\mathbb{C}}\mathbb{C}[\xi'],(x_j\otimes\xi_j/e_j-x_1\otimes\xi_1/e_1)_{j=2,\dots,r}\big).\end{aligned}$$

Since $V^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}}M$ is $\mathbb{C}[x']$-flat by Proposition 15.7.3(5), the sequence $(x_2,\dots,x_r)$ is regular on it, and the first part of Lemma 15.12.8, together with Exercise 15.2, shows that $\mathrm{gr}^FK^\bullet_\alpha$ is exact in negative degrees. The same property holds true for $K^\bullet_\alpha$ since the filtration $F_\bullet$ is bounded below.

(2) It remains to identify the kernel of the morphism (15.12.5 $*$), which is surjective according to the identification (15.12.1 $*$). Note first that every element of the form

$$m\otimes\delta_j-m\delta_j\otimes1$$

belongs to the kernel of this morphism, according to Formula (15.11.3 ∗∗), and the morphism

$$
\begin{aligned}
(V^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}}M\otimes_{\mathbb{C}[x]}A'_n)^{r-1} &\longrightarrow V^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}}M\otimes_{\mathbb{C}[x]}A'_n\\
(m_j\otimes P_j)_{j=2,\dots,r} &\longmapsto \sum_j(m_j\otimes\delta_jP_j-m_j\delta_j\otimes P_j)
\end{aligned}
\tag{15.12.9}
$$

has image contained in the kernel of the morphism (15.12.5 ∗). We can write

$$V^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}}M\otimes_{\mathbb{C}[x]}A'_n=\bigoplus_{\boldsymbol{k}\in\mathbb{N}^r}V^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}}M\otimes\partial^{\boldsymbol{k}}_{x'},$$

with the $(\boldsymbol{A}'+\mathbb{Z}^r)$-grading such that $M_{\boldsymbol{a}}\otimes\partial^{\boldsymbol{k}}_{x'}$ is of degree $\boldsymbol{a}'+\boldsymbol{k}$ with $\boldsymbol{a}':=(a_1,\dots,a_r)$. We also consider the $(\boldsymbol{A}'+\mathbb{Z}^r)$-grading on $M_g=\bigoplus_{\boldsymbol{a},\ell}M_{\boldsymbol{a}}\otimes\partial_t^\ell$ such that $M_{\boldsymbol{a}}\otimes\partial_t^\ell$ is of degree $\boldsymbol{a}'+\ell\boldsymbol{e}$. Then (15.11.3 ∗) shows that the morphism $V^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}}M\otimes_{\mathbb{C}[x]}A'_n\to M_g$ is $(\boldsymbol{A}'+\mathbb{Z}^r)$-graded, hence so is its kernel.

We first find a simple representative, modulo the image of (15.12.9), of any homogeneous element of $V^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}}M\otimes_{\mathbb{C}[x]}A'_n$. Let $\mu=\sum_{\boldsymbol{k}\in\mathbb{N}^r}m_{\boldsymbol{k}}\otimes\partial^{\boldsymbol{k}}_{x'}$ be a homogeneous element of degree $\boldsymbol{a}'^o$, so that $0\neq m_{\boldsymbol{k}}\in M_{\boldsymbol{a}(\boldsymbol{k})}$ with $\boldsymbol{a}'(\boldsymbol{k})\leqslant\alpha\boldsymbol{e}$ and $\boldsymbol{a}'(\boldsymbol{k})+\boldsymbol{k}=\boldsymbol{a}'^o$. Let us set $\boldsymbol{k}^o=\max(0,\lceil\boldsymbol{a}'^o-\alpha\boldsymbol{e}\rceil)$ componentwise. Then $m_{\boldsymbol{k}}\neq0\Rightarrow\boldsymbol{k}\geqslant\boldsymbol{k}^o$ componentwise.

For each i, we have $a^o_i-k^o_i\leqslant\alpha e_i<0$, so that $a_i(\boldsymbol{k})+k_i-k^o_i<0$ and multiplication by $x_i^{k_i-k_i^o}:M_{\boldsymbol{a}(\boldsymbol{k})+(k_i-k_i^o)\mathbf{1}_i}\to M_{\boldsymbol{a}(\boldsymbol{k})}$ is bijective. We can thus divide $m_{\boldsymbol{k}}$ by $x_i^{k_i-k_i^o}$ for each $i=1,\dots,r$ and write

$$m_{\boldsymbol{k}}\otimes\partial^{\boldsymbol{k}}_{x'}=\mu_{\boldsymbol{k}}\otimes(x^{\boldsymbol{k}-\boldsymbol{k}^o}\partial^{\boldsymbol{k}-\boldsymbol{k}^o}_{x'})\partial^{\boldsymbol{k}^o}_{x'},$$

with $\mu_{\boldsymbol{k}}\in M_{\boldsymbol{a}'^o-\boldsymbol{k}^o}$ since $\boldsymbol{a}(k)+\boldsymbol{k}-\boldsymbol{k}^o=\boldsymbol{a}'^o-\boldsymbol{k}^o$. This can be rewritten as a sum of terms $\widetilde{\mu}_{\boldsymbol{j}}\otimes(x'\partial_{x'})^{\boldsymbol{j}}\partial^{\boldsymbol{k}^o}_{x'}$ with $\mu_{\boldsymbol{j}}\in M_{\boldsymbol{a}'^o-\boldsymbol{k}^o}$ and each component j_i varying from 0 to $k_i-k_i^o$. Iterating the equality

$$\widetilde{\mu}_{\boldsymbol{j}}\otimes(x_i\partial_{x_i})\partial^{\boldsymbol{k}^o}_{x'}=\widetilde{\mu}_{\boldsymbol{j}}\delta_i\otimes\partial^{\boldsymbol{k}^o}_{x'}+\frac{e_i}{e_1}\,\widetilde{\mu}_{\boldsymbol{j}}\otimes(x_1\partial_{x_1})\partial^{\boldsymbol{k}^o}_{x'}\quad\mathrm{mod}\ \mathrm{image}\,(15.12.9),$$

we see that $\widetilde{\mu}_{\boldsymbol{j}}\otimes(x'\partial_{x'})^{\boldsymbol{j}}\partial^{\boldsymbol{k}^o}_{x'}$ is equivalent, modulo the image of (15.12.9), to a sum of terms $\widehat{\mu}_\ell(x_1\partial_{x_1})^\ell\partial^{\boldsymbol{k}^o}_{x'}$ with $\widehat{\mu}_\ell\in M_{\boldsymbol{a}'^o-\boldsymbol{k}^o}$ for each ℓ. In conclusion, modulo the image of (15.12.9), μ is equivalent to an expression of the form

$$\sum_{\ell=0}^{\ell^o}\nu_\ell\otimes(x_1\partial_{x_1})^\ell\partial^{\boldsymbol{k}^o}_{x'},$$

for some $\ell^o\geqslant0$, with $\nu_\ell\in M_{\boldsymbol{a}'^o-\boldsymbol{k}^o}$ for each ℓ. If the image of the above element in N is zero, the coefficient of $\partial_t^{|\boldsymbol{k}^o|+\ell^o}$, up to a nonzero constant, which is equal to

$$\nu_{\ell^o}x'^{(|\boldsymbol{k}^o|+\ell^o)\boldsymbol{e}-\boldsymbol{k}^o},$$

is thus equal to zero. We notice that each component of $|\boldsymbol{k}^o|\boldsymbol{e}-\boldsymbol{k}^o$ is nonnegative. Since $a_i^o-k_i^o<0$ for each $i=1,\dots,r$, multiplication by $x'^{(|\boldsymbol{k}^o|+\ell^o)\boldsymbol{e}-\boldsymbol{k}^o}$ is injective on $M_{\boldsymbol{a}'^o-\boldsymbol{k}^o}$, so that this implies that $\nu_{\ell^o}=0$, and thus $\nu=0$, hence the desired surjectivity of (15.12.9) onto the kernel of the morphism in the proposition. □

15.12.c. Normal crossing type of $\mathrm{gr}_\alpha^V \mathcal{M}_g$

15.12.10. Corollary **(Normal crossing type of $\mathrm{gr}_\alpha^V \mathcal{M}_g$).** *For each $\alpha < 0$, $\mathrm{gr}_\alpha^V \mathcal{M}_g$ is of normal crossing type.*

Proof. We prove the analogous statement for a monodromic A_n-module, the case of a $\mathcal{D}_X$-module of normal crossing type begin obtained by tensoring with $\mathcal{O}_X$. Recall that N_i $(i = 1, \dots, n)$ denotes the action of $x_i\partial_{x_i} - a_i$ on $M_{\boldsymbol{a}}$ for $\boldsymbol{a} \in \boldsymbol{A} + \mathbb{Z}^n$, and N denotes the action of $t\partial_t - \alpha$ on $\mathrm{gr}_\alpha^V \mathcal{M}_g$. For $m \in M_{\boldsymbol{a}}$ and $m \otimes 1 \in N$, Formula (15.11.3 $**$) implies

$$(m \otimes 1)x_i\partial_{x_i} = (\mathrm{N}_i + a_i)m \otimes 1 - e_i m \otimes t\partial_t.$$

If $\boldsymbol{a}' \leqslant \alpha\boldsymbol{e}$, $(m \otimes 1)$ is a section of $V_\alpha M_g$ and its image $[m \otimes 1]$ in $\mathrm{gr}_\alpha^V M_g$ satisfies

$$[m \otimes 1]x_i\partial_{x_i} = [(\mathrm{N}_i + a_i)m \otimes 1] - e_i(\mathrm{N} + \alpha)[m \otimes 1]. \tag{15.12.11}$$

Since N_i and N are nilpotent, it follows that $[m \otimes 1](x_i\partial_{x_i} - a_i + \alpha e_i)^k = 0$ for $k \gg 0$. As a consequence, the image of $M_{\boldsymbol{a}} \otimes 1 \subset V_{\alpha\boldsymbol{e}}^{(\boldsymbol{r})} M \otimes_{\mathbb{C}[x]} A'_n$ in $\mathrm{gr}_\alpha^V M_g$ by the morphism (15.12.5 $*$) is contained in $(\mathrm{gr}_\alpha^V M_g)_{\boldsymbol{b}}$ with $\boldsymbol{b} = \boldsymbol{a} - \alpha\boldsymbol{e}$. More generally, the image in $\mathrm{gr}_\alpha^V M_g$ of $M_{\boldsymbol{a}} \otimes \partial_{x'}^{\boldsymbol{k}}$ is contained in $(\mathrm{gr}_\alpha^V M_g)_{\boldsymbol{b}}$ with $\boldsymbol{b} = \boldsymbol{a} + \boldsymbol{k} - \alpha\boldsymbol{e}$ (by setting $k_i = 0$ for $i > r$).

Since $\mathrm{gr}_\alpha^V M_g$ is of finite type over A_n, there exists a maximal finite subset $\boldsymbol{B} \subset [-1, 0)^n$ such that $\bigoplus_{\boldsymbol{b} \in \boldsymbol{B} + \mathbb{Z}^n} (\mathrm{gr}_\alpha^V M_g)_{\boldsymbol{b}} \to \mathrm{gr}_\alpha^V M_g$ is injective. Furthermore, by the above argument, the morphism $V_{\alpha\boldsymbol{e}}^{(\boldsymbol{r})} M \otimes_{\mathbb{C}[x]} A'_n \to \mathrm{gr}_\alpha^V M_g$ factorizes through $\bigoplus_{\boldsymbol{b} \in \boldsymbol{B} + \mathbb{Z}^n} (\mathrm{gr}_\alpha^V M_g)_{\boldsymbol{b}}$. Since this morphism is surjective by the monodromic analogue of Proposition 15.12.1, we deduce that

$$\bigoplus_{\boldsymbol{b} \in \boldsymbol{B} + \mathbb{Z}^n} (\mathrm{gr}_\alpha^V M_g)_{\boldsymbol{b}} = \mathrm{gr}_\alpha^V M_g.$$

In order to conclude that $\mathrm{gr}_\alpha^V M_g$ is monodromic, we are left with showing that, for each $\boldsymbol{b}$, $(\mathrm{gr}_\alpha^V M_g)_{\boldsymbol{b}}$ is finite-dimensional. By the above argument, the direct sum of the terms $M_{\boldsymbol{a}} \otimes \partial_{x'}^{\boldsymbol{k}}$ with $\boldsymbol{a}$ varying in $\boldsymbol{A} + \mathbb{Z}^n$ and $\boldsymbol{k}$ in $\mathbb{Z}^r$ such that $\boldsymbol{a} + \boldsymbol{k} = \boldsymbol{b} + \alpha\boldsymbol{e}$ maps onto $(\mathrm{gr}_\alpha^V M_g)_{\boldsymbol{b}}$. In particular, the components $a_{r+1}, \dots, a_n$ of $\boldsymbol{a}$ are fixed. Let us choose $\boldsymbol{k}_o$ big enough so that all components of $\boldsymbol{b}' + \alpha\boldsymbol{e} - \boldsymbol{k}_o$ are $\leqslant 0$. Then, for $i \in \{1, \dots, r\}$, Formula (15.12.11) implies that an element of $M_{\boldsymbol{a}-\boldsymbol{1}_i} \otimes \partial_{x'}^{\boldsymbol{k}_o+\boldsymbol{1}_i}$ has image contained in that of $M_{\boldsymbol{a}} \otimes \partial_{x'}^{\boldsymbol{k}_o}$ plus its image by N. In other words, $(\mathrm{gr}_\alpha^V M_g)_{\boldsymbol{b}}$ is equal to the sum of a finite number of finite-dimensional vector spaces (the images of $M_{\boldsymbol{a}} \otimes \partial_{x'}^{\boldsymbol{k}}$ for $\boldsymbol{a} + \boldsymbol{k} = \boldsymbol{b}$ and $0 \leqslant \boldsymbol{k} \leqslant \boldsymbol{k}_0$ componentwise) and their images by any power of N. Since N is nilpotent, the finite-dimensionality of $(\mathrm{gr}_\alpha^V M_g)_{\boldsymbol{b}}$ follows.

Moreover, we have an estimate for $\boldsymbol{B}$:

$$\boldsymbol{B} + \mathbb{Z}^n \subset \boldsymbol{A} - \alpha\boldsymbol{e} + \mathbb{Z}^n,$$

and we recall that $\alpha < 0$ is such that $\alpha\boldsymbol{e} \in \boldsymbol{A}' + \mathbb{Z}^r$. □

15.12.12. Corollary **($\mathbb{R}$-specializability of $\mathrm{gr}_\alpha^V \mathcal{M}_g$ along D_i $(i \in I)$)**
For each $i \in I$, $\mathrm{gr}_\alpha^V \mathcal{M}_g$ is $\mathbb{R}$-specializable along D_i and its $V^{(i)}$-filtration is the

image of the filtration

$$V^{(i)}_{a_i}\big(V^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}}\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{D}'_X\big)=\sum_{k\geqslant 0}V^{(i)}_{a_i-k}(V^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}}\mathcal{M})\otimes_{\mathcal{O}_X}V^{(i)}_k\mathcal{D}'_X,$$

with $V^{(i)}_{a_i-k}(V^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}}\mathcal{M}):=V^{(i)}_{a_i-k}\mathcal{M}\cap V^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}}\mathcal{M}$, *which is a* $V^{(i)}_\bullet\mathcal{D}_X$*-filtration with respect to the right trivial structure, and* $V^{(\boldsymbol{r})}_{\boldsymbol{0}}(\mathrm{gr}^V_\alpha\mathcal{M}_g)$ *(resp.* $V^{(\boldsymbol{n})}_{\boldsymbol{0}}(\mathrm{gr}^V_\alpha\mathcal{M}_g)$*) is the image of*

$$V^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}}\mathcal{M}\otimes_{\mathcal{O}_X}V^{(\boldsymbol{r})}_{\boldsymbol{0}}\mathcal{D}'_X,\quad \textit{resp. } V^{(\boldsymbol{n})}_{\alpha\boldsymbol{e}}\mathcal{M}\otimes_{\mathcal{O}_X}V^{(\boldsymbol{r})}_{\boldsymbol{0}}\mathcal{D}'_X.$$

Furthermore, for every $i,j\in I$*, the right tensor action of* δ_j *is of order* 0 *with respect to* $V^{(i)}_\bullet$*.*

Proof. This is a direct consequence of Corollary 15.12.10 and its proof. □

15.12.d. The monodromy filtration of $\mathrm{gr}^V_\alpha\mathcal{M}_g$. The nilpotent operator N on $\mathrm{gr}^V_\alpha\mathcal{M}_g$ defines an increasing filtration on $\mathrm{gr}^V_\alpha\mathcal{M}_g$: the monodromy filtration $\mathrm{M}(\mathrm{N})_\bullet(\mathrm{gr}^V_\alpha\mathcal{M}_g)$ (see Lemma 3.3.1).

15.12.13. Proposition. *If* $\mathcal{M}$ *is of normal crossing type, then for each* $\alpha<0$ *and each* $\ell\in\mathbb{Z}$*, the* $\mathcal{D}_X$*-module* $\mathrm{gr}^{\mathrm{M}}_\ell\mathrm{gr}^V_\alpha\mathcal{M}_g$ *is also of normal crossing type. Furthermore, the filtrations* $M(\mathrm{N})_\bullet$ *and* $V^{(i)}_\bullet$ *(*$i\in I$*) are compatible and for each* $\boldsymbol{b}\leqslant\boldsymbol{0}$*, denoting* $\mathrm{N}_{\boldsymbol{b}}=\mathrm{gr}^{V^{(\boldsymbol{n})}}_{\boldsymbol{b}}\mathrm{N}$*, we have*

$$\mathrm{gr}^{V^{(\boldsymbol{n})}}_{\boldsymbol{b}}\mathrm{M}(\mathrm{N})_\ell\,(\mathrm{gr}^V_\alpha\mathcal{M}_g)=\mathrm{M}(\mathrm{N}_{\boldsymbol{b}})_\ell\,\mathrm{gr}^{V^{(\boldsymbol{n})}}_{\boldsymbol{b}}(\mathrm{gr}^V_\alpha\mathcal{M}_g).$$

Proof. We first notice that the analytification of $\mathrm{M}(\mathrm{N})_\bullet\,\mathrm{gr}^V_\alpha M_g$ is the monodromy filtration $\mathrm{M}(\mathrm{N})_\bullet\,\mathrm{gr}^V_\alpha\mathcal{M}_g$: this follows from the characteristic properties of the monodromy filtration, which are preserved by analytification (due to $\mathbb{C}[x]$-flatness of $\mathcal{O}_X$). The properties of the lemma are also preserved by analytification. It follows that we only need to consider the case of monodromic A_n-modules. Since N commutes with $x_i\partial_{x_i}$ for each $i\in I$, it preserves each $(\mathrm{gr}^V_\alpha M_g)_{\boldsymbol{b}}$ and the decomposition of $\mathrm{gr}^V_\alpha M_g$. We thus obtain a corresponding decomposition for each $\ell\in\mathbb{Z}$:

$$\mathrm{M}(\mathrm{N})_\ell\Big(\bigoplus_{\boldsymbol{b}}(\mathrm{gr}^V_\alpha M_g)_{\boldsymbol{b}}\Big)=\bigoplus_{\boldsymbol{b}}\mathrm{M}(\mathrm{N}_{\boldsymbol{b}})_\ell(\mathrm{gr}^V_\alpha M_g)_{\boldsymbol{b}}.$$ □

15.13. An explicit expression of nearby cycles

We restrict our computation to the case of a monodromic A_n-module $M=\bigoplus_{\boldsymbol{a}\in\boldsymbol{A}+\mathbb{Z}^n}M_{\boldsymbol{a}}$. The case of a $\mathcal{D}_X$-module of normal crossing type can be obtained by tensoring with $\mathcal{O}_X$. Compared with the presentation of Section 15.12.a, we emphasize the nilpotent operator N induced by $t\partial_t-\alpha$ on $\mathrm{gr}^V_\alpha M_g$ ($\alpha<0$), in relation with the nilpotent operators N_i acting by $x_i\partial_{x_i}-a_i$ on $M_{\boldsymbol{a}}$.

Let M be a monodromic A_n-module which is a middle extension along $D_{i\in I}$, i.e., satisfying the assumption of Theorem 15.11.1 in the monodromic situation.

15.13.a. Computation of nearby cycles. We revisit Corollary 15.12.10 a little differently. From Proposition 15.12.5 we obtain a surjective A'_n-linear morphism:

$$V^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}} M \otimes_{\mathbb{C}[x]} A'_n \longrightarrow \mathrm{gr}^V_\alpha M_g.$$

In order to obtain an A_n-linear morphism, we note the natural surjective morphism

$$V^{(\boldsymbol{n})}_{\alpha\boldsymbol{e}} M \otimes_{\mathbb{C}[x]} A_n \longrightarrow V^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}} M \otimes_{\mathbb{C}[x]} A'_n,$$

since $V^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}} M = \sum_{\boldsymbol{k}''\in\mathbb{Z}^{n-r}} V^{(\boldsymbol{n})}_{\alpha\boldsymbol{e}} M \cdot \partial_{x''}^{\boldsymbol{k}''}$, where $x'' = (x_{r+1}, \dots, x_n)$. Let us equip $M_{\boldsymbol{b}+\alpha\boldsymbol{e}}[\mathrm{N}] := M_{\boldsymbol{b}+\alpha\boldsymbol{e}} \otimes_{\mathbb{C}} \mathbb{C}[\mathrm{N}]$ with the $\mathbb{C}[\mathrm{N}_1, \dots, \mathrm{N}_n, \mathrm{N}]$-module structure such that

- N_i acts by $\mathrm{N}_i \otimes \mathrm{Id} - e_i \mathrm{Id} \otimes \mathrm{N}$, and
- N acts by $\mathrm{Id} \otimes \mathrm{N}$ (see (15.12.3)),

and $(\mathrm{gr}^V_\alpha M_g)_{\boldsymbol{b}}$ with its natural $\mathbb{C}[\mathrm{N}_1, \dots, \mathrm{N}_n, \mathrm{N}]$-module structure (see §15.7.a). The reason for twisting the action of N_i comes from Formula (15.11.3 $**$).

15.13.1. Proposition. *For $\boldsymbol{b} \leqslant 0$, we have a surjective $\mathbb{C}[\mathrm{N}_1, \dots, \mathrm{N}_n, \mathrm{N}]$-linear morphism*

$$M_{\boldsymbol{b}+\alpha\boldsymbol{e}}[\mathrm{N}] \longrightarrow (\mathrm{gr}^V_\alpha M_g)_{\boldsymbol{b}}$$

that takes $m \otimes \mathrm{N}^k$ to the class of $m \otimes (t\partial_t - \alpha)^k \in V_\alpha M_g$ modulo $V_{<\alpha} M_g$.

Let us start with a lemma valid for any $\boldsymbol{b}$.

15.13.2. Lemma. *For every $\boldsymbol{b} \in \mathbb{R}^n$, $(\mathrm{gr}^V_\alpha M_g)_{\boldsymbol{b}}$ is the image of*

$$V_\alpha M_g \cap \big(\textstyle\bigoplus_j M_{\boldsymbol{b}+(\alpha-j)\boldsymbol{e}} \otimes \partial_t^j\big)$$

in $\mathrm{gr}^V_\alpha M_g$.

Proof. Let us consider an arbitrary element of $V_\alpha M_g$, expressed as a finite sum

$$\sum_{\boldsymbol{a}\in\mathbb{R}^n} \sum_{j\in\mathbb{N}} m_{\boldsymbol{a},j} \otimes \partial_t^j,$$

with $m_{\boldsymbol{a},j} \in M_{\boldsymbol{a}}$. Assume that its image in $\mathrm{gr}^V_\alpha M_g$ belongs to $(\mathrm{gr}^V_\alpha M_g)_{\boldsymbol{b}}$, i.e.,

$$\Big(\sum_{\boldsymbol{a}\in\mathbb{R}^n} \sum_{j\in\mathbb{N}} m_{\boldsymbol{a},j} \otimes \partial_t^j\Big) \cdot (x_i\partial_{x_i} - b_i)^k \in V_{<\alpha} M_g$$

for every $i \in \{1, \dots, n\}$ and some $k \gg 0$. Our aim is to prove that, modulo $V_{<\alpha} M_g$, only those terms with $\boldsymbol{a} = \boldsymbol{b} + (\alpha - j)\boldsymbol{e}$ matter.

15.13.3. Lemma. *In the situation considered above, one has*

$$\sum_{\boldsymbol{a}\in\mathbb{R}^n} \sum_{j\in\mathbb{N}} m_{\boldsymbol{a},j} \otimes \partial_t^j = \sum_{j\in\mathbb{N}} m_{\boldsymbol{b}+(\alpha-j)\boldsymbol{e}} \otimes \partial_t^j \mod V_{<\alpha} M_g.$$

Proof. Let us start with an elementary lemma of linear algebra.

15.13.4. Lemma. *Let T be an endomorphism of a complex vector space V, and $W \subset V$ a linear subspace with $TW \subset W$. Suppose that $v_1, \dots, v_k \in V$ satisfy*

$$T^\mu(v_1 + \cdots + v_k) \in W$$

for some $\mu \geqslant 0$. If there are pairwise distinct complex numbers $\lambda_1, \dots, \lambda_k$ with $v_h \in E_{\lambda_h}(T)$, then one has $\lambda_h v_h \in W$ for every $h = 1, \dots, k$.

Proof. Choose a sufficiently large integer $\mu \in \mathbb{N}$ such that $(T - \lambda_h)^\mu v_h = 0$ for $h = 1, \dots, k$, and such that $T^\mu(v_1 + \cdots + v_k) \in W$. Assume that $\lambda_k \neq 0$. Setting $Q(T) = T^\mu (T - \lambda_1)^\mu \cdots (T - \lambda_{k-1})^\mu$, we have by assumption

$$Q(T)(v_1 + \cdots + v_k) \in W$$

The left-hand side equals $Q(T)v_k$. Since $Q(T)$ and $T - \lambda_k$ are coprime, Bézout's theorem implies that $v_k \in W$. At this point, we are done by induction. $\square$

We now go back to the proof of Lemma 15.13.3. Let us consider an element as in the lemma. As we have seen before,

$$(m_{\boldsymbol{a},j} \otimes \partial_t^j) \cdot \big((x_i \partial_{x_i} - b_i) + e_i(t\partial_t - \alpha)\big) = (m_{\boldsymbol{a},j} \otimes \partial_t^j) \cdot \big(\mathrm{D}_i - b_i - e_i(\alpha - j)\big),$$

and since some power of $t\partial_t - \alpha$ also send this element in $V_{<\alpha} M_g$, we may conclude that

$$\sum_{\boldsymbol{a} \in \mathbb{R}^n} \sum_{j \in \mathbb{N}} \Big(m_{\boldsymbol{a},j} \otimes \partial_t^j \cdot \big(\mathrm{D}_i - b_i - e_i(\alpha - j)\big)^k \Big) \in V_{<\alpha} M_g \tag{15.13.5}$$

for every $i \in I$ and $k \gg 0$.

In order to apply Lemma 15.13.4 to our situation, let us set $V = N$ and $W = V_{<\alpha} M_g$, and for a fixed choice of $i = 1, \dots, n$, let us consider the endomorphism

$$T_i = (x_i \partial_{x_i} - b_i) + e_i(t\partial_t - \alpha);$$

Evidently, $T_i W \subset W$. Since we have

$$T_i(m_{\boldsymbol{a},j} \otimes \partial_t^j) = (m_{\boldsymbol{a},j} \otimes \partial_t^j) \cdot \big((\mathrm{D}_i - a_i) + a_i - b_i - e_i(\alpha - j)\big),$$

it is clear that $m_{\boldsymbol{a},j} \otimes \partial_t^j$ is annihilated by a large power of $T_i - (a_i - b_i - e_i(\alpha - j))$. Grouping terms according to the value of $a_i - b_i - e_i(\alpha - j)$, we obtain

$$\sum_{\boldsymbol{a} \in \mathbb{R}^n} \sum_{j \in \mathbb{N}} m_{\boldsymbol{a},j} \otimes \partial_t^j = v_1 + \cdots + v_k$$

with $v_k \in E_{\lambda_k}(T_i)$ and $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ are pairwise distinct. According to Lemma 15.13.4, we have $v_h \in W$ whenever $\lambda_h \neq 0$; what this means is that the sum of all $m_{\boldsymbol{a},j} \otimes \partial_t^j$ with $a_i - b_i - e_i(\alpha - j) \neq 0$ belongs to $V_{<\alpha} M_g$. After subtracting this sum from our original element, we may therefore assume that $a_i = b_i - e_i(\alpha - j)$ for every term. We obtain the asserted congruence by performing this procedure for $T_1, \dots, T_n$. This ends the proof of Lemma 15.13.3 and at the same time that of Lemma 15.13.2. $\square$

***Proof of Proposition* 15.13.1.** Suppose now that $b_1, \dots, b_n \leqslant 0$, that we shall abbreviate as $\boldsymbol{b} \leqslant 0$ (recall also that we assume $\alpha \in [-1, 0)$). Let $j \in \mathbb{N}$. We observe that

$$e_i \neq 0 \implies b_i + (\alpha - j)e_i = (b_i + \alpha e_i) - je_i < -je_i.$$

Given a vector $m_j \in M_{\boldsymbol{b}+(\alpha-j)\boldsymbol{e}}$, this means that m_j is divisible by $x_i^{je_i}$. Consequently, $m_j = mx^{j\boldsymbol{e}}$ for a unique m in $M_{\boldsymbol{b}+\alpha\boldsymbol{e}}$, and therefore

$$m_j \otimes \partial_t^j = (m \otimes 1) \cdot t^j \partial_t^j$$

is a linear combination of $(m \otimes 1)(t\partial_t)^k$ for $k = 1, \dots, j$. Since $m \otimes 1 \in V_\alpha M_g$ and $V_\alpha M_g$ is stable by $t\partial_t$, we conclude that

$$\bigoplus_j M_{\boldsymbol{b}+(\alpha-j)\boldsymbol{e}} \otimes \partial_t^j = M_{\boldsymbol{b}+\alpha\boldsymbol{e}}[t\partial_t] \subset V_\alpha M_g,$$

and, by Lemma 15.13.2, $(\mathrm{gr}_\alpha^V M_g)_{\boldsymbol{b}}$ is the image of $M_{\boldsymbol{b}+\alpha\boldsymbol{e}}[t\partial_t] \bmod V_{<\alpha} M_g$. □

In order to have an explicit expression of $(\mathrm{gr}_\alpha^V M_g)_{\boldsymbol{b}}$ ($\boldsymbol{b} \leqslant 0$), it remains to find the kernel of the morphism in Proposition 15.13.1. For $\boldsymbol{b} \leqslant 0$, let us set

$$I_{\boldsymbol{e}}(\boldsymbol{b}) = \{i \mid e_i \neq 0 \text{ and } b_i = 0\}.$$

Given $m \in M_{\boldsymbol{b}+\alpha\boldsymbol{e}}$, we have $\big(m \prod_{i \in I_{\boldsymbol{e}}(\boldsymbol{b})} x_i\big) \otimes 1 = m \otimes t \in V_{<\alpha} M_g$ and therefore also

$$(m \otimes 1) \textstyle\prod_{i \in I_{\boldsymbol{e}}(\boldsymbol{b})} x_i \partial_{x_i} = (m \otimes 1) \cdot \prod_{i \in I_{\boldsymbol{e}}(\boldsymbol{b})} (\mathrm{N}_i - e_i \mathrm{N}) \in V_{<\alpha} M_g.$$

In this way, we obtain a large collection of elements in the kernel.

15.13.6. Corollary. *If $\alpha < 0$ and $\boldsymbol{b} \leqslant 0$, $(\mathrm{gr}_\alpha^V M_g)_{\boldsymbol{b}}$ is isomorphic to the cokernel of the injective morphism*

$$\varphi_{\boldsymbol{b}} := \prod_{i \in I_{\boldsymbol{e}}(\boldsymbol{b})} ((\mathrm{N}_i \otimes 1)/e_i - (1 \otimes \mathrm{N})) \in \mathrm{End}(M_{\boldsymbol{b}+\alpha\boldsymbol{e}}[\mathrm{N}]). \tag{15.13.6 $*$}$$

15.13.7. Remark. We have assumed, as in Theorem 15.11.1, that M is a middle extension along the normal crossing divisor $D_{i\in I}$. However, the previous expression shows that, for $\alpha < 0$ and $\boldsymbol{b} \leqslant 0$, $(\mathrm{gr}_\alpha^V M_g)_{\boldsymbol{b}}$ only depends on the $M_{\boldsymbol{a}}$'s with $a_i < 0$ if $i \in \{1, \dots, r\}$. For such an α, we conclude that $\mathrm{gr}_\alpha^V M_g$ only depends on the localized module $M(*g)$.

Moreover, by definition, the action of N_i (resp. N) on $(\mathrm{gr}_\alpha^V M_g)_{\boldsymbol{b}}$ is that induced by $\mathrm{N}_i \otimes 1 - e_i \mathrm{N}$ (resp. N). We thus find that $\prod_{i=1}^r \mathrm{N}_i$ acts by zero on $(\mathrm{gr}_\alpha^V M_g)_{\boldsymbol{b}}$.

If $\alpha < 0$ and $\boldsymbol{b} \leqslant 0$, set $b = |I_{\boldsymbol{e}}(\boldsymbol{b})|$. Corollary 15.13.6 implies that the natural $\mathbb{C}$-linear morphism

$$\bigoplus_{k=0}^{b-1} M_{\boldsymbol{b}+\alpha\boldsymbol{e}} \mathrm{N}^k \longrightarrow (\mathrm{gr}_\alpha^V M_g)_{\boldsymbol{b}} \tag{15.13.8}$$

is an isomorphism. Note also that the action of N on $(\mathrm{gr}_\alpha^V M_g)_{\boldsymbol{b}}$ is easily described on the expression (15.13.8):

$$m\mathrm{N}^k \cdot \mathrm{N} = \begin{cases} m\mathrm{N}^{k+1} & \text{if } k < b-1, \\ m\big[\mathrm{N}^b - \prod_{i \in I_{\boldsymbol{e}}(\boldsymbol{b})} (\mathrm{N} - \mathrm{N}_i/e_i)\big] & \text{if } k = b-1. \end{cases}$$

***Proof of Corollary* 15.13.6.** The injectivity of $\varphi_{\boldsymbol{b}}$ is clear by considering the effect of $\varphi_{\boldsymbol{b}}$ on the term of highest degree with respect to N. On the other hand, we already know that every element of $(\mathrm{gr}^V_\alpha M_g)_{\boldsymbol{b}}$ is the image of some $m=\sum_k(m_k\otimes 1)\mathrm{N}^k$ with $m_k\in M_{\boldsymbol{b}+\alpha\boldsymbol{e}}$ for every k. If we expand this using $\mathrm{N}=t\partial_t-\alpha$, we find

$$m\in\bigoplus_{j\in\mathbb{N}} M_{\boldsymbol{b}+(\alpha-j)\boldsymbol{e}}\otimes\partial_t^j. \tag{15.13.9}$$

Now suppose that m actually lies in $V_{<\alpha}M_g$. It can then be written as (see (15.12.1 $*$))

$$m=\sum_{\substack{\boldsymbol{a}'\leqslant(\alpha-\varepsilon)\boldsymbol{e}\\ \boldsymbol{k}\in\mathbb{N}^r}}(m_{\boldsymbol{a},\boldsymbol{k}}\otimes 1)\partial_{x'}^{\boldsymbol{k}},\quad m_{\boldsymbol{a},\boldsymbol{k}}\in M_{\boldsymbol{a}}. \tag{15.13.10}$$

If we expand the expression $(m_{\boldsymbol{a},\boldsymbol{k}}\otimes 1)\partial_{x'}^{\boldsymbol{k}}$ according to (15.11.3 $*$), all the terms that appear belong to $M_{\boldsymbol{a}+\boldsymbol{k}-j\boldsymbol{e}}\otimes\partial_t^j$ for some $j\leqslant|\boldsymbol{k}|$ (we identify $\boldsymbol{k}$ with $(\boldsymbol{k},0)\in\mathbb{Z}^n$). Comparing with (15.13.9), we can therefore discard those summands in (15.13.10) with $\boldsymbol{a}+\boldsymbol{k}\neq\boldsymbol{b}+\alpha\boldsymbol{e}$ without changing the value of the sum. The sum in (15.13.10) is thus simply indexed by those $\boldsymbol{k}\in\mathbb{N}^r$ such that $k_i>b_i$ for all $i\in\{1,\dots,r\}$ and the index $\boldsymbol{a}$ is replaced with $\boldsymbol{b}+\alpha\boldsymbol{e}-\boldsymbol{k}$.

Now, if $e_i\neq 0$ then $a_i=(b_i+\alpha e_i)-k_i<-k_i$ since we assume that $b_i\leqslant 0$ and $\alpha<0$, and so $m_{\boldsymbol{a},\boldsymbol{k}}$ is divisible by $x_i^{k_i}$. This means that we can write

$$m_{\boldsymbol{a},\boldsymbol{k}}=m'_{\boldsymbol{k}}x'^{\boldsymbol{k}}$$

for some $m'_{\boldsymbol{k}}\in M_{\boldsymbol{b}+\alpha\boldsymbol{e}}$. Therefore, (15.13.10) reads

$$m=\sum_{\substack{\boldsymbol{k}\in\mathbb{N}^r\\ k_i>b_i\,\forall\, i\in\{1,\dots,r\}}}(m'_{\boldsymbol{k}}\otimes 1)x'^{\boldsymbol{k}}\partial_{x'}^{\boldsymbol{k}},\quad m'_{\boldsymbol{k}}\in M_{\boldsymbol{b}+\alpha\boldsymbol{e}}.$$

If $m'_{\boldsymbol{k}}\neq 0$, then $k_i\geqslant 1$ for $i\in I_{\boldsymbol{e}}(\boldsymbol{b})$ (since $b_i=0$), and consequently, $x'^{\boldsymbol{k}}\partial_{x'}^{\boldsymbol{k}}$ is forced to be a multiple of

$$\prod_{i\in I_{\boldsymbol{e}}(\boldsymbol{b})}x_i\partial_{x_i}=\prod_{i\in I_{\boldsymbol{e}}(\boldsymbol{b})}(\mathrm{D}_i-e_i\,\mathrm{E}),$$

which acts on $M_{\boldsymbol{b}+\alpha\boldsymbol{e}}[\mathrm{N}]$ as $\prod_{i\in I_{\boldsymbol{e}}(\boldsymbol{b})}((\mathrm{N}_i\otimes 1)-e_i(1\otimes\mathrm{N}))$. As a consequence,

$$\begin{aligned}m\in\sum_{\boldsymbol{\ell}\in\mathbb{N}^r}(M_{\boldsymbol{b}+\alpha\boldsymbol{e}}\otimes 1)x'^{\boldsymbol{\ell}}\partial_{x'}^{\boldsymbol{\ell}}\cdot\prod_{i\in I_{\boldsymbol{e}}(\boldsymbol{b})}((\mathrm{N}_i\otimes 1)-e_i(1\otimes\mathrm{N}))\\ =\sum_{\boldsymbol{\ell}\in\mathbb{N}^{I_{\boldsymbol{e}}}}(M_{\boldsymbol{b}+\alpha\boldsymbol{e}}\otimes 1)(\mathrm{D}'-\boldsymbol{e}t\partial_t)^{\boldsymbol{\ell}}\cdot\prod_{i\in I_{\boldsymbol{e}}(\boldsymbol{b})}((\mathrm{N}_i\otimes 1)-e_i(1\otimes\mathrm{N}))\\ \subset M_{\boldsymbol{b}+\alpha\boldsymbol{e}}[\mathrm{E}]\cdot\prod_{i\in I_{\boldsymbol{e}}(\boldsymbol{b})}((\mathrm{N}_i\otimes 1)-e_i(1\otimes\mathrm{N})).\end{aligned}$$

□

15.13.b. The quiver of $\mathrm{gr}^V_\alpha M_g$. We give the explicit description of the quiver of $\mathrm{gr}^V_\alpha M_g$ for $\alpha<0$ (see Proposition 15.7.5). We thus consider the vector spaces

$(\mathrm{gr}_\alpha^V M_g)_{\boldsymbol{b}}$ for $\boldsymbol{b} \in [-1,0]^n$, and the morphisms

$$(\mathrm{gr}_\alpha^V M_g)_{\boldsymbol{b}-\mathbf{1}_i} \underset{\mathrm{var}_i(\boldsymbol{b})}{\overset{\mathrm{can}_i(\boldsymbol{b})}{\rightleftarrows}} (\mathrm{gr}_\alpha^V M_g)_{\boldsymbol{b}} \tag{15.13.11}$$

for every i such that $b_i = 0$. We know from that Corollary 15.13.6 that $(\mathrm{gr}_\alpha^V M_g)_{\boldsymbol{b}} \neq 0$ only if $b_i = 0$ for some $i \in I_{\boldsymbol{e}}$ (i.e., such that $e_i \neq 0$). Moreover, the description of $(\mathrm{gr}_\alpha^V M_g)_{\boldsymbol{b}}$ given in this corollary enables one to define a natural quiver as follows.

(1) If $i \notin I_{\boldsymbol{e}}$ and $b_i = 0$, we also have $(\boldsymbol{b} + \alpha\boldsymbol{e})_i = 0$, and we will see that the diagram

$$M_{\boldsymbol{b}+\alpha\boldsymbol{e}-\mathbf{1}_i}[\mathrm{N}] \underset{\mathrm{var}_i \otimes 1}{\overset{\mathrm{can}_i \otimes 1}{\rightleftarrows}} M_{\boldsymbol{b}+\alpha\boldsymbol{e}}[\mathrm{N}]$$

commutes with $\varphi_{\boldsymbol{b}}$ (which only involves indices $j \in I_{\boldsymbol{e}}$), inducing therefore in a natural way a diagram

$$(\mathrm{gr}_\alpha^V M_g)_{\boldsymbol{b}-\mathbf{1}_i} \underset{\mathrm{v}_i(\boldsymbol{b})}{\overset{\mathrm{c}_i(\boldsymbol{b})}{\rightleftarrows}} (\mathrm{gr}_\alpha^V M_g)_{\boldsymbol{b}}$$

We notice moreover that the middle extension property for M is preserved for this diagram, that is, $\mathrm{c}_i(\boldsymbol{b})$ remains surjective and $\mathrm{v}_i(\boldsymbol{b})$ remains injective.

(2) If $i \in I_{\boldsymbol{e}}$, we set $\varphi_{\mathbf{1}_i} = (\mathrm{N}_i \otimes 1)/e_i - \mathrm{N}$ so that, with obvious notation, $\varphi_{\boldsymbol{b}} = \varphi_{\mathbf{1}_i}\varphi_{\boldsymbol{b}-\mathbf{1}_i} = \varphi_{\boldsymbol{b}-\mathbf{1}_i}\varphi_{\mathbf{1}_i}$, and we can regard $\varphi_{\boldsymbol{b}}, \varphi_{\mathbf{1}_i}, \varphi_{\boldsymbol{b}-\mathbf{1}_i}$ as acting (injectively) both on $M_{\boldsymbol{b}+\alpha\boldsymbol{e}}[\mathrm{N}]$ and $M_{\boldsymbol{b}-\mathbf{1}_i+\alpha\boldsymbol{e}}[\mathrm{N}]$. Moreover, the multiplication by x_i, which is an isomorphism $M_{\boldsymbol{b}+\alpha\boldsymbol{e}} \xrightarrow{\sim} M_{\boldsymbol{b}-\mathbf{1}_i+\alpha\boldsymbol{e}}$, is such that $x_i \otimes 1$ commutes with $\varphi_{\boldsymbol{b}-\mathbf{1}_i}$. In such a way, we can regard $(\mathrm{gr}_\alpha^V M_g)_{\boldsymbol{b}-\mathbf{1}_i}$ as the cokernel of $\varphi_{\boldsymbol{b}-\mathbf{1}_i}$ acting on $M_{\boldsymbol{b}+\alpha\boldsymbol{e}}[\mathrm{N}]$. We can then define c_i and v_i as naturally induced by the following commutative diagrams:

$$\begin{array}{ccccc}
M_{\boldsymbol{b}+\alpha\boldsymbol{e}}[\mathrm{N}] & \xrightarrow{\varphi_{\boldsymbol{b}-\mathbf{1}_i}} & M_{\boldsymbol{b}+\alpha\boldsymbol{e}}[\mathrm{N}] & \twoheadrightarrow & (\mathrm{gr}_\alpha^V M_g)_{\boldsymbol{b}-\mathbf{1}_i} \\
\| & & \downarrow{\scriptstyle \varphi_{\mathbf{1}_i}} & & \downarrow{\scriptstyle \mathrm{c}_i(\boldsymbol{b})} \\
M_{\boldsymbol{b}+\alpha\boldsymbol{e}}[\mathrm{N}] & \xrightarrow[\varphi_{\boldsymbol{b}}]{} & M_{\boldsymbol{b}+\alpha\boldsymbol{e}}[\mathrm{N}] & \twoheadrightarrow & (\mathrm{gr}_\alpha^V M_g)_{\boldsymbol{b}}
\end{array}$$

resp.

$$\begin{array}{ccccc}
M_{\boldsymbol{b}+\alpha\boldsymbol{e}}[\mathrm{N}] & \xrightarrow{\varphi_{\boldsymbol{b}-\mathbf{1}_i}} & M_{\boldsymbol{b}+\alpha\boldsymbol{e}}[\mathrm{N}] & \twoheadrightarrow & (\mathrm{gr}_\alpha^V M_g)_{\boldsymbol{b}-\mathbf{1}_i} \\
{\scriptstyle \varphi_{\mathbf{1}_i}}\uparrow & & \| & & \uparrow{\scriptstyle \mathrm{v}_i(\boldsymbol{b})} \\
M_{\boldsymbol{b}+\alpha\boldsymbol{e}}[\mathrm{N}] & \xrightarrow[\varphi_{\boldsymbol{b}}]{} & M_{\boldsymbol{b}+\alpha\boldsymbol{e}}[\mathrm{N}] & \twoheadrightarrow & (\mathrm{gr}_\alpha^V M_g)_{\boldsymbol{b}}
\end{array}$$

In other words, $\mathrm{c}_i(\boldsymbol{b})$ is the natural morphism

$$M_{\boldsymbol{b}+\alpha\boldsymbol{e}}[\mathrm{N}]/\operatorname{Im}\varphi_{\boldsymbol{b}-\mathbf{1}_i} \xrightarrow{\varphi_{\mathbf{1}_i}} M_{\boldsymbol{b}+\alpha\boldsymbol{e}}[\mathrm{N}]/\operatorname{Im}\varphi_{\boldsymbol{b}},$$

and $\mathrm{v}_i(\boldsymbol{b})$ is the natural morphism induced by the inclusion $\operatorname{Im}\varphi_{\boldsymbol{b}} \subset \operatorname{Im}\varphi_{\boldsymbol{b}-\mathbf{1}_i}$:

$$M_{\boldsymbol{b}+\alpha\boldsymbol{e}}[\mathrm{N}]/\operatorname{Im}\varphi_{\boldsymbol{b}} \longrightarrow M_{\boldsymbol{b}+\alpha\boldsymbol{e}}[\mathrm{N}]/\operatorname{Im}\varphi_{\boldsymbol{b}-\mathbf{1}_i}.$$

We note that $\mathrm{v}_i(\boldsymbol{b})$ is *surjective*. Moreover,

15.13.12. Proposition. *For $\alpha < 0$, the quiver of $\mathrm{gr}_\alpha^V M_g$ has vertices $(\mathrm{gr}_\alpha^V M_g)_{\boldsymbol{b}} = \operatorname{Coker}\varphi_{\boldsymbol{b}}$ for $\boldsymbol{b} \in [-1,0]^n$ such that*

(1) $\boldsymbol{b} = \boldsymbol{a} - \alpha\boldsymbol{e}$ for some $\boldsymbol{a} \in A + \mathbb{Z}$,
(2) $b_i = 0$ for some $i \in I_{\boldsymbol{e}}$.

It is isomorphic to the quiver defined by the morphisms $\mathrm{c}_i(\boldsymbol{b}), \mathrm{v}_i(\boldsymbol{b})$ as described above.

15.13.c. Induced sesquilinear pairing on nearby cycles. We aim at computing the behaviour of a sesquilinear pairing with respect to the nearby cycle functor along a monomial function. We now consider the setting of Section 15.12 and switch back to the right setting. Suppose we have a sesquilinear pairing $\mathfrak{s} : \mathcal{M}' \otimes_{\mathbb{C}} \overline{\mathcal{M}''} \to \mathfrak{C}_{\Delta^n}$. We still denote by $\mathfrak{s}$ the pushforward sesquilinear pairing $\mathcal{M}_g{}' \otimes \overline{\mathcal{M}_g{}''} \to \mathfrak{C}_{\Delta^{n+1}}$ by the inclusion defined by the graph of $g(x) = x^{\boldsymbol{e}}$.

The purpose of this section is to find a formula (see Proposition 15.13.13 below) for the induced pairing, as defined by (12.5.10 $**$),

$$\mathrm{gr}_\alpha^V \mathfrak{s} : \mathrm{gr}_\alpha^V \mathcal{M}_g{}' \otimes \overline{\mathrm{gr}_\alpha^V \mathcal{M}_g{}''} \longrightarrow \mathfrak{C}_{\Delta^n}$$

for $\alpha \in [-1,0)$ that we fix below. Since we already know that $\mathrm{gr}_\alpha^V \mathcal{M}_g{}', \mathrm{gr}_\alpha^V \mathcal{M}_g{}''$ are of normal crossing type, $\mathrm{gr}_\alpha^V \mathfrak{s}$ is uniquely determined by the pairings

$$(\mathrm{gr}_\alpha^V \mathfrak{s})_{\boldsymbol{b}} : (\mathrm{gr}_\alpha^V M_g')_{\boldsymbol{b}} \otimes \overline{(\mathrm{gr}_\alpha^V M_g'')_{\boldsymbol{b}}} \longrightarrow \mathbb{C}$$

for $\boldsymbol{b} \leqslant 0$. What we have to do then is to derive a formula for $(\mathrm{gr}_\alpha^V \mathfrak{s})_{\boldsymbol{b}}$ in terms of the original pairing $\mathfrak{s}_{\boldsymbol{b}+\alpha\boldsymbol{e}}$. Any element of $(\mathrm{gr}_\alpha^V M_g')_{\boldsymbol{b}}$ can be expanded as $\sum_j n_j' \mathrm{N}^j$, where n_j' is in the image by the morphism in Proposition 15.13.1 of $m_j' \in M'_{\boldsymbol{b}+\alpha\boldsymbol{e}}$, and similarly with $M''_{\boldsymbol{b}+\alpha\boldsymbol{e}}$.

15.13.13. Proposition. *We have*

$$(\mathrm{gr}_\alpha^V \mathfrak{s})_{\boldsymbol{b}}\Big(\sum_{j\geqslant 0} n_j' \mathrm{N}^j, \overline{\sum_{k\geqslant 0} n_k'' \mathrm{N}^k}\Big) = \sum_{j,k\in\mathbb{N}} \mathfrak{s}_{\boldsymbol{b}+\alpha\boldsymbol{e}}\Big(m_j' \operatorname{Res}_{s=0}\Big(\prod_{i\in I_{\boldsymbol{e}}(\boldsymbol{b})} \frac{s^{j+k}}{\mathrm{N}_i - e_i s}\Big), \overline{m_k''}\Big).$$

The residue simply means here the coefficient of $1/s$. Explicitly:

$$\operatorname{Res}_{s=0}\Big(\prod_{i\in I_{\boldsymbol{e}}(\boldsymbol{b})} \frac{s^{j+k}}{\mathrm{N}_i - e_i s}\Big) = \prod_{i\in I_{\boldsymbol{e}}(\boldsymbol{b})} (-1/e_i) \cdot \prod_{\substack{\boldsymbol{\ell}\in\mathbb{N}^{I_{\boldsymbol{e}}(\boldsymbol{b})} \\ \sum_i \ell_i = j+k+1-\#I_{\boldsymbol{e}}(\boldsymbol{b})}} (N_i/e_i)^{\ell_i}. \tag{15.13.14}$$

Proof. Let us fix $m' \in M'_{\boldsymbol{b}+\alpha\boldsymbol{e}} \subset M'_{\boldsymbol{b}+\alpha\boldsymbol{e}}[\mathrm{N}]$ and $m'' \in M''_{\boldsymbol{b}+\alpha\boldsymbol{e}} \subset M''_{\boldsymbol{b}+\alpha\boldsymbol{e}}[\mathrm{N}]$, and let us consider their images n', n'' by the morphism in Proposition 15.13.1. It is enough to prove that, for any $\ell \geqslant 0$,

$$(\mathrm{gr}^V_\alpha \mathfrak{s})_{\boldsymbol{b}}\big(n'\mathrm{N}^\ell, \overline{n''}\big) = \mathfrak{s}_{\boldsymbol{b}+\alpha\boldsymbol{e}}\Big(m' \operatorname{Res}_{s=0}\Big(\textstyle\prod_{i\in I_{\boldsymbol{e}}(\boldsymbol{b})} \dfrac{s^\ell}{\mathrm{N}_i - e_i s}\Big), \overline{m''}\Big). \tag{15.13.15}$$

The induced pairing is given by the formula below, for $\eta_o \in C^\infty_{\mathrm{c}}(\Delta^n)$ and a cut-off function $\chi \in C^\infty_{\mathrm{c}}(\Delta)$ (see (12.5.10 $**$)):

$$\begin{aligned}\langle(\mathrm{gr}^V_\alpha \mathfrak{s})_{\boldsymbol{b}}(n'\mathrm{N}^\ell, \overline{n''}), \eta_o\rangle &= \operatorname{Res}_{s=\alpha}\langle \mathfrak{s}_{\boldsymbol{b}+\alpha\boldsymbol{e}}(m'\otimes 1, \overline{m''\otimes 1}), (t\partial_t - \alpha)^\ell \eta_o |t|^{2s}\chi(t)\rangle\\ &= \operatorname{Res}_{s=\alpha}(s-\alpha)^\ell \langle \mathfrak{s}_{\boldsymbol{b}+\alpha\boldsymbol{e}}(m', \overline{m''}), \eta_o |g|^{2s}\chi(g)\rangle.\end{aligned}$$

Using the symbolic notation of Remark 15.8.5, the current $\mathfrak{s}_{\boldsymbol{b}+\alpha\boldsymbol{e}}(m', \overline{m''})$ is equal to

$$\Omega_n \mathfrak{s}_{\boldsymbol{b}+\alpha\boldsymbol{e}}\bigg(m' \prod_{i\mid b_i+\alpha e_i<0} |x_i|^{-2(1+b_i+\alpha e_i+\mathrm{N}_i)} \prod_{i\mid b_i=e_i=0} \frac{|x_i|^{-2\mathrm{N}_i}-1}{\mathrm{N}_i}, \overline{m''}\bigg) \cdot \prod_{i\mid b_i=e_i=0} \partial_{x_i}\partial_{\overline{x}_i}.$$

The factor $\chi(g)$ does not affect the residue, and $|g|^{2s} = |x|^{2\boldsymbol{e}s}$. If we now define $F(s)$ as the result of pairing the current (renaming $s-\alpha$ by s)

$$s^\ell \cdot \Omega_n \mathfrak{s}_{\boldsymbol{b}+\alpha\boldsymbol{e}}\bigg(\prod_{i\mid b_i+\alpha e_i<0} |x_i|^{2e_i s-2(1+b_i+N_i)} \prod_{i\mid b_i=e_i=0} \frac{|x_i|^{-2N_i}-1}{N_i}\, m', \overline{m''}\bigg)$$

against the test function $\prod_{i\mid b_i=e_i=0} \partial_{x_i}\partial_{\overline{x}_i}\eta_o(x)$, then $F(s)$ is holomorphic on the half-space $\operatorname{Re} s > 0$, and

$$\langle(\mathrm{gr}^V_\alpha \mathfrak{s})_{\boldsymbol{b}}(n'\mathrm{N}^\ell, \overline{n''}), \eta_o\rangle = \operatorname{Res}_{s=0} F(s).$$

Recall the notation $I_{\boldsymbol{e}} = \{i \in I \mid e_i \neq 0\}$ and $I_{\boldsymbol{e}}(\boldsymbol{b}) = \{i \in I_{\boldsymbol{e}} \mid b_i = 0\}$. Looking at

$$\prod_{i\in I_{\boldsymbol{e}}(\boldsymbol{b})} |x_i|^{2e_i s-2-2\mathrm{N}_i} \prod_{i\in I_{\boldsymbol{e}}\smallsetminus I_{\boldsymbol{e}}(\boldsymbol{b})} |x_i|^{2e_i s-2(1+b_i)-2\mathrm{N}_i} \prod_{i\mid b_i=e_i=0} \frac{|x_i|^{-2\mathrm{N}_i}-1}{\mathrm{N}_i},$$

we notice that the second factor is holomorphic near $s=0$; the problem is therefore the behavior of the first factor near $s=0$. To understand what is going on, we apply integration by parts, in the form of the identity (6.8.6 $**$); the result is that $F(s)$ is equal to the pairing between the current

$$\Omega_n \mathfrak{s}_{\boldsymbol{b}+\alpha\boldsymbol{e}}\bigg(s^\ell \prod_{i\in I_{\boldsymbol{e}}(\boldsymbol{b})} \frac{|x_i|^{2e_i s-2\mathrm{N}_i}-1}{(\mathrm{N}_i - e_i s)^2} \prod_{i\mid b_i<0} |x_i|^{2e_i s-2(1+b_i+\mathrm{N}_i)} \prod_{i\mid b_i=e_i=0} \frac{|x_i|^{-2\mathrm{N}_i}-1}{\mathrm{N}_i}\, m', \overline{m''}\bigg)$$

and the test function

$$\prod_{i\mid b_i=0} \partial_{x_i}\partial_{\overline{x}_i}\eta_o(x).$$

The new function is meromorphic on a half-space of the form $\operatorname{Re} s > -\varepsilon$, with a unique pole of some order at the point $s=0$. We know a priori (Proposition 15.8.3) that $\operatorname{Res}_{s=0} F(s)$ can be expanded into a linear combination of $\langle u_{\boldsymbol{b},\boldsymbol{p}}, \eta_o\rangle$ for certain $\boldsymbol{p} \in \mathbb{N}^n$, and that $(\mathrm{gr}^V_\alpha \mathfrak{s})_{\boldsymbol{b}}(n'\mathrm{N}^\ell, \overline{n''})$ is the coefficient of $u_{\boldsymbol{b},0}$ in this expansion; here

$$u_{\boldsymbol{b},0} = \Big[\Omega_n \textstyle\prod_{i\mid b_i<0} |x|^{-2(1+b_i)} \prod_{i\in I_{\boldsymbol{e}}(\boldsymbol{b})} \mathrm{L}(x_i)\Big] \cdot \partial_{x_i}\partial_{\overline{x}_i}.$$

Throwing away all the terms that cannot contribute to $\langle u_{\boldsymbol{b},0}, \eta_o\rangle$, we eventually arrive at (15.13.15). In particular, we see from Formula (15.13.14) that $(\mathrm{gr}_\alpha^V \mathfrak{s})_{\boldsymbol{b}}(n', \overline{n''}) = 0$ if $\#I_{\boldsymbol{e}}(\boldsymbol{b}) \geqslant 2$. $\square$

15.14. End of the proof of Theorem 15.11.1

We now include the F-filtration in the picture, and we will of course make strong use of the distributivity property of the family $(F, V^{(1)}, \dots, V^{(n)})$.

15.14.a. Strict $\mathbb{R}$-specializability along (g). We first enhance the surjective morphism (15.12.5 $*$) of Proposition 15.12.5 to a filtered surjective morphism. For that purpose, we equip $\mathcal{K}_\alpha^0 = V_{\alpha\boldsymbol{e}}^{(\boldsymbol{r})}\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{D}'_X$ with the following $F_\bullet\mathcal{D}'_X$-filtration:

$$F_p\mathcal{K}_\alpha^0 = \sum_{q+k=p} F_q(V_{\alpha\boldsymbol{e}}^{(\boldsymbol{r})}\mathcal{M}) \otimes_{\mathcal{O}_X} F_k\mathcal{D}'_X, \tag{15.14.1}$$

with $F_q(V_{\alpha\boldsymbol{e}}^{(\boldsymbol{r})}\mathcal{M}) := F_q\mathcal{M} \cap V_{\alpha\boldsymbol{e}}^{(\boldsymbol{r})}\mathcal{M}$, and with respect to which the operators $\cdot_{\mathrm{tens}}\delta_j$ are of order one. We then set

$$F'_pV_\alpha\mathcal{M}_g = \mathrm{image}\big[F_p\mathcal{K}_\alpha^0 \longrightarrow V_\alpha\mathcal{M}_g\big].$$

On the other hand, we set as usual

$$F_pV_\alpha\mathcal{M}_g = F_p\mathcal{M}_g \cap V_\alpha\mathcal{M}_g.$$

15.14.2. Proposition. *For $\alpha < 0$ and any $p \in \mathbb{Z}$, the filtrations $F_pV_\alpha\mathcal{M}_g$ and $F'_pV_\alpha\mathcal{M}_g$ coincide.*

Proof. The inclusion $F'_pV_\alpha\mathcal{M}_g \subset F_pV_\alpha\mathcal{M}_g$ is clear. For the reverse inclusion, it is enough to prove that, for any $p \in \mathbb{Z}$, we have

$$F_p\mathcal{M}_g \cap F'_{p+1}V_\alpha\mathcal{M}_g \subset F'_pV_\alpha\mathcal{M}_g.$$

Indeed, by an easy induction, this implies the inclusion $F_p\mathcal{M}_g \cap F'_{p+\ell}V_\alpha\mathcal{M}_g \subset F'_pV_\alpha\mathcal{M}_g$ for any $\ell \geqslant 1$, and thus, letting $\ell \to \infty$, $F_pV_\alpha\mathcal{M}_g \subset F'_pV_\alpha\mathcal{M}_g$.

On the other hand, the above inclusion is equivalent to the injectivity of

$$\mathrm{gr}^{F'}V_\alpha\mathcal{M}_g \longrightarrow \mathrm{gr}^F\mathcal{M}_g. \tag{15.14.3}$$

By Proposition 15.12.5, the surjective morphism $\mathcal{K}_\alpha^0 \to V_\alpha\mathcal{M}_g$ factorizes as

$$\mathcal{K}_\alpha^0 \longrightarrow\!\!\!\!\!\rightarrow H^0(\mathcal{K}_\alpha^\bullet) \xrightarrow{\ \sim\ } V_\alpha\mathcal{M}_g, \tag{15.14.4}$$

and by definition the morphism $F_p\mathcal{K}_\alpha^0 \to F'_pV_\alpha\mathcal{M}_g$ is surjective. As the differentials of the Koszul complex are filtered up to a shift, it follows that we have a commutative diagram

$$\begin{array}{ccc} H^0(\mathrm{gr}^F\mathcal{K}_\alpha^\bullet) & \longrightarrow\!\!\!\!\!\rightarrow & \mathrm{gr}^{F'}V_\alpha\mathcal{M}_g \\ & \searrow & \Big\downarrow{\scriptstyle (15.14.3)} \\ & & \mathrm{gr}^F\mathcal{M}_g \end{array}$$

and it is thus enough to prove:

15.14.5. Lemma. *The natural morphism* $H^0(\mathrm{gr}^F\mathcal{K}_\alpha^\bullet)\to\mathrm{gr}^F\mathcal{M}_g$ *is injective.*

***Proof of Lemma* 15.14.5.** In order to manipulate the filtration $F_\bullet\mathcal{K}_\alpha^0$ and its graded objects, it is convenient to introduce the auxiliary filtration

$$G_q\mathcal{K}_\alpha^0 := V_{\alpha\boldsymbol{e}}^{(\boldsymbol{r})}\mathcal{M}\otimes_{\mathcal{O}_X}F_q\mathcal{D}'_X,$$

and correspondingly,

$$G_q\mathcal{M}_g=\bigoplus_{j\leqslant q}\mathcal{M}\otimes\partial_t^j$$

which induces in a natural way a filtration on $\mathrm{gr}^F\mathcal{M}_g$, so that, denoting as usual by $G_qH^0(\mathrm{gr}^F\mathcal{K}_\alpha^\bullet)$ the image of $H^0(G_q\mathrm{gr}^F\mathcal{K}_\alpha^\bullet)$ in $H^0(\mathrm{gr}^F\mathcal{K}_\alpha^\bullet)$, it is sufficient to prove the injectivity of

$$\mathrm{gr}^GH^0(\mathrm{gr}^F\mathcal{K}_\alpha^\bullet)\longrightarrow\mathrm{gr}^G\mathrm{gr}^F\mathcal{M}_g.$$

We will prove:

15.14.6. Lemma. *The complex* $\mathrm{gr}^G\mathrm{gr}^F\mathcal{K}_\alpha^\bullet=\mathrm{gr}^F\mathrm{gr}^G\mathcal{K}_\alpha^\bullet$ *has nonzero cohomology in degree* 0 *at most and the natural morphism*

$$H^0(\mathrm{gr}^F\mathrm{gr}^G\mathcal{K}_\alpha^\bullet)\longrightarrow\mathrm{gr}^F\mathrm{gr}^G\mathcal{M}_g\tag{15.14.6 $*$}$$

is injective.

From the first part of Lemma 15.14.6 we only make use of the vanishing of $H^{-1}(\mathrm{gr}^G\mathrm{gr}^F\mathcal{K}_\alpha^\bullet)$, which implies that $H^0(G_{j-1}\mathrm{gr}^F\mathcal{K}_\alpha^\bullet)\to H^0(G_j\mathrm{gr}^F\mathcal{K}_\alpha^\bullet)$ is injective for every j. Therefore (degeneration at E_1 of the spectral sequence),

$$\mathrm{gr}^GH^0(\mathrm{gr}^F\mathcal{K}_\alpha^\bullet)=H^0(\mathrm{gr}^G\mathrm{gr}^F\mathcal{K}_\alpha^\bullet)=H^0(\mathrm{gr}^F\mathrm{gr}^G\mathcal{K}_\alpha^\bullet),$$

so the injectivity of (15.14.6 $*$) concludes the proof of Lemma 15.14.5 and thus that of Proposition 15.14.2. □

***Proof of Lemma* 15.14.6.** If we omit the F-filtration, we have proved the corresponding statement in Proposition 15.12.5 by reducing the proof to the monodromic case, a strategy which does not apply in the presence of F.

In the following, we make use of the identifications, using the notation of Proposition 15.7.13(6) and omitting the functor p'^{-1} in the notation for the sake of simplicity,

$$\mathcal{K}_\alpha^0=V_{\alpha\boldsymbol{e}}^{(\boldsymbol{r})}\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{D}'_X\simeq V_{\alpha\boldsymbol{e}}^{(\boldsymbol{r})}\mathcal{M}\otimes_{\mathcal{O}_{X'}}\mathcal{D}_{X'}\simeq V_{\alpha\boldsymbol{e}}^{(\boldsymbol{r})}\mathcal{M}\otimes_{\mathbb{C}}\mathbb{C}[\partial_{x'}],$$

and correspondingly for the F- and the G-filtrations.

On the one hand, we have

$$F_p\mathrm{gr}_q^G\mathcal{K}_\alpha^0=F_{p-q}V_{\alpha\boldsymbol{e}}^{(\boldsymbol{r})}\mathcal{M}\otimes_{\mathbb{C}}\mathbb{C}[\xi']_q,$$

where $\mathbb{C}[\xi']_q$ consists of polynomials of degree $\leqslant q$ in $\xi'=(\xi_i)_{i\in I_{\boldsymbol{e}}}$ (class of ∂_{x_i}), and thus[(2)]

$$\mathrm{gr}^F\mathrm{gr}^G\mathcal{K}_\alpha^0\simeq(\mathrm{gr}^FV_{\alpha\boldsymbol{e}}^{(\boldsymbol{r})}\mathcal{M})\otimes_{\mathcal{O}_{X'}}\mathcal{O}_{X'}[\xi']\simeq(\mathrm{gr}^FV_{\alpha\boldsymbol{e}}^{(\boldsymbol{r})}\mathcal{M})\otimes_{\mathbb{C}}\mathbb{C}[\xi'].$$

[(2)] In the following, we do not make precise the bi-grading of the objects and how the isomorphisms are bi-graded, as it is straightforward.

The bi-graded endomorphism corresponding to $\cdot_{\rm tens}\delta_j$ is $(x_j\otimes\xi_j/e_j-x_1\otimes\xi_1/e_1)$. Since $\operatorname{gr}^FV^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}}\mathcal{M}$ is $\mathcal{O}_{X'}$-flat (see Proposition 15.9.4(3)), the sequence $(x_2,\dots,x_r)$ is regular on $\operatorname{gr}^FV^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}}\mathcal{M}$, and Lemma 15.12.8 yields the first part of the lemma.

On the other hand, $\operatorname{gr}^G\mathcal{M}_g=\mathcal{M}[\tau]$, where τ is the class of ∂_t, and $\operatorname{gr}^F\operatorname{gr}^G\mathcal{M}_g=(\operatorname{gr}^F\mathcal{M})[\tau]$. The morphism $\operatorname{gr}^F\operatorname{gr}^G\mathcal{K}^0_\alpha\to\operatorname{gr}^F\operatorname{gr}^G\mathcal{M}_g$ is the morphism

$$(\operatorname{gr}^FV^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}}\mathcal{M})[\xi']\longrightarrow(\operatorname{gr}^F\mathcal{M})[\tau]$$

induced by the natural morphism $\operatorname{gr}^FV^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}}\mathcal{M}\to\operatorname{gr}^F\mathcal{M}$ and sending ξ_i to $\partial g/\partial x_i\cdot\tau$. It factorizes through the inclusion $(\operatorname{gr}^FV^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}}\mathcal{M})[\tau]\to(\operatorname{gr}^F\mathcal{M})[\tau]$. Let us also recall that the localization morphism $\operatorname{gr}^FV^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}}\mathcal{M}\to(\operatorname{gr}^FV^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}}\mathcal{M})(g^{-1})$ is injective (as follows from the first line of (15.9.7) for any $i\in\{1,\dots,r\}$).

15.14.7. Assertion. *The Koszul complex*

$$K\Big(\big((\operatorname{gr}^FV^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}}\mathcal{M})(g^{-1})/(\operatorname{gr}^FV^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}}\mathcal{M})\big)[\xi'],(x_j\xi_j/e_j-x_1\xi_1/e_1)_{j=2,\dots,r}\Big)$$

has zero cohomology in negative degrees.

Before proving the assertion, let us check that the assertion implies the injectivity of (15.14.6 $*$). We wish to prove the injectivity of

$$(\operatorname{gr}^FV^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}}\mathcal{M})[\xi']/(x_j\xi_j/e_j-x_1\xi_1/e_1)_{j=2,\dots,r}\longrightarrow(\operatorname{gr}^FV^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}}\mathcal{M})[\tau]\qquad\xi_i\longmapsto\partial g/\partial x_i\cdot\tau.\tag{15.14.8}$$

It is easy to see that its localization by g is an isomorphism. It is therefore enough to prove that the localization morphism for the left-hand side of (15.14.8) is injective. This is the natural morphism

$$H^0(\operatorname{gr}^F\operatorname{gr}^G\mathcal{K}^\bullet_\alpha)\longrightarrow H^0(\operatorname{gr}^F\operatorname{gr}^G\mathcal{K}^\bullet_\alpha(*g)),$$

so it is enough to check that $H^{-1}\big((\operatorname{gr}^F\operatorname{gr}^G\mathcal{K}^\bullet_\alpha(*g))/(\operatorname{gr}^F\operatorname{gr}^G\mathcal{K}^\bullet_\alpha)\big)=0$. This in turn follows from the assertion.

In order to end the proof of Lemma 15.14.5, we are left with proving the assertion. Let us set $h=x_1\cdots x_r$. Since

$$h^k:(\operatorname{gr}^FV^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}}\mathcal{M})h^{-k}/(\operatorname{gr}^FV^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}}\mathcal{M})h^{-k+1}\longrightarrow(\operatorname{gr}^FV^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}}\mathcal{M})/(\operatorname{gr}^FV^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}}\mathcal{M})h,\quad k\geqslant0$$

is an isomorphism, an easy induction reduces to proving that the Koszul complex of $\big((\operatorname{gr}^FV^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}}\mathcal{M})/(\operatorname{gr}^FV^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}}\mathcal{M})h\big)[\xi']$ with respect to $(x_j\xi_j/e_j-x_1\xi_1/e_1)_{j=2,\dots,r}$ has zero cohomology in negative degrees. It is therefore enough to prove that the Koszul complex of $(\operatorname{gr}^FV^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}}\mathcal{M})[\xi']$ with respect to $\big(h,(x_j\xi_j/e_j-x_1\xi_1/e_1)_{j=2,\dots,r}\big)$ has zero cohomology in negative degrees, and furthermore (see Exercise 15.2), it is enough to check that $\big(h,(x_j\xi_j/e_j-x_1\xi_1/e_1)_{j=2,\dots,r}\big)$ is a regular sequence on $(\operatorname{gr}^FV^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}}\mathcal{M})\otimes_{\mathcal{O}_{X'}}\mathcal{O}_{X'}[\xi']=(\operatorname{gr}^FV^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}}\mathcal{M})[\xi']$. Lastly, since $\operatorname{gr}^FV^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}}\mathcal{M}$ is $\mathcal{O}_{X'}$-flat (see Proposition 15.9.4(3)), it is enough to check that it is a regular sequence on $\mathcal{O}_{X'}[\xi']$, equivalently, the sequence $\big((x_j\xi_j/e_j-x_1\xi_1/e_1)_{j=2,\dots,r}\big)$ is regular on $(\mathcal{O}_{X'}/(h))[\xi']$.

For that purpose, we identify $\mathcal{O}_{X'}/(h)$ with $\bigoplus_{i=1}^r \mathcal{O}_{D'_i}$ with $\mathcal{O}_{D'_i} = \mathcal{O}_{X'}/(x_i)$ and we consider each term independently. Let us fix $i_o \in \{1,\dots,r\}$. Then, on $\mathcal{O}_{D'_i}[\xi']$, the sequence can be replaced with $\big((x_i\xi_i/e_i - x_{i_o}\xi_{i_o}/e_{i_o})_{i\in\{1,\dots,\widehat{i_o},\dots r\}}\big)$, for which the regularity follows from Lemma 15.12.8. □

We can now prove the first part of Theorem 15.11.1, namely:

***15.14.9. Corollary* (ℝ-specializability and middle extension along (g))**
Let $(\mathcal{M}, F_\bullet\mathcal{M})$ be a coherently F-filtered $\mathcal{D}_X$-module of normal crossing type along D. Assume that $(\mathcal{M}, F_\bullet\mathcal{M})$ is a middle extension along $D_{i\in I}$. Then $(\mathcal{M}, F_\bullet\mathcal{M})$ is $\mathbb{R}$-specializable and a middle extension along (g).

***Proof*.** We refer to Definition 10.5.1 for the notion of filtered ℝ-specializability and middle extension of $(\mathcal{M}, F_\bullet\mathcal{M})$ along (g), that is, of $(\mathcal{M}_g, F_\bullet\mathcal{M}_g)$ along (t).

We first wish to prove that the multiplication by t induces an isomorphism $F_pV_\alpha\mathcal{M}_g \xrightarrow{\sim} F_pV_{\alpha-1}\mathcal{M}_g$ if $\alpha < 0$. Since we already know that it is injective by definition of the Kashiwara-Malgrange filtration, it suffices to prove that it is onto. By the formulas (15.11.3 ∗) and (15.12.1 ∗), the multiplication by t is induced by $g\otimes 1$ on $\mathcal{K}^0_\alpha$. Since $g : F_pV^{(\boldsymbol{r})}_{\alpha\boldsymbol{e}}\mathcal{M} \to F_pV^{(\boldsymbol{r})}_{(\alpha-1)\boldsymbol{e}}\mathcal{M}$ is an isomorphism according to (15.9.7), it follows that $g\otimes 1 : F_p\mathcal{K}^0_\alpha \to F_p\mathcal{K}^0_\alpha$ is also an isomorphism and we deduce from Proposition 15.14.2 that $t : F_pV_\alpha\mathcal{M}_g \xrightarrow{\sim} F_pV_{\alpha-1}\mathcal{M}_g$ is onto.

We next aim at proving that, for $\alpha \geqslant 0$ and any $p\in\mathbb{Z}$,

$$F_pV_\alpha\mathcal{M}_g := F_p\mathcal{M}_g \cap V_\alpha\mathcal{M}_g = (F_p\mathcal{M}_g \cap V_{<\alpha}\mathcal{M}_g) + (F_{p-1}V_{\alpha-1}\mathcal{M}_g)\cdot\partial_t,$$

and since we already know that $\mathcal{M}_g$ is an intermediate extension along (t), we are left with proving the inclusion ⊂. By definition, $F_p\mathcal{M}_g = \bigoplus_{\ell\geqslant 0} F_{p-k}\mathcal{M}\otimes\partial_t^k$. On the other hand,

$$F_{p-k}\mathcal{M} = \sum_{\boldsymbol{\ell}\in\mathbb{N}^n} F_{p-k-|\boldsymbol{\ell}|}V^{(\boldsymbol{n})}_{<\boldsymbol{0}}\mathcal{M}\cdot\partial_x^{\boldsymbol{\ell}},$$

according to Proposition 15.9.11(3) and Remark 15.9.13. Then, if $m = \sum_{k\geqslant 0} m_k\otimes\partial_t^k$ belongs to $F_p\mathcal{M}_g\cap V_\alpha\mathcal{M}_g$, and if we set $m_0 = \sum_{\boldsymbol{\ell}} m_{0,\boldsymbol{\ell}}\partial_x^{\boldsymbol{\ell}}$ with $m_{0,\boldsymbol{\ell}} \in F_{p-|\boldsymbol{\ell}|}V^{(\boldsymbol{n})}_{<\boldsymbol{0}}\mathcal{M}$, the second line of (15.11.3 ∗) shows that we can write

$$m = \sum_{\boldsymbol{\ell}}(m_{0,\boldsymbol{\ell}}\otimes 1)\partial_x^{\boldsymbol{\ell}} + m', \quad\text{with}\ \begin{cases}\sum_{\boldsymbol{\ell}}(m_{0,\boldsymbol{\ell}}\otimes 1)\partial_x^{\boldsymbol{\ell}} \in F_pV_{<0}\mathcal{M}_g \subset F_pV_\alpha\mathcal{M}_g,\\ m' \in F_p\mathcal{M}_g\cap V_\alpha\mathcal{M}_g\cap(\mathcal{M}_g\cdot\partial_t).\end{cases}$$

Now, by definition, $F_p\mathcal{M}_g\cap(\mathcal{M}_g\cdot\partial_t) = F_{p-1}\mathcal{M}_g\cdot\partial_t$. Moreover, since $\partial_t : \mathrm{gr}^V_a\mathcal{M}_g \to \mathrm{gr}^V_{a+1}\mathcal{M}_g$ is injective for $a \neq -1$, we deduce easily that, for $\alpha\geqslant 0$, $V_\alpha\mathcal{M}_g\cap(\mathcal{M}_g\cdot\partial_t) = V_{\alpha-1}\mathcal{M}_g\cdot\partial_t$. In conclusion,

$$F_p\mathcal{M}_g\cap V_\alpha\mathcal{M}_g\cap(\mathcal{M}_g\cdot\partial_t) = (F_{p-1}\mathcal{M}_g\cdot\partial_t)\cap(V_{\alpha-1}\mathcal{M}_g\cdot\partial_t) = (F_{p-1}\mathcal{M}_g\cap V_{\alpha-1}\mathcal{M}_g)\cdot\partial_t,$$

where the latter equality follows from the injectivity of ∂_t on $\mathcal{M}_g$, and so

$$F_pV_\alpha\mathcal{M}_g \subset (F_p\mathcal{M}_g\cap V_{<0}\mathcal{M}_g) + (F_{p-1}V_{\alpha-1}\mathcal{M}_g)\cdot\partial_t,$$

as desired. □

15.14.b. Normal crossing properties of $(\psi_{g,\lambda}\mathcal{M}, F_\bullet\psi_{g,\lambda}\mathcal{M})$ along D. In this section, we fix $\alpha \in [-1, 0)$. As we already know that $\mathrm{gr}_\alpha^V\mathcal{M}_g$ is of normal crossing type along D and $\mathbb{R}$-specializable along each D_i $(i = 1, \ldots, r)$ by Corollaries 15.12.10 and 15.12.12, it remains to prove the $\mathbb{R}$-specializability of $(\mathrm{gr}_\alpha^V\mathcal{M}_g, F_\bullet\mathrm{gr}_\alpha^V\mathcal{M}_g)$ along each D_i and the distributivity of the family $(F_\bullet\mathrm{gr}_\alpha^V\mathcal{M}_g, V_\bullet^{(1)}\mathrm{gr}_\alpha^V\mathcal{M}_g, \ldots, V_\bullet^{(n)}\mathrm{gr}_\alpha^V\mathcal{M}_g)$. Furthermore, as we already know that $\mathrm{gr}_\alpha^V\mathcal{M}_g$ is of normal crossing type, Proposition 15.9.14 prompts us to consider the logarithmic module $(\mathrm{gr}_\alpha^V\mathcal{M}_g)_{\leqslant\mathbf{0}} = V_{\mathbf{0}}^{(\boldsymbol{n})}(\mathrm{gr}_\alpha^V\mathcal{M}_g)$ and its induced filtrations

$$(F_\bullet(\mathrm{gr}_\alpha^V\mathcal{M}_g)_{\leqslant\mathbf{0}}, V_\bullet^{(1)}(\mathrm{gr}_\alpha^V\mathcal{M}_g)_{\leqslant\mathbf{0}}, \ldots, V_\bullet^{(n)}(\mathrm{gr}_\alpha^V\mathcal{M}_g)_{\leqslant\mathbf{0}}).$$

This approach will prove effective to obtain an explicit expression of the filtration on $\mathrm{gr}_{\boldsymbol{b}}^{V^{(\boldsymbol{n})}}\mathrm{gr}_\alpha^V\mathcal{M}_g$ in terms of the presentation of Corollary 15.13.6.

We recall the notation:

- $g = x^{\boldsymbol{e}}$, $r = \#I_{\boldsymbol{e}} = \#\{i \in I \mid e_i \neq 0\}$,
- $\mathcal{D}'_X = \mathcal{O}_X\langle\partial_{x_1}, \ldots, \partial_{x_r}\rangle$,
- $V_{\mathbf{0}}^{(\boldsymbol{r})}(\mathcal{D}'_X) = \mathcal{O}_X\langle x_1\partial_{x_1}, \ldots, x_r\partial_{x_r}\rangle$.

We now emphasize $V_{\alpha\boldsymbol{e}}^{(\boldsymbol{n})}\mathcal{M}$ (considering $\alpha\boldsymbol{e}$ as an n-multi-index with entries equal to 0 if $i \notin I_{\boldsymbol{e}}$), which is a coherent $V_{\mathbf{0}}^{(\boldsymbol{n})}(\mathcal{D}_X)$-module and that we will also consider as a $V_{\mathbf{0}}^{(\boldsymbol{r})}(\mathcal{D}'_X)$-module (by forgetting the action of $x_i\partial_{x_i}$ for $i \notin I_{\boldsymbol{e}}$).

In a way similar to that of Section 15.12.b, we set $\mathcal{K}^0_{\alpha,\leqslant\mathbf{0}} = V_{\alpha\boldsymbol{e}}^{(\boldsymbol{n})}\mathcal{M} \otimes_{\mathcal{O}_X} V_{\mathbf{0}}^{(\boldsymbol{r})}(\mathcal{D}'_X)$ that we regard with its two structures of a $V_{\mathbf{0}}^{(\boldsymbol{n})}(\mathcal{D}_X)$-module (the trivial one and the tensor one). For each $i \in I$ and $b_i \leqslant 0$, we set

$$V_{b_i}^{(i)}\mathcal{K}^0_{\alpha,\leqslant\mathbf{0}} = (V_{\alpha\boldsymbol{e}+b_i\mathbf{1}_i}^{(\boldsymbol{n})}\mathcal{M}) \otimes_{\mathcal{O}_X} V_{\mathbf{0}}^{(\boldsymbol{r})}(\mathcal{D}'_X),$$

so that, for $\boldsymbol{b} \leqslant 0$,

$$V_{\boldsymbol{b}}^{(\boldsymbol{n})}\mathcal{K}^0_{\alpha,\leqslant\mathbf{0}} = (V_{\alpha\boldsymbol{e}+\boldsymbol{b}}^{(\boldsymbol{n})}\mathcal{M}) \otimes_{\mathcal{O}_X} V_{\mathbf{0}}^{(\boldsymbol{r})}(\mathcal{D}'_X),$$

and in particular, $V_{\mathbf{0}}^{(\boldsymbol{n})}\mathcal{K}^0_{\alpha,\leqslant\mathbf{0}} = \mathcal{K}^0_{\alpha,\leqslant\mathbf{0}}$.

According to Corollary 15.12.12, the composed morphism $\mathcal{K}^0_\alpha \to V_\alpha\mathcal{M}_g \to \mathrm{gr}_\alpha^V\mathcal{M}_g$ sends $\mathcal{K}^0_{\alpha,\leqslant\mathbf{0}}$ *onto* $(\mathrm{gr}_\alpha^V\mathcal{M}_g)_{\leqslant\mathbf{0}}$ and, arguing similarly, we find that for each $\boldsymbol{b} \leqslant 0$, $V_{\boldsymbol{b}}^{(\boldsymbol{n})}(\mathrm{gr}_\alpha^V\mathcal{M}_g)_{\leqslant\mathbf{0}}$ is the image of $V_{\boldsymbol{b}}^{(\boldsymbol{n})}\mathcal{K}^0_{\alpha,\leqslant\mathbf{0}}$.

We denote by $(V_\alpha\mathcal{M}_g)_{\leqslant\mathbf{0}}$ the image of $\mathcal{K}^0_{\alpha,\leqslant\mathbf{0}}$ in $V_\alpha\mathcal{M}_g$, so that its image in $\mathrm{gr}_\alpha^V\mathcal{M}$ is nothing but $(\mathrm{gr}_\alpha^V\mathcal{M}_g)_{\leqslant\mathbf{0}}$. Arguing as in Corollary 15.12.12, we find that $(\mathrm{gr}_\alpha^V\mathcal{M}_g)_{\leqslant\mathbf{0}}$ is also equal to $(V_\alpha\mathcal{M}_g)_{\leqslant\mathbf{0}}/(V_{<\alpha}\mathcal{M}_g)_{\leqslant\mathbf{0}}$.

We consider the complex $(\mathcal{K}^\bullet_{\alpha,\leqslant\mathbf{0}}, (\cdot_{\mathrm{tens}}\delta_j)_{j=2,\ldots,r})$, which is a complex of right $V_{\mathbf{0}}^{(\boldsymbol{n})}(\mathcal{D}_X)$-modules with the trivial structure, and the quotient complex $\mathcal{K}^\bullet_{[\alpha],\leqslant\mathbf{0}} := \mathcal{K}^\bullet_{\alpha,\leqslant\mathbf{0}}/\mathcal{K}^\bullet_{\alpha-\varepsilon,\leqslant\mathbf{0}}$ ($\varepsilon > 0$ small enough). Let us first check the logarithmic analogue of Proposition 15.12.5.

15.14.10. Lemma. *The Koszul complex $\mathcal{K}^\bullet_{[\alpha],\leqslant\mathbf{0}}$ is a resolution of $(\mathrm{gr}_\alpha^V\mathcal{M}_g)_{\leqslant\mathbf{0}}$.*

Proof. It similar, but simpler, than that of Proposition 15.12.5. It is enough to prove that for each $\alpha < 0$, the complex $\mathcal{K}^\bullet_{\alpha,\leqslant\mathbf{0}}$ is a resolution of $(V_\alpha\mathcal{M}_g)_{\leqslant\mathbf{0}}$.

For the exactness in negative degree, we filter $V_{\mathbf{0}}^{(\boldsymbol{n})}\mathcal{D}'_X$ by the degree of differential operators so that the graded complex reads

$$\big(V_{\alpha\boldsymbol{e}}^{(\boldsymbol{n})}\mathcal{M}\otimes_{\mathbb{C}}\mathbb{C}[\eta_1,\dots,\eta_n],\mathrm{Id}\otimes(\eta_j/e_j-\eta_1/e_1)_{j=2,\dots,r}\big),$$

where η_i is the class of $x_i\partial_{x_i}$, and the sequence $(\eta_j/e_j-\eta_1/e_1)_{j=2,\dots,r}$ is clearly regular, hence the exactness.

As in Proposition 15.12.5 we see that the image Im of $\mathcal{K}^{-1}_{\alpha,\leqslant\mathbf{0}}\to\mathcal{K}^{0}_{\alpha,\leqslant\mathbf{0}}$ is contained in the kernel of $\mathcal{K}^{0}_{\alpha,\leqslant\mathbf{0}}\to(V_\alpha\mathcal{M}_g)_{\leqslant\mathbf{0}}$. Passing to the monodromic setting, one checks that $\mathcal{K}^{0}_{\alpha,\leqslant\mathbf{0}}=\mathrm{Im}\oplus\bigoplus_{\ell\geqslant0}\bigoplus_{\boldsymbol{b}\leqslant0}M_{\alpha\boldsymbol{e}+\boldsymbol{b}}\otimes(x_1\partial_{x_1})^\ell$. Assume that an element $\sum_{\ell=0}^{\ell_o}\sum_{\boldsymbol{b}\leqslant\mathbf{0}}m_{\boldsymbol{b},\ell}\otimes(x_1\partial_{x_1})^\ell$ of the second term with $m_{\boldsymbol{b},\ell_o}\neq0$ for some $\boldsymbol{b}\leqslant\mathbf{0}$ is sent to zero in $(V_\alpha M_g)_{\leqslant\mathbf{0}}\subset M_g$. The term of maximal degree in ∂_t of its image reads $\sum_{\boldsymbol{b}\leqslant\mathbf{0}}m_{\boldsymbol{b},\ell_o}g^{\ell_o}\otimes\partial_t^{\ell_o}$, so $m_{\boldsymbol{b},\ell_o}g^{\ell_o}$ must be zero for each $\boldsymbol{b}\leqslant\mathbf{0}$. As $\alpha\boldsymbol{e}+\boldsymbol{b}\leqslant\mathbf{0}$ and M is a middle extension of normal crossing type along each D_i, this implies $m_{\boldsymbol{b},\ell_o}=0$ for each $\boldsymbol{b}\leqslant\mathbf{0}$, a contradiction. In conclusion, $\mathrm{Ker}[\mathcal{K}^{0}_{\alpha,\leqslant\mathbf{0}}\to(V_\alpha\mathcal{M}_g)_{\leqslant\mathbf{0}}]$ is equal to $\mathrm{Im}[\mathcal{K}^{-1}_{\alpha,\leqslant\mathbf{0}}\to\mathcal{K}^{0}_{\alpha,\leqslant\mathbf{0}}]$. ∎

We also equip $\mathcal{K}^{0}_{\alpha,\leqslant\mathbf{0}}$ with the filtration

$$F_p\mathcal{K}^{0}_{\alpha,\leqslant\mathbf{0}}=\sum_{q\leqslant p}F_qV_{\alpha\boldsymbol{e}}^{(\boldsymbol{n})}\mathcal{M}\otimes_{\mathcal{O}_X}F_{p-q}V_{\mathbf{0}}^{(\boldsymbol{r})}(\mathcal{D}'_X).$$

The right $\mathcal{D}_X$-module $\mathcal{K}^0_\alpha$ (with its trivial structure) contains $\mathcal{K}^{0}_{\alpha,\leqslant\mathbf{0}}$ and is equal to the $\mathcal{D}_X$-submodule generated by it. Correspondingly we have

$$F_p\mathcal{K}^0_\alpha=\sum_{q\leqslant p}F_q\mathcal{K}^{0}_{\alpha,\leqslant\mathbf{0}}\cdot F_{p-q}\mathcal{D}_X.$$

Indeed, this follows from the property that $V_{\alpha\boldsymbol{e}}^{(\boldsymbol{r})}\mathcal{M}=\sum_{\boldsymbol{k}\in\mathbb{N}^{I\smallsetminus I_{\boldsymbol{e}}}}V_{\alpha\boldsymbol{e}}^{(\boldsymbol{n})}\mathcal{M}\cdot\partial_x^{\boldsymbol{k}}$, as a consequence of (15.9.7).

The surjective map $\mathcal{K}^{0}_{\alpha,\leqslant\mathbf{0}}\to(\mathrm{gr}^V_\alpha\mathcal{M}_g)_{\leqslant\mathbf{0}}$ sends the filtration $F_\bullet\mathcal{K}^{0}_{\alpha,\leqslant\mathbf{0}}$ to a coherent F-filtration that we denote $F'_\bullet(\mathrm{gr}^V_\alpha\mathcal{M}_g)_{\leqslant\mathbf{0}}$. By the previous considerations, the latter filtration generates the filtration $F'_\bullet\mathrm{gr}^V_\alpha\mathcal{M}_g$ (i.e., $F'_p\mathrm{gr}^V_\alpha\mathcal{M}_g=\sum_{q\leqslant p}F'_q(\mathrm{gr}^V_\alpha\mathcal{M}_g)_{\leqslant\mathbf{0}}\cdot F_{p-q}\mathcal{D}_X$), that we know, by Proposition 15.14.2, to be equal to the filtration $F_\bullet\mathrm{gr}^V_\alpha\mathcal{M}_g$. The preceding discussion justifies that, with Proposition 15.9.14, the proof of Theorem 15.11.1 will be achieved with the next proposition.

15.14.11. Proposition. *The family*

$$(F'_\bullet(\mathrm{gr}^V_\alpha\mathcal{M}_g)_{\leqslant\mathbf{0}},V^{(1)}_\bullet(\mathrm{gr}^V_\alpha\mathcal{M}_g)_{\leqslant\mathbf{0}},\dots,V^{(n)}_\bullet(\mathrm{gr}^V_\alpha\mathcal{M}_g)_{\leqslant\mathbf{0}})$$

is distributive and satisfies

- $F'_pV^{(i)}_{b_i}(\mathrm{gr}^V_\alpha\mathcal{M}_g)_{\leqslant\mathbf{0}}\cdot x_i=F'_pV^{(i)}_{b_i-1}(\mathrm{gr}^V_\alpha\mathcal{M}_g)_{\leqslant\mathbf{0}}$ *for every* $i\in I$ *and* $b_i<0$.
- $F'_pV^{(i)}_{-1}(\mathrm{gr}^V_\alpha\mathcal{M}_g)_{\leqslant\mathbf{0}}\cdot\partial_{x_i}\subset F'_{p+1}V^{(i)}_0(\mathrm{gr}^V_\alpha\mathcal{M}_g)_{\leqslant\mathbf{0}}$ *for every* $i\in I$.

Proof of distributivity. For the sake of simplicity, we will give the proof for any family of $\mathbb{Z}$-indexed V-filtrations $V^{(i)}_{\beta_i+\mathbb{Z}}\mathrm{gr}^V_\alpha\mathcal{M}_g$ with fixed $\beta_i\in A_i\subset[-1,0)$ $(i=1,\dots,n)$, so that we can easily interpret distributivity in terms of flatness over a polynomial ring. The general case would need that we replace each $V^{(i)}_{\beta_i+\mathbb{Z}}\mathrm{gr}^V_\alpha\mathcal{M}_g$ by various $V^{(i)}_{\beta_{ij}+\mathbb{Z}}$

with β_{ij} varying in A_i for each $i = 1, \dots, n$ (see (9.3.5)). Distributivity amounts to $\mathbb{C}[z_0, z_1, \dots, z_n]$-flatness of the Rees module $R_{F'V}(\mathrm{gr}_\alpha^V \mathcal{M}_g)$, which is a module over the ring $R = \mathbb{C}[z_0, z_1, \dots, z_n]$, where z_0 resp. z_i $(i = 1, \dots, n)$ is the Rees variable of the filtration $F_\bullet$, resp. $V_\bullet^{(i)}$.

We enhance the complexes $\mathcal{K}^\bullet_{\alpha,\leqslant \mathbf{0}}$ and $\mathcal{K}^\bullet_{[\alpha],\leqslant \mathbf{0}}$ by taking into account the filtrations. We have already defined the filtrations $(F_\bullet, (V_\bullet^{(i)})_{i=1,\dots,n})$ on $\mathcal{K}^0_{\alpha,\leqslant \mathbf{0}}$, hence on each term of the Koszul complex $\mathcal{K}^\bullet_{\alpha,\leqslant \mathbf{0}}$ and on the quotient complex $\mathcal{K}^\bullet_{[\alpha],\leqslant \mathbf{0}} := \mathcal{K}^\bullet_{\alpha,\leqslant \mathbf{0}} / \mathcal{K}^\bullet_{\alpha-\varepsilon,\leqslant \mathbf{0}}$. The isomorphism

$$H^0(\mathcal{K}^\bullet_{[\alpha],\leqslant \mathbf{0}}) \xrightarrow{\sim} (\mathrm{gr}_\alpha^V \mathcal{M}_g)_{\leqslant \mathbf{0}}$$

provided by Lemma 15.14.10 is strictly compatible with each of the filtrations $F_\bullet$, $(V_\bullet^{(i)})_{i=1,\dots,n}$.

With the multi-Rees construction, we focus on the complex $\widetilde{\mathcal{K}}^\bullet_{[\alpha],\leqslant \mathbf{0}} = R_{FV}(\mathcal{K}^\bullet_{[\alpha],\leqslant \mathbf{0}})$ of R-modules, which is a Koszul complex with respect to differentials deduced from $(\cdot_{\mathrm{tens}} \delta_j)_{j=2,\dots,r}$.

15.14.12. Lemma.

(1) *The natural morphism* $R_{FV} H^0(\mathcal{K}^\bullet_{[\alpha],\leqslant \mathbf{0}}) \to R_{F'V}(\mathrm{gr}_\alpha^V \mathcal{M}_g)_{\leqslant \mathbf{0}}$ *is an isomorphism.*

(2) *The Koszul complex* $\widetilde{\mathcal{K}}^\bullet_{[\alpha],\leqslant \mathbf{0}}$ *is exact in negative degrees.*

(3) *The R-module* $H^0(\widetilde{\mathcal{K}}^\bullet_{[\alpha],\leqslant \mathbf{0}})$ *is flat.*

We end the proof of the distributivity property by means of the flatness criterion of Proposition 15.2.6, applied to $H^0(\widetilde{\mathcal{K}}^\bullet_{[\alpha],\leqslant \mathbf{0}})$. Being R-flat by (3), $H^0(\widetilde{\mathcal{K}}^\bullet_{[\alpha],\leqslant \mathbf{0}})$ is nothing but the Rees module of an $(n+1)$-filtration $(F_\bullet, V_\bullet^{(1)}, \dots, V_\bullet^{(n)})$ on $H^0(\widetilde{\mathcal{K}}^\bullet_{[\alpha],\leqslant \mathbf{0}})$ (see Exercise 15.1). Furthermore, theses filtrations are those induced by the corresponding ones on $(\widetilde{\mathcal{K}}^0_{[\alpha],\leqslant \mathbf{0}})$. In other words, $H^0(\widetilde{\mathcal{K}}^\bullet_{[\alpha],\leqslant \mathbf{0}}) = R_{FV} H^0(\widetilde{\mathcal{K}}^\bullet_{[\alpha],\leqslant \mathbf{0}})$ and, by the first point of the lemma, $R_{F'V}(\mathrm{gr}_\alpha^V \mathcal{M}_g)$ is thus R-flat, which is the desired distributivity. □

***Proof of Lemma* 15.14.12(1).** According to our preliminary discussion, the natural morphism $H^0(\mathcal{K}^\bullet_{[\alpha],\leqslant \mathbf{0}}) \to (\mathrm{gr}_\alpha^V \mathcal{M}_g)_{\leqslant \mathbf{0}}$ is an isomorphism, and this isomorphism is strictly compatible with the filtrations $V_\bullet^{(i)}$ $(i = 1, \dots, n)$ and $F_\bullet, F'_\bullet$. This yields the first point. □

***Proof of Lemma* 15.14.12(2).** We write

$$\mathcal{K}^0_{[\alpha],\leqslant \mathbf{0}} = (V^{(\boldsymbol{n})}_{\alpha \boldsymbol{e}} \mathcal{M} / V^{(\boldsymbol{n})}_{(\alpha-\varepsilon)\boldsymbol{e}} \mathcal{M}) \otimes_{\mathbb{C}} \mathbb{C}[x' \partial_{x'}]$$

with $x' = (x_1, \dots, x_r)$. We have $R_{FV} \mathbb{C}[x' \partial_{x'}] = R[x' \widetilde{\partial}_{x'}]$. The complex of multi-Rees modules $R_{FV}(\mathcal{K}^\bullet_{[\alpha],\leqslant \mathbf{0}})$ has differentials given by $\cdot_{\mathrm{tens}}(x_j \widetilde{\partial}_{x_j} - x_1 \widetilde{\partial}_{x_1})$ $(j = 2, \dots, r)$. It also comes equip with the filtration $G_\bullet$ as in the proof of Lemma 15.14.10.

As G is bounded below, it is enough to show the exactness in negative degrees of the complex $R_{FV}\mathrm{gr}^G(\mathcal{K}^\bullet_{[\alpha],\leqslant\mathbf{0}})$, which is the Koszul complex of

$$\begin{aligned}R_{FV}\mathrm{gr}^G(\mathcal{K}^0_{[\alpha],\leqslant\mathbf{0}}) &= (R_{FV}\mathrm{gr}^{V^{(\boldsymbol{n})}}_{\alpha\boldsymbol{e}}\mathcal{M})\otimes_R R_{FV}(\mathbb{C}[\eta_1,\dots,\eta_r])\\ &= (R_{FV}\mathrm{gr}^{V^{(\boldsymbol{n})}}_{\alpha\boldsymbol{e}}\mathcal{M})\otimes_R R[\widetilde{\eta}_1,\dots,\widetilde{\eta}_r].\end{aligned}$$

In this presentation, the induced action of $\cdot_{\mathrm{tens}}\delta_j$ is by $1\otimes\widetilde{\eta}_j/e_j - 1\otimes\widetilde{\eta}_1/e_1$. The complex $R_{FV}(\mathrm{gr}^G\mathcal{K}^\bullet_{[\alpha],\leqslant\mathbf{0}})$ is thus identified with the Koszul complex

$$K\Big((R_{FV}\mathrm{gr}^{V^{(\boldsymbol{n})}}_{\alpha\boldsymbol{e}}\mathcal{M})\otimes_R R[\widetilde{\eta}_1,\dots,\widetilde{\eta}_r],\big(1\otimes\widetilde{\eta}_j/e_j - 1\otimes\widetilde{\eta}_1/e_1\big)_{j=2,\dots,r}\Big).$$

Since $R_{FV}\mathrm{gr}^{V^{(\boldsymbol{n})}}_{\alpha\boldsymbol{e}}\mathcal{M})$ is R-flat, due to the distributivity property of $F_\bullet,(V^{(i)}_\bullet)_{i\in I}$ on $\mathcal{M}$, this complex reads

$$R_{FV}(\mathrm{gr}^{V^{(\boldsymbol{n})}}_{\alpha\boldsymbol{e}}\mathcal{M})\otimes_R K\big(R[\widetilde{\eta}_1,\dots,\widetilde{\eta}_r],(\widetilde{\eta}_j/e_j-\widetilde{\eta}_1/e_1)_{j=2,\dots,r}\big).$$

Its is straightforward to check that the Koszul complex

$$K\big(R[\widetilde{\eta}_1,\dots,\widetilde{\eta}_r],(\widetilde{\eta}_j/e_j-\widetilde{\eta}_1/e_1)_{j=2,\dots,r}\big)$$

is a resolution of $R[\widetilde{\eta}_1]$, hence, using flatness of $R_{FV}(\mathrm{gr}^{V^{(\boldsymbol{n})}}_{\alpha\boldsymbol{e}}\mathcal{M})$ once more, we find that $R_{FV}(\mathrm{gr}^G\mathcal{K}^\bullet_{[\alpha],\leqslant\mathbf{0}})$ is a resolution of $R_{FV}(\mathrm{gr}^{V^{(\boldsymbol{n})}}_{\alpha\boldsymbol{e}}\mathcal{M})\otimes_R R[\widetilde{\eta}_1]$. In particular, its cohomology in negative degree is zero. □

***Proof of Lemma* 15.14.12(3).** From the previous computation one deduces that

$$H^0(R_{FV}(\mathrm{gr}^G\mathcal{K}^\bullet_{[\alpha],\leqslant\mathbf{0}}))\simeq(R_{FV}\mathrm{gr}^{V^{(\boldsymbol{n})}}_{\alpha\boldsymbol{e}}\mathcal{M})\otimes_{\mathbb{C}}\mathbb{C}[u_1],$$

with the R-module structure induced from that on $R_{FV}\mathrm{gr}^{V^{(\boldsymbol{n})}}_{\alpha\boldsymbol{e}}\mathcal{M}$. By the normal crossing type property of $(\mathcal{M},F_\bullet\mathcal{M})$, it is thus R-flat.

The proof of (2) also shows that each complex $R_{FV}(G_q\mathcal{K}^\bullet_{[\alpha],\leqslant\mathbf{0}})$ is acyclic in negative degrees for any q, and an easy induction implies flatness of the R-module $H^0(R_{FV}(G_q\mathcal{K}^\bullet_{[\alpha],\leqslant\mathbf{0}}))$ for any k, hence that of $H^0(R_{FV}(\mathcal{K}^\bullet_{[\alpha],\leqslant\mathbf{0}}))$. □

We now prove the last two properties of Proposition 15.14.11.

Proof that $F'_pV^{(i)}_{b_i}(\mathrm{gr}^V_\alpha\mathcal{M}_g)_{\leqslant\mathbf{0}}\cdot x_i = F'_pV^{(i)}_{b_i-1}(\mathrm{gr}^V_\alpha\mathcal{M}_g)_{\leqslant\mathbf{0}}$ ***if*** $b_i<0$**.** Due to the resolution of $R_{F'V^{(i)}}(\mathrm{gr}^V_\alpha\mathcal{M}_g)_{\leqslant\mathbf{0}}$ by $R_{FV^{(i)}}(\mathcal{K}^\bullet_{[\alpha],\leqslant\mathbf{0}})$, it is enough to check that

$$x_i: F_pV^{(i)}_{b_i}(\mathcal{K}^0_{[\alpha],\leqslant\mathbf{0}})\longrightarrow F_pV^{(i)}_{b_i-1}(\mathcal{K}^0_{[\alpha],\leqslant\mathbf{0}})$$

is an isomorphism for any p, any $i\in I$ and any $b_i<0$, and it is enough to prove the same property for $\mathcal{K}^0_{\alpha,\leqslant\mathbf{0}}$ for any $\alpha<0$, which amount to the inclusion $F_pV^{(i)}_{b_i-1}\mathcal{K}^0_{\alpha,\leqslant\mathbf{0}}\subset F_pV^{(i)}_{b_i}\mathcal{K}^0_{\alpha,\leqslant\mathbf{0}}\cdot x_i$.

On the one hand, by the logarithmic analogue of Corollary 15.12.12 we have, for $b_i\leqslant 0$,

$$V^{(i)}_{b_i}\mathcal{K}^0_{\alpha,\leqslant\mathbf{0}}=(V^{(\boldsymbol{n})}_{\alpha\boldsymbol{e}+b_i\mathbf{1}_i}\mathcal{M})\otimes\mathbb{C}[x'\partial_{x'}].$$

On the other hand, by definition,

$$F_p\mathcal{K}^0_{\alpha,\leqslant\mathbf{0}}=\bigoplus_{\boldsymbol{k}\geqslant\mathbf{0}}F_{p-|\boldsymbol{k}|}(V^{(\boldsymbol{n})}_{\alpha\boldsymbol{e}}\mathcal{M})\otimes_{\mathcal{O}_X}(x'\partial_{x'})^{\boldsymbol{k}},$$

so that, if $b_i \leqslant 0$,

$$F_pV^{(i)}_{b_i}\mathcal{K}^0_{\alpha,\leqslant\mathbf{0}} = \bigoplus_{\boldsymbol{k}\geqslant\mathbf{0}} F_{p-|\boldsymbol{k}|}(V^{(\boldsymbol{n})}_{\alpha\boldsymbol{e}+b_i\mathbf{1}_i}\mathcal{M}) \otimes_{\mathcal{O}_X} (x'\partial_{x'})^{\boldsymbol{k}}. \tag{15.14.13}$$

Since $(\mathcal{M}, F_\bullet\mathcal{M})$ is of normal crossing type, (15.9.7) yields, if $b_i < 0$,

$$F_{p-|\boldsymbol{k}|}(V^{(\boldsymbol{n})}_{\alpha\boldsymbol{e}+(b_i-1)\mathbf{1}_i}\mathcal{M}) = F_{p-|\boldsymbol{k}|}(V^{(\boldsymbol{n})}_{\alpha\boldsymbol{e}+b_i\mathbf{1}_i}\mathcal{M}) \cdot x_i,$$

and, on noting the inclusion

$$\big(F_{p-|\boldsymbol{k}|}(V^{(\boldsymbol{n})}_{\alpha\boldsymbol{e}+b_i\mathbf{1}_i}\mathcal{M}) \cdot x_i\big) \otimes (x_i\partial_{x_i})^{k_i} \subset \Big[\sum_{\ell_i=0}^{k_i} F_{p-|\boldsymbol{k}|}(V^{(\boldsymbol{n})}_{\alpha\boldsymbol{e}+b_i\mathbf{1}_i}\mathcal{M}) \otimes (x_i\partial_{x_i})^{\ell_i}\Big] \cdot x_i,$$

we deduce $F_pV^{(i)}_{b_i-1}K^0_{\alpha,\leqslant\mathbf{0}} \subset F_pV^{(i)}_{b_i}K^0_{\alpha,\leqslant\mathbf{0}} \cdot x_i$, as desired. □

Proof that $F'_pV^{(i)}_{-1}(\mathrm{gr}^V_\alpha\mathcal{M}_g)_{\leqslant\mathbf{0}} \cdot \partial_{x_i} \subset F'_{p+1}V^{(i)}_0(\mathrm{gr}^V_\alpha\mathcal{M}_g)_{\leqslant\mathbf{0}}$. As above, we argue with $\mathcal{K}^0_{\alpha,\leqslant\mathbf{0}}$ which is contained in $\mathcal{K}^0_\alpha$, and the action of ∂_{x_i} is that on $\mathcal{K}^0_\alpha$. We use the expression (15.14.13) and we are led to checking that $(F_{p-|\boldsymbol{k}|}V^{(\boldsymbol{n})}_{\alpha\boldsymbol{e}-\mathbf{1}_i}\mathcal{M}) \cdot \partial_{x_i} \subset F_{p+1-|\boldsymbol{k}|}V^{(\boldsymbol{n})}_{\alpha\boldsymbol{e}}\mathcal{M}$, which is by definition of the filtrations. □

15.14.c. Explicit expression of nearby cycles with filtration. We revisit the isomorphism of Corollary 15.13.6 for $(\mathcal{M}, F_\bullet\mathcal{M})$ satisfying the assumptions of Theorem 15.11.1. For $\boldsymbol{b} \leqslant \mathbf{0}$, we replace $(\mathrm{gr}^V_\alpha M_g)_{\boldsymbol{b}}$ of Corollary 15.13.6 with $\mathrm{gr}^{V^{(\boldsymbol{n})}}_{\boldsymbol{b}}(\mathrm{gr}^V_\alpha\mathcal{M}_g)$ and $M_{\alpha\boldsymbol{e}+\boldsymbol{b}}$ with $\mathrm{gr}^{V^{(\boldsymbol{n})}}_{\alpha\boldsymbol{e}+\boldsymbol{b}}\mathcal{M}$. We still denote by $\varphi_{\boldsymbol{b}}$ the morphism

$$V^{(\boldsymbol{n})}_{\alpha\boldsymbol{e}+\boldsymbol{b}}\mathcal{M} \otimes_{\mathbb{C}} \mathbb{C}[t\partial_t] \longrightarrow V^{(\boldsymbol{n})}_{\alpha\boldsymbol{e}+\boldsymbol{b}}\mathcal{M} \otimes_{\mathbb{C}} \mathbb{C}[t\partial_t]$$

defined by (15.13.6 ∗), with $\mathrm{N} = t\partial_t - \alpha$ and $\mathrm{N}_i = \mathrm{D}_i - \alpha e_i$. From the expression (15.12.1 ∗) it follows, since $\boldsymbol{b} \leqslant \mathbf{0}$, that $V^{(\boldsymbol{n})}_{\alpha\boldsymbol{e}+\boldsymbol{b}}\mathcal{M} \otimes_{\mathbb{C}} \mathbb{C}[t\partial_t]$ is sent into $V_\alpha\mathcal{M}_g$ via the isomorphism

$$\begin{aligned} V^{(\boldsymbol{n})}_{\alpha\boldsymbol{e}+\boldsymbol{b}}\mathcal{M} \otimes_{\mathbb{C}} \mathbb{C}[t\partial_t] &= \bigoplus_q V^{(\boldsymbol{n})}_{\alpha\boldsymbol{e}+\boldsymbol{b}}\mathcal{M} \otimes_{\mathbb{C}} t^q\partial_t^q \\ &\xrightarrow[\iota]{\sim} \bigoplus_q V^{(\boldsymbol{n})}_{\alpha\boldsymbol{e}+\boldsymbol{b}}\mathcal{M}g^q \otimes_{\mathbb{C}} \partial_t^q = \bigoplus_q V^{(\boldsymbol{n})}_{(\alpha-q)\boldsymbol{e}+\boldsymbol{b}}\mathcal{M} \otimes_{\mathbb{C}} \partial_t^q \\ &\subset \mathcal{M} \otimes_{\mathbb{C}} \mathbb{C}[\partial_t] = \mathcal{M}_g. \end{aligned}$$

From Proposition 15.13.1 we deduce that the image of $V^{(\boldsymbol{n})}_{\alpha\boldsymbol{e}+\boldsymbol{b}}\mathcal{M} \otimes_{\mathbb{C}} \mathbb{C}[t\partial_t]$ in $\mathrm{gr}^V_\alpha\mathcal{M}_g$ is equal to $V^{(\boldsymbol{n})}_{\boldsymbol{b}}(\mathrm{gr}^V_\alpha\mathcal{M}_g)$ and Corollary 15.13.6 identifies more precisely $\mathrm{gr}^{V^{(\boldsymbol{n})}}_{\boldsymbol{b}}(\mathrm{gr}^V_\alpha\mathcal{M}_g)$ with the cokernel of

$$\varphi_{\boldsymbol{b}} : \mathrm{gr}^{V^{(\boldsymbol{n})}}_{\alpha\boldsymbol{e}+\boldsymbol{b}}\mathcal{M} \otimes_{\mathbb{C}} \mathbb{C}[t\partial_t] \longrightarrow \mathrm{gr}^{V^{(\boldsymbol{n})}}_{\alpha\boldsymbol{e}+\boldsymbol{b}}\mathcal{M} \otimes_{\mathbb{C}} \mathbb{C}[t\partial_t].$$

We equip $V^{(\boldsymbol{n})}_{\alpha\boldsymbol{e}+\boldsymbol{b}}\mathcal{M} \otimes_{\mathbb{C}} \mathbb{C}[t\partial_t]$ with the filtration

$$\begin{aligned} F_p(V^{(\boldsymbol{n})}_{\alpha\boldsymbol{e}+\boldsymbol{b}}\mathcal{M} \otimes_{\mathbb{C}} \mathbb{C}[t\partial_t]) &= \bigoplus_{q\geqslant 0} F_{p-q}V^{(\boldsymbol{n})}_{\alpha\boldsymbol{e}+\boldsymbol{b}}\mathcal{M} \otimes t^q\partial_t^q \\ &\xrightarrow[\iota]{\sim} \bigoplus_{q\geqslant 0} F_{p-q}V^{(\boldsymbol{n})}_{(\alpha-q)\boldsymbol{e}+\boldsymbol{b}}\mathcal{M} \otimes \partial_t^q \subset F_p\mathcal{M}_g \cap V_\alpha\mathcal{M}_g = F_pV_\alpha\mathcal{M}_g. \end{aligned}$$

The image of the induced morphism

$$F_p(V^{(\boldsymbol{n})}_{\alpha\boldsymbol{e}+\boldsymbol{b}}\mathcal{M}\otimes_{\mathbb{C}}\mathbb{C}[t\partial_t])\longrightarrow \operatorname{gr}^V_\alpha\mathcal{M}_g$$

is thus contained in $F_p(\operatorname{gr}^V_\alpha\mathcal{M}_g)\cap V^{(\boldsymbol{n})}_{\boldsymbol{b}}(\operatorname{gr}^V_\alpha\mathcal{M}_g)$.

15.14.14. Proposition. *This inclusion is an equality.*

The main application of the proposition is the next corollary, which extends the isomorphism of Corollary 15.13.6 to the filtered setting, and thus yields an explicit expression for the F-filtration on $\operatorname{gr}^{V^{(\boldsymbol{n})}}_{\boldsymbol{b}}(\operatorname{gr}^V_\alpha\mathcal{M}_g)$.

15.14.15. Corollary. *The filtration of $\operatorname{gr}^{V^{(\boldsymbol{n})}}_{\boldsymbol{b}}(\operatorname{gr}^V_\alpha\mathcal{M}_g)$ naturally induced from $F_\bullet(\operatorname{gr}^V_\alpha\mathcal{M}_g)$ (taking into account that $(\operatorname{gr}^V_\alpha\mathcal{M}_g,F_\bullet\operatorname{gr}^V_\alpha\mathcal{M}_g)$ is of normal crossing type along D) is equal to the image of the filtration $F_\bullet(\operatorname{gr}^{V^{(\boldsymbol{n})}}_{\alpha\boldsymbol{e}+\boldsymbol{b}}\mathcal{M}\otimes_{\mathbb{C}}\mathbb{C}[t\partial_t])$ by the morphism*

$$\operatorname{gr}^{V^{(\boldsymbol{n})}}_{\alpha\boldsymbol{e}+\boldsymbol{b}}\mathcal{M}\otimes_{\mathbb{C}}\mathbb{C}[t\partial_t]\longrightarrow \operatorname{Coker}\varphi_{\boldsymbol{b}}.$$

Proof of the corollary. Recall that $\operatorname{gr}^{V^{(\boldsymbol{n})}}_{\alpha\boldsymbol{e}+\boldsymbol{b}}\mathcal{M}=V^{(\boldsymbol{n})}_{\alpha\boldsymbol{e}+\boldsymbol{b}}\mathcal{M}/\sum_{\boldsymbol{b}'\lneqq\boldsymbol{b}}V^{(\boldsymbol{n})}_{\alpha\boldsymbol{e}+\boldsymbol{b}'}\mathcal{M}$ and similarly $\operatorname{gr}^{V^{(\boldsymbol{n})}}_{\boldsymbol{b}}\operatorname{gr}^V_\alpha\mathcal{M}_g=V^{(\boldsymbol{n})}_{\boldsymbol{b}}\operatorname{gr}^V_\alpha\mathcal{M}_g/\sum_{\boldsymbol{b}'\lneqq\boldsymbol{b}}V^{(\boldsymbol{n})}_{\boldsymbol{b}'}\operatorname{gr}^V_\alpha\mathcal{M}_g$.

On the one hand, the filtration $F_p(\operatorname{gr}^{V^{(\boldsymbol{n})}}_{\alpha\boldsymbol{e}+\boldsymbol{b}}\mathcal{M}\otimes_{\mathbb{C}}\mathbb{C}[t\partial_t])$ is defined as

$$\bigoplus_{q\geqslant 0}F_{p-q}\operatorname{gr}^{V^{(\boldsymbol{n})}}_{\alpha\boldsymbol{e}+\boldsymbol{b}}\mathcal{M}\otimes t^q\partial_t^q,$$

and is equal to the image of $F_p(V^{(\boldsymbol{n})}_{\alpha\boldsymbol{e}+\boldsymbol{b}}\mathcal{M}\otimes_{\mathbb{C}}\mathbb{C}[t\partial_t])$ in $\operatorname{gr}^{V^{(\boldsymbol{n})}}_{\alpha\boldsymbol{e}+\boldsymbol{b}}\mathcal{M}\otimes_{\mathbb{C}}\mathbb{C}[t\partial_t]$.

On the other hand, $F_p\operatorname{gr}^{V^{(\boldsymbol{n})}}_{\boldsymbol{b}}\operatorname{gr}^V_\alpha\mathcal{M}_g$ is equal, since $(\operatorname{gr}^V_\alpha\mathcal{M}_g,F_\bullet\operatorname{gr}^V_\alpha\mathcal{M}_g)$ is of normal crossing type along D, to the image of $F_pV^{(\boldsymbol{n})}_{\boldsymbol{b}}\operatorname{gr}^V_\alpha\mathcal{M}_g$ by the projection $V^{(\boldsymbol{n})}_{\boldsymbol{b}}\operatorname{gr}^V_\alpha\mathcal{M}_g\to\operatorname{gr}^{V^{(\boldsymbol{n})}}_{\boldsymbol{b}}\operatorname{gr}^V_\alpha\mathcal{M}_g$.

The assertion then follows from the commutative diagram below, where the upper horizontal morphism is onto according to the proposition:

$$\begin{array}{ccc}F_p(V^{(\boldsymbol{n})}_{\alpha\boldsymbol{e}+\boldsymbol{b}}\mathcal{M}\otimes_{\mathbb{C}}\mathbb{C}[t\partial_t]) & \twoheadrightarrow & F_pV^{(\boldsymbol{n})}_{\boldsymbol{b}}\operatorname{gr}^V_\alpha\mathcal{M}_g\\ \big\downarrow\!\!\!\downarrow & & \big\downarrow\!\!\!\downarrow\\ F_p(\operatorname{gr}^{V^{(\boldsymbol{n})}}_{\alpha\boldsymbol{e}+\boldsymbol{b}}\mathcal{M}\otimes_{\mathbb{C}}\mathbb{C}[t\partial_t]) & \longrightarrow & F_p\operatorname{gr}^{V^{(\boldsymbol{n})}}_{\boldsymbol{b}}\operatorname{gr}^V_\alpha\mathcal{M}_g\end{array}$$

□

***Proof of Proposition* 15.14.14.** We observe that the image of $F_p(V^{(\boldsymbol{n})}_{\alpha\boldsymbol{e}+\boldsymbol{b}}\mathcal{M}\otimes_{\mathbb{C}}\mathbb{C}[t\partial_t])$ in $\operatorname{gr}^V_\alpha\mathcal{M}_g$, for $\boldsymbol{b}\leqslant 0$, is contained in $(\operatorname{gr}^V_\alpha\mathcal{M}_g)_{\leqslant\boldsymbol{0}}$ and is equal to the image of $F_p(V^{(\boldsymbol{n})}_{\alpha\boldsymbol{e}+\boldsymbol{b}}\mathcal{M}\otimes_{\mathcal{O}_X}V^{(\boldsymbol{r})}_{\boldsymbol{0}}\mathcal{D}'_X)=F_pV^{(\boldsymbol{n})}_{\alpha\boldsymbol{e}+\boldsymbol{b}}(V^{(\boldsymbol{n})}_{\alpha\boldsymbol{e}}\mathcal{M}\otimes_{\mathcal{O}_X}V^{(\boldsymbol{r})}_{\boldsymbol{0}}\mathcal{D}'_X)$, according to the relation (15.11.3 $**$) between the action of $x_i\partial_{x_i}$ and that of $t\partial_t$ on $\operatorname{gr}^V_\alpha\mathcal{M}_g$.

By Lemma 15.14.12, $R_{FV}\mathcal{K}^0_{\alpha,\leqslant\boldsymbol{0}}$ surjects onto $R_{F'V}(\operatorname{gr}^V_\alpha\mathcal{M}_g)_{\leqslant\boldsymbol{0}}$, which implies in particular that $F_pV^{(\boldsymbol{n})}_{\boldsymbol{b}}(\mathcal{K}^0_{\alpha,\leqslant\boldsymbol{0}})$ has image $F'_pV^{(\boldsymbol{n})}_{\boldsymbol{b}}(\operatorname{gr}^V_\alpha\mathcal{M}_g)_{\leqslant\boldsymbol{0}}$, where $F'_\bullet(\operatorname{gr}^V_\alpha\mathcal{M}_g)_{\leqslant\boldsymbol{0}}$ is the filtration used in Proposition 15.14.11. In conclusion, the image in $(\operatorname{gr}^V_\alpha\mathcal{M}_g)_{\leqslant\boldsymbol{0}}$ of $F_p(V^{(\boldsymbol{n})}_{\alpha\boldsymbol{e}+\boldsymbol{b}}\mathcal{M}\otimes_{\mathbb{C}}\mathbb{C}[t\partial_t])$ is equal to $F'_pV^{(\boldsymbol{n})}_{\alpha\boldsymbol{e}+\boldsymbol{b}}(\operatorname{gr}^V_\alpha\mathcal{M}_g)_{\leqslant\boldsymbol{0}}$.

We recall that the filtration $F_\bullet\mathrm{gr}_\alpha^V\mathcal{M}_g$ is that generated by $F'_\bullet(\mathrm{gr}_\alpha^V\mathcal{M}_g)_{\leqslant\boldsymbol{0}}$. Then, Proposition 15.9.14(1), together with Proposition 15.14.11, implies in particular that $F_p\mathrm{gr}_\alpha^V\mathcal{M}_g\cap V_{\boldsymbol{0}}^{(\boldsymbol{n})}\mathrm{gr}_\alpha^V\mathcal{M}_g = F'_pV_{\boldsymbol{0}}^{(\boldsymbol{n})}\mathrm{gr}_\alpha^V\mathcal{M}_g$. Intersecting both terms with $V_{\boldsymbol{b}}^{(\boldsymbol{n})}\mathrm{gr}_\alpha^V\mathcal{M}_g$ for $\boldsymbol{b}\leqslant\boldsymbol{0}$ yields that the image in $(\mathrm{gr}_\alpha^V\mathcal{M}_g)_{\leqslant\boldsymbol{0}}$ of $F_p(V_{\alpha\boldsymbol{e}+\boldsymbol{b}}^{(\boldsymbol{n})}\mathcal{M}\otimes_{\mathbb{C}}\mathbb{C}[t\partial_t])$ is equal to $F_p\mathrm{gr}_\alpha^V\mathcal{M}_g\cap V_{\boldsymbol{b}}^{(\boldsymbol{n})}\mathrm{gr}_\alpha^V\mathcal{M}_g$. □

15.14.d. A criterion for the existence of the filtered monodromy filtration

In order to settle the question, we switch to the setting of $\widetilde{\mathcal{D}}_X$-modules as in Chapter 9, so that $\widetilde{\mathcal{M}}$ denotes the Rees module $R_F\mathcal{M}$. Our previous results can be expressed by saying that, under the filtered normal crossing type assumption, $\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along (g). We still let N denote the nilpotent endomorphism on $\mathrm{gr}_\alpha^V\widetilde{\mathcal{M}}_g$, which admits a monodromy filtration $\mathrm{M}(\mathrm{N})_\bullet$. In general, one cannot ensure that each graded module $\mathrm{gr}_\ell^{\mathrm{M}}(\mathrm{gr}_\alpha^V\widetilde{\mathcal{M}}_g)$ is strict, equivalently, each primitive submodule $\mathrm{P}_\ell(\mathrm{gr}_\alpha^V\widetilde{\mathcal{M}}_g)$ is strict. A criterion for strictness has been given in Proposition 9.4.10: any power of N should be a strict endomorphism of $\mathrm{gr}_\alpha^V\widetilde{\mathcal{M}}_g$.

15.14.16. Proposition. *Let $(\mathcal{M},F_\bullet\mathcal{M})$ be of normal crossing type along D. Assume that for each $\boldsymbol{b}\leqslant\boldsymbol{0}$ and for a fixed $\alpha<0$, the filtered vector space $(\mathrm{gr}_{\alpha\boldsymbol{e}+\boldsymbol{b}}^{V^{(\boldsymbol{n})}}\mathcal{M},F_\bullet\mathrm{gr}_{\alpha\boldsymbol{e}+\boldsymbol{b}}^{V^{(\boldsymbol{n})}}\mathcal{M})$ underlies a mixed Hodge structure such that each N_i is a morphism of mixed Hodge structures $\mathrm{gr}_{\alpha\boldsymbol{e}+\boldsymbol{b}}^{V^{(\boldsymbol{n})}}\mathcal{M}\to\mathrm{gr}_{\alpha\boldsymbol{e}+\boldsymbol{b}}^{V^{(\boldsymbol{n})}}\mathcal{M}(-1)$. Then any power of $\mathrm{N}:\mathrm{gr}_\alpha^V\widetilde{\mathcal{M}}_g\to\mathrm{gr}_\alpha^V\widetilde{\mathcal{M}}_g$ is strict.*

Proof. Recall the notation $\mathrm{N}_{\boldsymbol{b}}$ for $\mathrm{gr}_{\boldsymbol{b}}^{V^{(\boldsymbol{n})}}\mathrm{N}$ on $\mathrm{gr}_{\boldsymbol{b}}^{V^{(\boldsymbol{n})}}\mathrm{gr}_\alpha^V\widetilde{\mathcal{M}}_g$. We first claim that it is enough to prove strictness for any power of $\mathrm{N}_{\boldsymbol{b}}$ for any $\boldsymbol{b}\leqslant\boldsymbol{0}$. Indeed, assuming this property, we argue by induction on $\#I$:

Let us fix $i\in I$. Since $(\mathrm{gr}_{b_i}^{V^{(i)}}(\mathrm{gr}_\alpha^V\widetilde{\mathcal{M}}_g),F_\bullet\mathrm{gr}_{b_i}^{V^{(i)}}(\mathrm{gr}_\alpha^V\widetilde{\mathcal{M}}_g))$ is of normal crossing type on $(D_i,\bigcup_{j\neq i}D_j)$ for any $b_i\in[-1,0]$ (see Proposition 15.9.4(1)), we deduce from the assumption on $\mathrm{N}_{\boldsymbol{b}}$, by induction on $\#I$, that $\mathrm{gr}_{b_i}^{V^{(i)}}\mathrm{N}^\ell$ is strict for any $\ell\geqslant1$. This means, by definition, that N^ℓ $(\ell\geqslant1)$ is strictly $\mathbb{R}$-specializable along D_i. Corollary 10.7.6 implies then that N^ℓ is strict in some neighborhood of D_i, as desired.

For the strictness of $\mathrm{N}_{\boldsymbol{b}}^\ell$, it is enough to check that $\mathrm{Coker}\,\varphi_{\boldsymbol{b}}$ underlies a mixed Hodge structure and that $\mathrm{N}_{\boldsymbol{b}}$ (hence any $\mathrm{N}_{\boldsymbol{b}}^\ell$) is a morphism of mixed Hodge structures (see Proposition 2.6.8). This is precisely Example 2.6.10(4). □

15.15. Exercises

Exercise 15.15 **(Proof of Lemma 15.12.8).**

(1) Prove that $(a_2u_2-a_1u_1)$ is injective on $M[u_1,\dots,u_n]:=M\otimes_A A[u_1,\dots,u_n]$ by using that a_2 is injective on M.

(2) Show that the natural map

$$F_0^{(2)}M[u_1,\dots,u_n]:=M[u_1,u_3,\dots,u_n]\longrightarrow M[u_1,\dots,u_n]/(a_2u_2-a_1u_1)$$

is injective.

(3) Show inductively that

$$\mathrm{gr}_k^{F^{(2)}}\big(M[u_1,\dots,u_n]/(a_2u_2-a_1u_1)\big)=(M/(a_2))u_2^k[u_1,u_3,\dots,u_n].$$

(4) Conclude the proof by induction on n.

***Exercise 15.16* (Comptibility of $(F,V^{(1)},\dots,V^{(n)})$ on $\mathcal{D}_X$).**

(1) Consider first the ring $\mathbb{C}[x]\langle\partial_x\rangle$.

(a) Show that $\mathbb{C}[x]$ decomposes as a $\mathbb{C}$-vector space as the direct sum, indexed by subsets $I\subset\{1,\dots,n\}$ with complement I^{c}, of the spaces $\mathbb{C}[x_{I^{\mathrm{c}}}]$, and thus

$$\begin{aligned}V_0^{(\boldsymbol{n})}\mathbb{C}[x]\langle\partial_x\rangle&=\mathbb{C}[x]\langle x\partial_x\rangle=\bigoplus_I\mathbb{C}[x_{I^{\mathrm{c}}}]\langle x\partial_x\rangle,\\ V_{\boldsymbol{k}}^{(\boldsymbol{n})}\mathbb{C}[x]\langle\partial_x\rangle&=\bigoplus_I\bigoplus_{\boldsymbol{\ell}_I\leqslant\boldsymbol{k}_I}\mathbb{C}[x_{I^{\mathrm{c}}}]\langle x\partial_x\rangle\partial_{x_I}^{\boldsymbol{\ell}_I},\\ F_pV_{\boldsymbol{k}}^{(\boldsymbol{n})}\mathbb{C}[x]\langle\partial_x\rangle&=\bigoplus_I\bigoplus_{\boldsymbol{\ell}_I\leqslant\boldsymbol{k}_I}\bigoplus_{\substack{\boldsymbol{m}\in\mathbb{N}^n\\|\boldsymbol{\ell}_I|+|\boldsymbol{m}|\leqslant p}}\mathbb{C}[x_{I^{\mathrm{c}}}](x\partial_x)^{\boldsymbol{m}}\partial_{x_I}^{\boldsymbol{\ell}_I}.\end{aligned}$$

(b) Use this decomposition to show that the ring $R_{FV}\mathbb{C}[x]\langle\partial_x\rangle$ is free over $R=\mathbb{C}[z_0,\dots,z_n]$.

(2) Show that $R_{FV}\mathcal{D}_X=\mathcal{O}_X[z_0,\dots,z_n]\otimes_R R_{FV}\mathbb{C}[x]\langle\partial_x\rangle$, and conclude that $R_{FV}\mathcal{D}_X$ is R-flat.

15.16. Comments

This chapter is quite technical. This is mainly due to the nature of the problems considered. Dealing with many filtrations on an object and understanding their relations is intrinsically complicated. It is intended to be an expanded version of the part of Section 3 in [**Sai90**] which is concerned with filtered $\mathcal{D}$-modules. As already explained, we do not refer to perverse sheaves, so the perverse sheaf version, which is present in loc. cit., is not considered here.

The main theme for us is the notion of "transversality" between filtrations and its behavior under the nearby cycle functor. The notion of compatibility of filtrations has been analyzed in a very general setting in Section 1 of [**Sai88**]. We have chosen here to emphasize a more explicit approach in the framework of abelian categories, and even in the more restrictive framework of categories of sheaves of modules on a topological space. Furthermore, we mainly focus on the notion distributive families of filtrations, although we relate it to that of compatible families of filtrations considered in [**Sai88**]. We interpret these notions in algebraic terms, that is, in terms of flatness of the associated multi-Rees module, which is a multi-graded module over the polynomial ring of its parameters. This approach goes back at least to [**Sab87b**]. When omitting the F-filtration, the theory mainly reduces to that of monodromic modules over the Weyl algebra in n variables and is equivalent to that of monodromic perverse sheaves as considered by Verdier in [**Ver83**].