CHAPTER 14

POLARIZABLE HODGE MODULES AND THEIR DIRECT IMAGES

Summary. This chapter contains the definition of polarizable Hodge modules. The actual presentation justifies the introduction of the language of triples. The main properties are abelianity and semi-simplicity of the category of polarizable pure Hodge modules of weight w. It is convenient to also introduce polarizable Hodge-Lefschetz modules, as they appear in many intermediate steps of various proofs, due to the very definition of a polarizable Hodge module. We also give the proof of one of the two main important results concerning polarizable Hodge modules, namely, the decomposition theorem. The proof of the structure theorem will be given in Chapter 16. Here, we will use the machinery of filtered $\widetilde{\mathcal{D}}$-module theory and sesquilinear pairings to reduce the proof to the case of the map from a compact Riemann surface to a point, that we have analyzed in Chapter 7, according to the results of Schmid and Zucker developed in Chapter 6. This strategy justifies the somewhat complicated and recursive definition of the category $\mathsf{pHM}(X, w)$ of polarizable Hodge modules.

14.1. Introduction

Polarizable Hodge modules on a complex analytic manifold X are supposed to play the role of polarizable Hodge structures with a multi-dimensional parameter. These objects can acquire singularities. The way each characteristic property of a Hodge structure is translated in higher dimension of the parameter space is given by the table below.

dimension 0	dimension $n \geqslant 1$
$\mathcal{H}$ a $\mathbb{C}$-vector space	$\mathcal{M}$ a holonomic $\mathcal{D}$-module
$F^\bullet\mathcal{H}$ a filtration	$F_\bullet\mathcal{M}$ a coherent filtration
$\widetilde{\mathcal{H}} = R_F\mathcal{H}$ a strict graded $\mathbb{C}[z]$-module	$\widetilde{\mathcal{M}} = R_F\mathcal{M}$ a strict graded $R_F\mathcal{D}$-module
$H = (\widetilde{\mathcal{H}}', \widetilde{\mathcal{H}}'', \mathfrak{s})$ a triple with sesquilinear pairing $\mathfrak{s}$	$M = (\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}'', \mathfrak{s})$ a triple with sesquilinear pairing $\mathfrak{s}$
$\mathrm{S} : H \to H^*(-w)$ a polarization	$\mathrm{S} : M \to M^*(-w)$ a polarization

Why choosing holonomic $\mathcal{D}$-modules as analogues of $\mathbb{C}$-vector space? The reason is that the category of holonomic $\mathcal{D}$-modules is Artinian, that is, any holonomic

$\mathcal{D}$-module has finite length (locally on the underlying manifold). A related reason is that its de Rham complex has constructible cohomology, generalizing the notion of local system attached to a flat bundle. Moreover, the property of holonomicity is preserved by various operations (proper pushforward, pullback by a holomorphic map), and the nearby/vanishing cycle theory (the V-filtration) is well-defined for holonomic $\mathcal{D}$-modules without any other assumption, so that the issue concerning nearby/vanishing cycles of filtered holonomic $\mathcal{D}$-modules only comes from the filtration.

In order to define the Hodge properties, we use the same method as in dimension 1 (see Chapter 7):

• as in Section 7.4.a, we work in the ambient abelian category $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$, which has been defined in Section 12.7;

• the definition of the category $\mathsf{pHM}(X, w)$ of polarizable Hodge modules of weight w is obtained by induction on the dimension of the support of the triples entering the definition; contrary to dimension 1, many steps may be needed before reaching the case of polarizable Hodge structures.

The definition of a polarizable Hodge module can look frightening: in order to check that an object $M = (\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}'', \mathfrak{s})$ belongs to $\mathsf{pHM}(X, w)$, we have to consider in an inductive way nearby cycles with respect to *all* germs of holomorphic functions.

That the category of polarizable Hodge modules is non-empty is a non trivial fact. Already, it is not obvious at all that $\mathcal{O}_X$ underlies a polarizable Hodge module when $\dim X \geqslant 2$. The reason is that the definition involves considering nearby and vanishing cycles along *any* germ of holomorphic function, whose singularities can be arbitrarily complicated. In dimension 1, holomorphic functions are just powers of coordinates, and this explains why the property is easier to check. The higher-dimensional case will be proved in Theorem 14.6.1.

The question should however be considered the other way round. Once we know at least one polarizable Hodge module (e.g. a polarizable variation of Hodge structure, according to Theorem 14.6.1), we automatically know an infinity of them, by considering (monodromy-graded) nearby or vanishing cycles with respect to *any* holomorphic function and pushforward by any projective morphism, by the Hodge-Saito theorem 14.3.1.

In the same vein, due to this inductive definition, the proof of many properties of polarizable Hodge modules can be done by induction on the dimension of the support, and this reduces to checking the property for polarizable Hodge structures.

14.2. Definition and first properties of polarizable Hodge modules

The notion of a polarizable Hodge module will be defined by induction on the dimension of the support, and we will make extensive use of the properties of the abelian category $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$ introduced in Section 12.7, in particular the definitions relative to nearby/vanishing cycles (Section 12.7.14). We mimic the definitions in dimension 1.

14.2.1. Definition **(of a polarizable Hodge module of weight w)**
The category $\mathsf{pHM}(X, w)$ of polarizable Hodge modules of weight w on X is the full subcategory of $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$ whose objects $\widetilde{\mathcal{T}}$ are holonomic and for which there exists a morphism $\mathrm{S} : \widetilde{\mathcal{T}} \to \widetilde{\mathcal{T}}^*(-w)$ such that $(\widetilde{\mathcal{T}}, \mathrm{S})$ is a polarized Hodge module of weight w on X in the sense of Definition 14.2.2 below.

We will denote by M a triple which is a polarizable Hodge module and by $\mathsf{pHM}(X, w)$ the full subcategory of the category of holonomic $\widetilde{\mathcal{D}}_X$-triples whose objects are polarizable Hodge modules of weight w. Objects of $\mathsf{pHM}(X, w)$ can be represented either by left or right triples, by using the corresponding definition for the functors in the left or right case. The definition below has to be understood in an inductive way, with respect to the dimension of the support of a triple.

14.2.2. Definition **(of a polarized Hodge module of weight w)**
Let $\widetilde{\mathcal{T}}$ be an object of $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$ which is holonomic, and let $\mathrm{S} : \mathcal{T} \to \widetilde{\mathcal{T}}^*(-w)$ be a morphism ($w \in \mathbb{Z}$).

(0) If $\dim \operatorname{Supp} \widetilde{\mathcal{T}} = 0$ and ι denotes the inclusion $\operatorname{Supp} \widetilde{\mathcal{T}} \hookrightarrow X$, we say that $(\widetilde{\mathcal{T}}, \mathrm{S})$ is a *polarized Hodge module of weight* w on X if

$$(\widetilde{\mathcal{T}}, \mathrm{S}) \simeq \bigoplus_{x \in \operatorname{Supp} \widetilde{\mathcal{T}}} {}_{\mathrm{T}}\iota_*(H_x, \mathrm{S}_x),$$

where each (H_x, S_x) is a polarized Hodge structure of weight w.

(>0) For $d \geqslant 1$, assume we have defined polarized Hodge module of weight w having support of dimension $< d$, and let $(\widetilde{\mathcal{T}}, \mathrm{S})$ be such that $\dim \operatorname{Supp} \widetilde{\mathcal{T}} = d$. We say that $(\widetilde{\mathcal{T}}, \mathrm{S})$ is a *polarized Hodge module of weight* w on X if $\widetilde{\mathcal{T}}$ is *strict* and for any open set $U \subset X$ and any holomorphic function $g : U \to \mathbb{C}$,

$(1)_g$ $\widetilde{\mathcal{T}}$ is strictly $\mathbb{R}$-specializable along (g);

$(2)_g$ if moreover $g^{-1}(0) \cap \operatorname{Supp} \widetilde{\mathcal{T}}$ has everywhere codimension 1 in $\operatorname{Supp} \widetilde{\mathcal{T}}$, then for every $\ell \geqslant 0$ and every $\lambda \in \mathbb{S}^1$,

(a) $\mathrm{P}_\ell \psi_{g,\lambda}(\widetilde{\mathcal{T}}, \mathrm{S})$ is a polarized Hodge module of weight $w + \ell - 1$ on U,

(b) $\mathrm{P}_\ell \phi_{g,1}(\widetilde{\mathcal{T}}, \mathrm{S})$ is a polarized Hodge module of weight $w + \ell$ on U.

(See (12.7.17 $*$) for the objects considered in $(2)_g$.) Note that, by the strictness assumption, $\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}''$ correspond to coherently F-filtered holonomic $\mathcal{D}_X$-modules $(\mathcal{M}', F_\bullet\mathcal{M}')$ and $(\mathcal{M}'', F_\bullet\mathcal{M}'')$.

14.2.3. Remarks. Let us already emphasize some properties that will be proved in Theorem 14.2.17 below, or are a consequence of this theorem.

(1) The restriction on g in $(2)_g$ can be relaxed, and in fact $(2)_g$ holds for any g.

(2) The morphism S, that we call a *polarization of* $\widetilde{\mathcal{T}}$ is in fact a pre-polarization of weight w of the triple $\widetilde{\mathcal{T}}$, that is, a Hermitian isomorphism.

(3) If Properties 14.2.2$(1)_g$ and $(2)_g$ are satisfied, then so are 14.2.2$(1)_{g^r}$ and $(2)_{g^r}$ for any $r \geqslant 2$. This follows from Section 12.7.21.

(4) If $(\widetilde{\mathcal{T}}, \mathrm{S})$ satisfies $(1)_g$, $(2\mathrm{a})_g$ and is a middle extension along (g), then it also satisfies $(2\mathrm{b})_g$. This follows from the vanishing cycle theorem 14.2.22.

14.2.a. First properties of $\mathsf{pHM}(X, w)$

14.2.4. Hermitian duality. Hermitian duality in $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$ exchanges $\mathsf{pHM}(X, w)$ with $\mathsf{pHM}(X, -w)^{\mathrm{op}}$.

14.2.5. Tate twist. The Tate twist (ℓ) in $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$ sends the category $\mathsf{pHM}(X, w)$ to $\mathsf{pHM}(X, w + 2\ell)$. More precisely, if S is a polarization of M, then $(-1)^{\ell}\mathrm{S}$ is a polarization of $M(\ell)$.

***14.2.6. Strictness of* N.** We also note that, for an object M of $\mathsf{pHM}(X, w)$ and for any function $g : U \to \mathbb{C}$ such that $g^{-1}(0) \cap \operatorname{Supp} M$ has everywhere codimension 1 in $\operatorname{Supp} M$, the morphism N is strict on $\psi_{g,\lambda}M$ and $\phi_{g,1}M$: this follows from Proposition 9.4.10. We will relax below the restriction on g.

14.2.7. Stability by direct sums and isomorphisms. The category $\mathsf{pHM}(X, w)$ is stable by direct sums in $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$: this is clear for polarizable Hodge structures of weight w in the category $\widetilde{\mathbb{C}}\text{-}\mathsf{Triples}$ (see Section 5.2), and the general case follows by induction on the dimension of the support. Similarly, we obtain that any object of $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$ which is isomorphic of an object of $\mathsf{pHM}(X, w)$ is an object of $\mathsf{pHM}(X, w)$.

14.2.8. Stability by direct summands. The category $\mathsf{pHM}(X, w)$ is stable by direct summand in $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$. More precisely, if $\widetilde{\mathcal{T}}_1 \oplus \widetilde{\mathcal{T}}_2 = M$ is in $\mathsf{pHM}(X, w)$ and if S is a polarization of M, then $\widetilde{\mathcal{T}}_1, \widetilde{\mathcal{T}}_2$ are in $\mathsf{pHM}(X, w)$ and S induces a polarization on each of them. Indeed, the property of coherence and holonomicity restricts to direct summands, as well as strictness and the property of strict $\mathbb{R}$-specializability along any g (see Exercise 9.20(1)). We then argue by induction on the dimension of the support, the case of dimension zero reducing to Lemma 5.2.8 and Exercise 2.12(1). If the support has dimension $\geqslant 1$, let S_1 the morphism $\widetilde{\mathcal{T}}_1 \to \widetilde{\mathcal{T}}_1^*(-w)$ induced by S. Then, for any g such that $g^{-1}(0) \cap \operatorname{Supp} M$ has everywhere codimension 1 in $\operatorname{Supp} M$, $\mathrm{P}_{\ell}\psi_{g,\lambda}\mathrm{S}$ induces $\mathrm{P}_{\ell}\psi_{g,\lambda}\mathrm{S}_1$ on $\mathrm{P}_{\ell}\psi_{g,\lambda}M_1$, and this is a polarization by the induction hypothesis. A similar property holds for $\phi_{g,1}$, showing that $(\widetilde{\mathcal{T}}_1, \mathrm{S}_1)$ satisfies $(2)_g$.

***14.2.9. Proposition* (Kashiwara's equivalence).** *Let $Y \overset{\iota}{\hookrightarrow} X$ be a closed analytic submanifold of the analytic manifold X. The functor ${}_{\mathrm{T}}\iota_*$ induces an equivalence between $\mathsf{pHM}(Y, w)$ and $\mathsf{pHM}_Y(X, w)$ (objects supported on Y).*

Proof. Full faithfulness follows from Section 12.7.29. It follows that essential surjectivity is a local question, and more precisely, if essential surjectivity holds locally for polarized objects (M, S), it holds globally. In the local setting, we can argue by induction and assume that $Y = H$ is a smooth hypersurface. Then Proposition 9.6.6 (and its obvious variant for sesquilinear pairings, hence for objects in $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$), implies the assertion by induction on $\dim X$. ∎

14.2.10. Proposition **(Generic structure of polarizable Hodge modules)**
Let M be an object of $\mathsf{pHM}(X,w)$ *with support on an irreducible closed analytic set* $Z \overset{\iota}{\hookrightarrow} X$. *Then there exists an open dense set* $Z^o \subset S$ *and a smooth Hodge triple* H *of weight* w *on* Z^o, *such that* $M_{|Z^o} = {}_{\mathrm{T}}\iota_* H$. *In particular, if* $Z = X$, *then* $M_{|X^o}$ *is a smooth Hodge triple of weight* w.

Note that we use Definition 5.4.7 for a smooth Hodge triple, in order to have an object similar to the object M. By definition, it corresponds to a polarizable variation of Hodge structure of weight $w - \dim Z$ on Z^o.

Proof. Set $M = (\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}'', \mathfrak{s})$ and let S be a polarization. We first restrict to the smooth locus of S and apply Kashiwara's equivalence 14.2.9 to reduce to the case when $S = X$. By Corollary 9.7.13 (that we can apply since M is strict), there exists a dense open subset X^o of X such that $\widetilde{\mathcal{M}}'_{|X^o}$ and $\widetilde{\mathcal{M}}''_{|X^o}$ are $\widetilde{\mathcal{O}}_{X^o}$-locally free of finite rank. Then $\mathfrak{s}_{|X^o}$ takes values in $\widetilde{\mathcal{C}}^\infty_{|X^o}$ (see Lemma 12.3.6). We now restrict to X^o and argue by induction on $\dim X$. It will be convenient to use the left setting.

Let t be a local coordinate and set $H = \{t = 0\}$. We have seen in the proof of Proposition 9.7.10 that $\mathrm{gr}^0_V \widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}/t\widetilde{\mathcal{M}}$ for $\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}''$. After Remark 12.5.19 and Example 12.5.18, $\mathrm{gr}^0_V \mathfrak{s}$ is the restriction of $\mathfrak{s}$ to $t = 0$ as a C^∞ function. We conclude that $\psi_{t,1} M$ is the pushforward ${}_{\mathrm{T}}\iota_*^{-1} M_{|t=0}$. It is also pure of weight $w - 1$ since N is easily seen to be zero. Therefore, $M_{|t=0}$ is pure of weight $w - 1$ and, by induction on $\dim X$, is a smooth Hodge triple of weight $w - 1$. Since this holds for any H and since $\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}''$ are $\widetilde{\mathcal{O}}_X$-locally free, it is clear that M is a smooth Hodge triple of weight w. A similar argument shows that S is a polarization of this smooth Hodge triple. □

14.2.11. Caveat. At this point, we do not know the converse property that a polarizable smooth Hodge triple of weight w on X is an object of $\mathsf{pHM}(X,w)$, since we have not checked that $(2)_g$ holds for *any* nonzero g for such a triple. This will be done in Theorem 14.6.1.

14.2.b. Abelianity and the S-decomposition theorem

Before proving the main properties of $\mathsf{pHM}(X,w)$, we introduce other categories which will prove useful at some intermediate steps.

14.2.12. The category of W-filtered Hodge modules. As a first approximation of the category of mixed Hodge modules, we consider the category $\mathsf{WHM}(X)$: this is the full subcategory of $\mathsf{W}\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$ (see Section 2.6.b) such that, for each object $(\widetilde{\mathcal{T}}, W_\bullet \widetilde{\mathcal{T}})$, the graded object $\mathrm{gr}^W_\ell \widetilde{\mathcal{T}}$ belongs to $\mathsf{pHM}(X,\ell)$. We denote the objects of $\mathsf{WHM}(X)$ as $(M, W_\bullet M)$. We can regard each $\mathsf{pHM}(X,w)$ as a full subcategory of $\mathsf{WHM}(X)$ by considering on M the filtration $W_\bullet$ which jumps at w only.

14.2.13. The category of polarizable Hodge-Lefschetz modules. We also consider the category $\mathsf{pHLM}(X,w)$ of *polarizable Hodge-Lefschetz modules with central weight* w. An object in this category consists of a Lefschetz triple $(\widetilde{\mathcal{T}}, \mathrm{N})$ (see Section 12.7.11), that is, $\widetilde{\mathcal{T}}$ is an object of $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$ and N is a nilpotent endomorphism of $\widetilde{\mathcal{T}}$, such that there exists a pre-polarization $\mathrm{S} : (\widetilde{\mathfrak{T}}, \mathrm{N}) \to (\widetilde{\mathfrak{T}}, \mathrm{N})^*(-w)$ of weight w satisfying (see Section 5.3)

- $(\mathrm{P}_\ell\widetilde{\mathfrak{T}}, (-1)^\ell \mathrm{P}_\ell\mathrm{S})$ is a polarized Hodge module of weight $w + \ell$ for every $\ell \geqslant 0$, where $\mathrm{P}_\ell\mathrm{S}$ is the morphism $\mathrm{P}_\ell\mathrm{S} : \mathrm{P}_\ell\widetilde{\mathfrak{T}} \to (\mathrm{P}_\ell\widetilde{\mathfrak{T}})^*(-(w+\ell))$ defined in a way similar to that of Sections 3.2.11 and 3.4.c.

We denote an object of $\mathsf{pHLM}(X, w)$ as (M, N) and we also say that $(M, \mathrm{N}, \mathrm{S})$ is a *polarized Hodge-Lefschetz module with central weight w*. From the Lefschetz decomposition, we deduce that, setting $W_kM := \mathrm{M}_{k-w}M$ (i.e., $\mathrm{M}_\ell = W_{w+\ell}$), $(M, W_\bullet)$ is an object of $\mathsf{WHM}(X)$ (but $\mathsf{pHLM}(X, w)$ is not a full subcategory of $\mathsf{WHM}(X)$, since morphisms have to commute with N).

14.2.14. Caveat. We do not claim that objects and morphisms in $\mathsf{WHM}(X)$ or $\mathsf{pHLM}(X, w)$ are strictly specializable along any (g). On the other hand, objects and morphisms in the graded category $\mathsf{p}\mathfrak{sl}_2\mathsf{HM}(X, w)$ defined below are so, since they are graded with respect to the weight or monodromy filtration.

14.2.15. The category of polarizable Hodge-Lefschetz quivers. In a way similar to that of Definition 3.4.19, we also define the notion of *polarized/polarizable Hodge-Lefschetz quiver with central weight w*, starting from a Lefschetz quiver in $\widetilde{\mathfrak{D}}\text{-}\mathsf{Triples}(X)$ (defined in a way similar to what is done in Section 5.3.6): such an object consists of the data $\big((M, \mathrm{N}, \mathrm{S}), (M_1, \mathrm{N}_1, \mathrm{S}_1), \mathrm{c}, \mathrm{v}\big)$, where the first terms are polarized Hodge-Lefschetz modules of weight $w - 1$ and w respectively, and $\mathrm{c} : M \to M_1$ and $\mathrm{v} : M_1 \to M(-1)$ are morphisms in $\widetilde{\mathfrak{D}}\text{-}\mathsf{Triples}(X)$ such that $\mathrm{v} \circ \mathrm{c} = \mathrm{N}$, $\mathrm{c} \circ \mathrm{v} = \mathrm{N}_1$ and the following diagram commutes (see (3.2.14)):

$$\begin{array}{ccc} M_1 & \xrightarrow{\ \mathrm{S}_1\ } & M_1^*(-w) \\ {\scriptstyle \mathrm{v}}\big\downarrow & & \big\downarrow{\scriptstyle -\mathrm{c}^*} \\ M(-1) & \xrightarrow{\ \mathrm{S}\ } & M^*(-w) \end{array}$$

The corresponding category is denoted by $\mathsf{pHLQ}(X, w)$.

We can rephrase Condition $(2)_g$ of Theorem 14.2.2 as follows:

$(2')_g$ if moreover $g^{-1}(0) \cap \operatorname{Supp}\widetilde{\mathfrak{T}}$ has everywhere codimension 1 in $\operatorname{Supp}\widetilde{\mathfrak{T}}$, then for every $\ell \geqslant 0$ and every $\lambda \in \mathbb{S}^1$,

(a) for each $\lambda \in \mathbb{S}^1 \smallsetminus \{1\}$, $(\psi_{g,\lambda}\widetilde{\mathfrak{T}}, \mathrm{N}, \psi_{g,\lambda}\mathrm{S})$ is an object of $\mathsf{pHLM}(X, w-1)$,

(b) the set of data $\big((\psi_{g,1}\widetilde{\mathfrak{T}}, \psi_{g,1}\mathrm{S}), (\phi_{g,1}\widetilde{\mathfrak{T}}, \phi_{g,1}\mathrm{S}), \mathrm{can}, \mathrm{var}\big)$ is a polarized object of $\mathsf{pHLQ}(X, w)$.

Indeed, the only properties which need a check are those for can and var, and they have been proved in 12.7.16.

14.2.16. The category of polarizable $\mathfrak{sl}_2$-Hodge modules. The category $\mathsf{p}\mathfrak{sl}_2\mathsf{HM}(X, w)$ of *polarizable $\mathfrak{sl}_2$-Hodge modules with central weight w* consists of objects of $\mathsf{pHLM}(X, w)$ which are *graded* with respect to their monodromy filtration $\mathrm{M}_\bullet$. Morphisms should also be graded. A polarization of an object $(M_\bullet, \mathrm{N})$ of $\mathsf{p}\mathfrak{sl}_2\mathsf{HM}(X, w)$ is by definition a polarization of $(M_\bullet, \mathrm{N})$ as an object of $\mathsf{pHLM}(X, w)$ which is *graded*. This is not a restrictive condition since the conditions on the polarization S in $\mathsf{pHLM}(X,w)$ concern grS (see Section 3.4.c).

Therefore, given an object (M,N) of $\mathsf{pHLM}(X,w)$, the graded object $(\mathrm{gr}^{\mathrm{M}}_{\bullet}M,\mathrm{grN})$ is an object of $\mathsf{p}\mathfrak{sl}_2\mathsf{HM}(X,w)$ and, conversely, any object of $\mathsf{p}\mathfrak{sl}_2\mathsf{HM}(X,w)$ is an object of $\mathsf{pHLM}(X,w)$ (by forgetting the grading). On the other hand, morphisms in $\mathsf{p}\mathfrak{sl}_2\mathsf{HM}(X,w)$ are graded with respect the given grading. So, by definition, there is a functor gr^{M} from $\mathsf{pHLM}(X,w)$ to $\mathsf{p}\mathfrak{sl}_2\mathsf{HM}(X,w)$. We denote by $(M_\bullet,\rho)$ an object of $\mathsf{p}\mathfrak{sl}_2\mathsf{HM}(X,w)$, where ρ is meant for the corresponding $\mathfrak{sl}_2$-representation with $\rho(\mathrm{H})$ defined by means of the grading and $\rho(\mathrm{Y})=\mathrm{N}$.

One can set $\mathrm{H}=\ell\,\mathrm{Id}$ on M_ℓ and, due to Proposition 3.1.6 applied with the category $\mathsf{A}=\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$, one can extend uniquely $\mathrm{Y}=\mathrm{N},\mathrm{H}$ as an $\mathfrak{sl}_2$-triple $\mathrm{X},\mathrm{Y},\mathrm{H}$. Then X induces morphisms $M_\ell\to M_{\ell+2}(1)$ in $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$.

On the other hand, given a finitely $\mathbb{Z}$-graded object $M_\bullet$ of $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$ endowed with an endomorphism X which satisfies the Lefschetz property, there is a unique action of Y defining a representation ρ of $\mathfrak{sl}_2$ on $M_\bullet$ such that H is defined by means of the grading. We then say that $(M_\bullet,\mathrm{X})$ is an object of $\mathsf{p}\mathfrak{sl}_2\mathsf{HM}(X,w)$ if $(M_\bullet,\rho)$ is so.

We set $(M^*)_\ell=(M_{-\ell})^*$. Then $M^*(-w)$ is also an object of $\mathsf{p}\mathfrak{sl}_2\mathsf{HM}(X,w)$. By definition, a polarization S of M is a (graded, by definition) morphism $\mathrm{S}:M\to M^*(-w)$ such that $(-1)^\ell\mathrm{P}_\ell\mathrm{S}$ is a polarization of $\mathrm{P}_\ell M$ for every $\ell\geqslant 0$.

14.2.17. Theorem (Main properties of polarizable Hodge modules)

(1) *Any object* $M=(\widetilde{\mathcal{M}}',\widetilde{\mathcal{M}}'',\mathfrak{s})$ *of* $\mathsf{pHM}(X,w)$ *is S-decomposable in* $\mathsf{pHM}(X,w)$, *and the components of the pure support of* $\widetilde{\mathcal{M}}'$ *and* $\widetilde{\mathcal{M}}''$ *are the same.*

(2) *There is no nonzero morphism (in* $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$*) from an object in* $\mathsf{pHM}(X,w_1)$ *to an object in* $\mathsf{pHM}(X,w_2)$ *if* $w_1>w_2$.

(3) *Property* 14.2.2$(2)_g$ *holds without any restriction on* g.

(4) *The category* $\mathsf{pHM}(X,w)$ *is abelian. Any morphism is strict and strictly specializable along any* (g).

(5) *Any polarization of an object of* $\mathsf{pHM}(X,w)$ *or* $\mathsf{pHLM}(X,w)$ *is a Hermitian isomorphism (i.e., a pre-polarization of weight* w *of the corresponding triple).*

(6) *If* M_1 *is a subobject of* M *in* $\mathsf{pHM}(X,w)$*, then it is a direct summand and a polarization* S *of* M *induces a polarization on each summand.*

(7) *The category* $\mathsf{p}\mathfrak{sl}_2\mathsf{HM}(X,w)$ *is abelian. Any morphism is strict and strictly specializable along any* (g)*. Any sub-object of an object* $(M_\bullet,\rho)$ *in* $\mathsf{p}\mathfrak{sl}_2\mathsf{HM}(X,w)$ *is a direct summand and a polarization of* $(M_\bullet,\rho)$ *induces a polarization on it.*

(8) *The category* $\mathsf{WHM}(X)$ *is abelian, and any morphism is strict and strictly compatible with* $W_\bullet$.

(9) *The category* $\mathsf{pHLM}(X,w)$ *is abelian. Any morphism is strict and strictly compatible with the monodromy filtration* $\mathrm{M}_\bullet$.

(10) *Any polarizable Hodge-Lefschetz quiver* $\big(M,M_1,\mathrm{c},\mathrm{v}\big)$ *with central weight* w *satisfies* $(M_1,\mathrm{N}_1)=\mathrm{Im}\,\mathrm{c}\oplus\mathrm{Ker}\,\mathrm{v}$ *in* $\mathsf{pHLM}(X,w)$.

Let us emphasize some direct consequences of the theorem.

14.2.18. Notation. If $Z \subset X$ is a closed irreducible analytic subset, we denote by $\mathsf{pHM}_Z(X,w)$ the full sub-category of $\mathsf{pHM}(X,w)$ whose objects have pure support Z. By the S-decomposition property 14.2.17(1), Any object of $\mathsf{pHM}(X,w)$ resp. any morphism between objects of $\mathsf{pHM}(X,w)$ decomposes as the direct sum of objects resp. morphisms in of $\mathsf{pHM}_{Z_i}(X,w)$ for a suitable locally finite family of closed irreducible analytic subsets $Z_i \subset X$.

14.2.19. Corollary. *Given two objects M_1, M_2 in $\mathsf{pHM}(X,w)$, any morphism between them (as objects of $\widetilde{\mathfrak{D}}$-$\mathsf{Triples}(X)$) has kernel, image and cokernel in $\mathsf{pHM}(X,w)$; and a corresponding statement for $\mathsf{pHLM}(X,w)$ and $\mathsf{p}\mathfrak{sl}_2\mathsf{HM}(X,w)$.* □

***14.2.20. Corollary* (S-decomposition theorem and semi-simplicity for $\mathsf{pHM}(X,w)$)**

(1) *Each object M decomposes uniquely into the direct sum of objects in $\mathsf{pHM}(X,w)$ having pure support a closed irreducible analytic subset of X.*

(2) *The category $\mathsf{pHM}(X,w)$ is semi-simple (all objects are semi-simple and morphisms between simple objects are zero or isomorphisms).*

(3) *The category $\mathsf{p}\mathfrak{sl}_2\mathsf{HM}(X,w)$ is semi-simple.* □

14.2.21. Corollary. *If M is an object of $\mathsf{pHM}(X,w)$ with polarization S, then for every open subset $U \subset X$ and every holomorphic function $g : U \to \mathbb{C}$,*

(1) *for every $\ell \geqslant 1$, $\mathrm{N}^\ell : \psi_{g,\lambda}M \to \psi_{g,\lambda}M(-\ell)$ and $\phi_{g,1}M \to \phi_{g,1}M(-\ell)$ are strict and strictly shift $\mathrm{M}_\bullet(\mathrm{N})$ by 2ℓ, and a similar property holds for grN^ℓ,*

(2) $\mathrm{can} : \psi_{g,1}M \to \phi_{g,1}M$ *and* $\mathrm{var} : \phi_{g,1}M \to \psi_{g,1}M(-1)$ *are strict and strictly shift $\mathrm{M}_\bullet$ by 1.* □

***14.2.22. Corollary* (The vanishing cycle theorem).** *Let $(M, \mathrm{N}, \mathrm{S})$ be a polarized object of $\mathsf{pHLM}(X, w-1)$. Let us endow $(\operatorname{Im}\mathrm{N}, \mathrm{N}_{|\operatorname{Im}\mathrm{N}})$ with the morphism*

$$\mathrm{S}_1 : (\operatorname{Im}\mathrm{N}, \mathrm{N}_{|\operatorname{Im}\mathrm{N}}) \longrightarrow (\operatorname{Im}\mathrm{N}, \mathrm{N}_{|\operatorname{Im}\mathrm{N}})(-w)$$

such that the following diagram commutes:

$$\begin{array}{ccc} \operatorname{Im}\mathrm{N} & \xrightarrow{\ \mathrm{S}_1\ } & (\operatorname{Im}\mathrm{N})^*(-w) \\ {\scriptstyle\text{incl.}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{N}^*} \\ M(-1) & \xrightarrow{\ \mathrm{S}\ } & M^*(-w) \end{array}$$

Then $(\operatorname{Im}\mathrm{N}, \mathrm{N}_{|\operatorname{Im}\mathrm{N}}, \mathrm{S}_1)$ a polarized object of $\mathsf{pHLM}(X,w)$.

Proof. We use the same argument as in the proof of Proposition 3.4.20. Strictness of $\mathrm{can} = \mathrm{N} : M \to \operatorname{Im}\mathrm{N}$, $\mathrm{var} = \mathrm{incl.} : \operatorname{Im}\mathrm{N} \to M(-1)$, $\mathrm{can}^* = \mathrm{N}^*$, and S (according to 14.2.17(9)) enables us to reduce the problem to the graded case. We note that, arguing as in (3.2.16), for $\ell \geqslant 0$, the isomorphism $\mathrm{can} : \mathrm{P}_{\ell+1}M \xrightarrow{\sim} \mathrm{P}_\ell M_1$ transports the polarization $(-1)^{\ell+1}\mathrm{P}_{\ell+1}\mathrm{S}$ to $(-1)^\ell \mathrm{P}_\ell \mathrm{S}_1$. □

14.2.23. Corollary. *Given any morphism $\varphi : M_1 \to M_2$ between objects of* $\mathsf{pHM}(X,w)$ *and any germ g of holomorphic function on X, then, the specialized morphisms $\psi_{g,\lambda}\varphi$ ($\lambda \in \mathbb{S}^1$) and $\phi_{g,1}\varphi$ are strictly compatible with the monodromy filtration $\mathrm{M}_\bullet$ and, for every $\ell \in \mathbb{Z}$, $\mathrm{gr}_\ell^{\mathrm{M}}\psi_{g,\lambda}\varphi$ (and similarly $\mathrm{gr}_\ell^{\mathrm{M}}\phi_{g,1}\varphi$) decomposes with respect to the Lefschetz decomposition, i.e.,*

$$\mathrm{gr}_\ell^{\mathrm{M}}\psi_{g,\lambda}\varphi = \begin{cases} \bigoplus_{k\geqslant 0} \mathrm{N}^k \mathrm{P}_{\ell+2k}\psi_{g,\lambda}\varphi & (\ell \geqslant 0), \\ \bigoplus_{k\geqslant 0} \mathrm{N}^{k-\ell} \mathrm{P}_{-\ell+2k}\psi_{g,\lambda}\varphi & (\ell \leqslant 0). \end{cases}$$

In particular we have

$$\mathrm{gr}_\ell^{\mathrm{M}}\psi_{g,\lambda}\operatorname{Ker}\varphi = \operatorname{Ker}\mathrm{gr}_\ell^{\mathrm{M}}\psi_{g,\lambda}\varphi$$

and similarly for Coker*, where, on the left side, the filtration $\mathrm{M}_\bullet$ is that induced naturally by $\mathrm{M}_\bullet\psi_{g,\lambda}M_1$ or, equivalently, the monodromy filtration of* N *acting on* $\psi_{g,\lambda}\operatorname{Ker}\varphi = \operatorname{Ker}\psi_{g,\lambda}\varphi$. ☐

14.2.24. Corollary. *If M is in* $\mathsf{pHM}(X,w)$*, then the Lefschetz decomposition for* $\mathrm{gr}_\ell^{\mathrm{M}}\psi_{g,\lambda}M$ *($\lambda \in \mathbb{S}^1$) resp.* $\mathrm{gr}_\ell^{\mathrm{M}}\phi_{g,1}M$ *holds in* $\mathsf{pHM}(X,w-1+\ell)$ *resp.* $\mathsf{pHM}(X,w+\ell)$.

Proof. Indeed, $\mathrm{N} : \mathrm{gr}_\ell^{\mathrm{M}}\psi_{g,\lambda}M \to \mathrm{gr}_{\ell-2}^{\mathrm{M}}\psi_{g,\lambda}M(-1)$ is a morphism in the category $\mathsf{pHM}(X,w-1+\ell)$, which is abelian, so the primitive part is an object of this category, and therefore each term of the Lefschetz decomposition is also an object of this category. ☐

Similarly to Proposition 7.4.9, we can simplify the data of a polarizable Hodge module.

14.2.25. Proposition **(Simplified form for an object of $\mathsf{pHM}(X,w)$ or $\mathsf{pHLM}(X,w)$)**
Any object M of $\mathsf{pHM}(X,w)$ *resp.* (M,N) *of* $\mathsf{pHLM}(X,w)$*, resp.* $(M_\bullet,\mathrm{N})$ *of* $\mathsf{p\mathfrak{sl}_2HM}(X,w)$*, is isomorphic to an object of the form*

$$\big((\mathcal{M},F^\bullet\mathcal{M}),(\mathcal{M},F^\bullet\mathcal{M})(w),\mathcal{S}\big)$$

(resp. ...) such that $\mathcal{S}^*=\mathcal{S}$ and with polarization $(\mathrm{Id},\mathrm{Id}) : M \to M^*(-w)$. ☐

We call the data $\big((\mathcal{M},F^\bullet\mathcal{M}),\mathcal{S}\big)$ a *Hodge-Hermitian pair of weight w* (resp. *Hodge-Lefschetz Hermitian pair with central weight w*, resp. $\mathfrak{sl}_2$*-Hodge Hermitian pair with central weight w*) if the corresponding triple $\big((\mathcal{M},F^\bullet\mathcal{M}),(\mathcal{M},F^\bullet\mathcal{M})(w),\mathcal{S}\big)$ with polarization $(\mathrm{Id},\mathrm{Id})$ is polarized Hodge module of weight w (resp. ...).

14.2.26. Example **(of filtered Hermitian pairs).** We consider the following corresponding filtered Hermitian pairs (see Example 12.3.5)

$${}_{\mathrm{H}}\mathcal{O}_X := ((\mathcal{O}_X,F_\bullet\mathcal{O}_X),\mathfrak{s}_n^{\mathrm{left}}),\quad {}_{\mathrm{H}}\omega_X := ((\omega_X,F_\bullet\omega_X),\mathfrak{s}_n^{\mathrm{right}}).$$

We will prove in Theorem 14.6.1 that they are Hodge-Hermitian pairs of weight n. The case where $n=1$ is a consequence of the results in Chapter 7 (see Exercise 14.1).

***Proof of Theorem* 14.2.17.** It is done by induction on the dimension of the support. By the point (0) in Definition 14.2.2, the categories of objects with support equal to a point as considered in the theorem are equivalent to the corresponding categories for $X = \mathrm{pt}$. In such a case, the assertions of the theorem are proved in Chapters 2 and 3.

We will thus fix $d \geqslant 1$ and assume that the assertions are proved for the subcategories consisting of objects having support of dimension $< d$, in order to prove them when the dimension of the support of M is d.

$(10)_{<d} \implies (1)_d$. Let $x_o \in \operatorname{Supp} M$ and let g be a germ of holomorphic function at x_o such that $g^{-1}(0) \cap \operatorname{Supp} M$ has everywhere codimension 1 in M. By Condition 14.2.13$(2')_g$, the nearby/vanishing quiver of M along (g) satisfies the assumption of $(10)_{<d}$, hence its conclusion, so M is S-decomposable along (g) in $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$. By (14.2.8), the summands also belong to $\mathsf{pHM}(X, w)$. This proves S-decomposability in $\mathsf{pHM}(X, w)$.

We assume that there is a pure component Z' of $\operatorname{Supp} \widetilde{\mathcal{M}}'$ which is not a pure component of $\operatorname{Supp} \widetilde{\mathcal{M}}''$. Then we have a summand $(\widetilde{\mathcal{M}}'_{Z'}, 0, 0)$ of M in $\mathsf{pHM}(X, w)$, according to the previous argument. We wish to show that $\widetilde{\mathcal{M}}'_{Z'} = 0$, and it is enough, by the condition of the pure support, to show the vanishing on the smooth locus of Z'. We can thus reduce to the case where $Z' = X$, according to Proposition 9.7.10.

We now argue by induction on $\dim X$, the case $\dim X = 0$ reducing to the case of Hodge structures, which is easy. Let t be a local coordinate on X. Arguing as in Corollary 9.7.11, one checks that $\widetilde{\mathcal{M}}'_X / t\widetilde{\mathcal{M}}'_X = \psi_{t,1}\widetilde{\mathcal{M}}'_X$, and that $\psi_{t,\lambda}\widetilde{\mathcal{M}}'_X = 0$ for $\lambda \in \mathbb{S}^1 \smallsetminus \{1\}$, as well as $\phi_{t,1}\widetilde{\mathcal{M}}'_X = 0$. It follows that $\mathrm{N} = 0$, so $\psi_{t,1}\widetilde{\mathcal{M}}'_X$ is S-decomposable, according to Condition 14.2.2$(2)_t$. By induction, the object $\psi_{t,1}(\widetilde{\mathcal{M}}'_X, 0, 0)$ is zero. Hence $\widetilde{\mathcal{M}}'_X / t\widetilde{\mathcal{M}}'_X = 0$, and by applying Nakayama's lemma as in Corollary 9.7.11, we obtain $\widetilde{\mathcal{M}}_X = 0$.

$(1)_d \implies (2)_d$. Since any morphism between S-decomposable objects decomposes correspondingly, it is enough to consider a morphism $\varphi : M_1 \to M_2$ between polarizable Hodge modules of respective weights w_1, w_2 having pure support. Since the result is clear for polarizable variations of Hodge structure (see Proposition 2.5.6(2)), it follows from Proposition 14.2.10 that the support of $\operatorname{Im} \varphi$ is strictly smaller than Z. By definition of the pure support (see Definition 9.7.9), this implies that $\operatorname{Im} \varphi = 0$.

$(1)_d \implies (3)_d$. The question is local at x_o and by assumption we can assume that M_{x_o} has pure support a closed irreducible subset $Z_{x_o} \subset X_{x_o}$. Let $g : X_{x_o} \to \mathbb{C}$ be a germ of holomorphic function. If g is non-constant on Z_{x_o}, it satisfies the constraint in Definition 14.2.2$(2)_g$. Otherwise, $\operatorname{Supp} M_{x_o} \subset |g^{-1}(0)|$ and Proposition 12.7.15 implies that $M_{x_o} = \phi_{g,1} M_{x_o}$ (and similarly $\mathrm{S} = \phi_{g,1}\mathrm{S}$). Moreover, $\psi_{g,\lambda} M_{x_o} = 0$ for any $\lambda \in \mathbb{S}^1$, and $\mathrm{N} = 0$. Hence, if M_{x_o} is an object of $\mathsf{pHM}((X, x_o), w)$, 14.2.2$(2)_g$ obviously holds.

$(4)_{<d}$, $(6)_{<d}$ & $(8)_{<d} \implies (4)_d$. The question is local. Let $\varphi : M_1 \to M_2$ be a morphism in $\mathsf{pHM}(X, w)$ between objects having support in dimension d. Then, by

$(8)_{<d}$ applied to $\psi_{g,\lambda}\varphi, \phi_{g,1}\varphi$, for any germ g satisfying the constraint of Definition 14.2.2$(2)_g$, $\varphi : M_1 \to M_2$ is strictly $\mathbb{R}$-specializable along (g) and Corollary 10.7.6 implies that it is strict. Moreover, $\psi_{g,\lambda}\varphi$ and $\phi_{t,1}\varphi$ are strict with respect to the monodromy filtrations, since these are weight filtrations up to a shift.

It remains to check that $\operatorname{Ker}\varphi, \operatorname{Im}\varphi; \operatorname{Coker}\varphi$ belong to $\mathsf{pHM}(X, w)$. Let us check this for $\operatorname{Ker}\varphi$ for example. It follows from the M-strictness above that

$$\operatorname{gr}^{\mathrm{M}}_\ell \psi_{g,\lambda} \operatorname{Ker}\varphi = \operatorname{Ker}\operatorname{gr}^{\mathrm{M}}_\ell \psi_{g,\lambda}\varphi$$

and thus, for any $\ell \geqslant 0$, $\mathrm{P}_\ell\psi_{g,\lambda}\operatorname{Ker}\varphi = \operatorname{Ker}\mathrm{P}_\ell\psi_{g,\lambda}\varphi$. Since $\mathrm{P}_\ell\psi_{g,\lambda}\varphi$ is a morphism in $\mathsf{pHM}(X, w-1+\ell)$ between objects having support in dimension $< d$, $(4)_{<d}$ implies that $\operatorname{Ker}\mathrm{P}_\ell\psi_{g,\lambda}\varphi$ is an object of $\mathsf{pHM}(X, w-1+\ell)$ and, according to $(6)_{<d}$, is a direct summand of $\mathrm{P}_\ell\psi_{g,\lambda}M_1$. If S is a polarization of M_1, let S_φ denote the morphism induced by S on $\operatorname{Ker}\varphi$. On the one hand, the morphism induced by $-\mathrm{P}_\ell\psi_{g,\lambda}\mathrm{S}$ on $\operatorname{Ker}\mathrm{P}_\ell\psi_{g,\lambda}\varphi$ is a polarization, according to 14.2.8. On the other hand, it is equal to $-\mathrm{P}_\ell\psi_{g,\lambda}\mathrm{S}_\varphi$. We can argue similarly with $\phi_{g,1}$, by assumption on g. This shows that $(\operatorname{Ker}\varphi, \mathrm{S}_\varphi)$ satisfies 14.2.2$(2)_g$.

$(4)_d$ & $(5)_{<d} \implies (5)_d$. A polarization S of M is a morphism $M \to M^*(-w)$, hence it is strict and strictly specializable along any (g). Let g be a holomorphic function such that $g^{-1}(0) \cap \operatorname{Supp} M$ has everywhere codimension 1 in $\operatorname{Supp} M$. $(5)_{<d}$ implies that $\mathrm{P}_\ell\psi_{g,\lambda}\mathrm{S}$ and $\mathrm{P}_\ell\phi_{g,1}\mathrm{S}$ are isomorphisms for every $\ell \geqslant 0$, which implies the same property for $\operatorname{gr}^{\mathrm{M}}_\ell\psi_{g,\lambda}\mathrm{S}$ and $\operatorname{gr}^{\mathrm{M}}_\ell\phi_{g,1}\mathrm{S}$ and thus for $\psi_{g,\lambda}\mathrm{S}$ and $\phi_{g,1}\mathrm{S}$. By strict $\mathbb{R}$-specializability, $\psi_{g,\lambda}$ and $\phi_{g,1}$ commute with taking Ker and Coker on S. We conclude that $\psi_{g,\lambda}\operatorname{Ker}\mathrm{S} = 0$ and $\phi_{g,1}\operatorname{Ker}\mathrm{S} = 0$, and similarly with Coker. Since $\operatorname{Ker}\mathrm{S}$ and $\operatorname{Coker}\mathrm{S}$ are in $\mathsf{pHM}(X, w)$ by $(4)_d$, we can apply to them the regularity property along (g) of Corollary 10.7.5, which implies they both are zero.

That S is Hermitian is obtained similarly by applying the argument to $\operatorname{Im}(\mathrm{S} - \mathrm{S}^*)$.

$(1)_d \implies (6)_d$. A polarization of M decomposes with respect to the S-decomposition of M, and it is clear that it induces a polarization on each summand. We can thus restrict to considering objects M with pure support a closed irreducible analytic subset Z of X.

If $\dim Z = 0$, we apply Exercise 2.12. If $\dim Z \geqslant 1$, we consider the exact sequences (defining S_1)

$$\begin{array}{ccccccccc} 0 & \longleftarrow & M_1^*(-w) & \xleftarrow{\;i^*\;} & M^*(-w) & \longleftarrow & M_2^*(-w) & \longleftarrow & 0 \\ & & \uparrow{\scriptstyle \mathrm{S}_1} & & \uparrow{\scriptstyle \mathrm{S}\,\wr} & & & & \\ 0 & \longrightarrow & M_1 & \xrightarrow{\;i\;} & M & \longrightarrow & M_2 & \longrightarrow & 0 \end{array}$$

where M_2 is the cokernel, in the abelian category $\mathsf{pHM}(X, w)$, of $M_1 \hookrightarrow M$. We first show that S_1 is an isomorphism. It is enough to prove it on an open dense subset Z^o of Z. By Kashiwara's equivalence 14.2.9 and the generic structure 14.2.10, we are reduced to considering the case of polarizable variations of Hodge structure, which is follows from Exercise 4.2. We conclude that we have a projection

$p = \mathrm{S}_1^{-1} \circ i^* \circ \mathrm{S} : M \to M_1$ such that $p \circ i = \mathrm{Id}$, and a decomposition $M = M_1 \oplus \mathrm{S}^{-1} M_2^*(-w)$. By construction, S splits correspondingly, and it is then clear that each summand is a polarization.

$(4)_d$ & $(6)_d \implies (7)_d$. Abelianity and strictness resp. strict $\mathbb{R}$-specializability of morphisms follow from $(4)_d$ in a straightforward way by the grading property. In the same way, $(6)_d$ implies the similar property for $\mathsf{p}\mathfrak{sl}_2\mathsf{HM}(X, w)$.

$(4)_d$ & $(2)_d \implies (8)_d$. We note first that, since objects of $\mathsf{pHM}(X, w)$ are strict, Lemma 5.1.9(1) implies that the $\widetilde{\mathcal{D}}_X$-modules which are components of an object in $\mathsf{WHM}_{\leqslant d}(X)$ are strict. According to $(2)_d$ and Proposition 2.6.3, $(4)_d$ implies that the category $\mathsf{WHM}_{\leqslant d}(X)$ is abelian and that morphisms are strictly compatible with W. Using Lemma 5.1.9(2), we conclude that all morphisms are strict.

$(8)_d \implies (9)_d$. Since $\mathsf{pHLM}(X, w)$ is a subcategory of $\mathsf{WHM}(X)$ with the weight filtration given by the shifted monodromy filtration, strictness of morphisms and strict compatibility with $W_\bullet$ follow from $(8)_d$.

$(7)_d$, $(8)_d$ & $(10)_{<d} \implies (10)_d$. Since c, v are morphisms in $\mathsf{WHM}(X)$, they are strictly compatible with the weight filtration, due to $(8)_d$, hence strictly shift by -1 the monodromy filtrations. We then denote by $\mathrm{gr}\,\mathrm{c}, \mathrm{gr}\,\mathrm{v}$ the corresponding morphisms, graded of degree -1 with respect to $\mathrm{M}_\bullet$. We then have $\mathrm{gr}^{\mathrm{M}} \mathrm{Im}\,\mathrm{c} = \mathrm{Im}\,\mathrm{gr}\,\mathrm{c}$ and $\mathrm{gr}^{\mathrm{M}} \mathrm{Ker}\,\mathrm{v} = \mathrm{Ker}\,\mathrm{gr}\,\mathrm{v}$. Moreover, the natural morphism $\mathrm{Im}\,\mathrm{c} \oplus \mathrm{Ker}\,\mathrm{v} \to M_1$ is strict with respect to the weight filtration, hence to the monodromy filtrations. It follows that, if the graded morphism $\mathrm{Im}\,\mathrm{gr}\,\mathrm{c} \oplus \mathrm{Ker}\,\mathrm{gr}\,\mathrm{v} \to \mathrm{gr}^{\mathrm{M}} M_1$ is an isomorphism, then $M_1 = \mathrm{Im}\,\mathrm{c} \oplus \mathrm{Ker}\,\mathrm{v}$, as wanted. We are therefore reduced to proving the assertion in the category of polarizable graded Hodge-Lefschetz quivers.

In such a case, $M, M_1, \mathrm{c}, \mathrm{v}$ are strict and strictly $\mathbb{R}$-specializable along any (g), according to $(7)_d$, and by the regularity property (Corollary 10.7.5), it is enough to prove locally, for any holomorphic germ g, the decompositions

$$\begin{aligned}\psi_{g,\lambda} M_1 &= \mathrm{Im}\,\psi_{g,\lambda}\mathrm{c} \oplus \mathrm{Ker}\,\psi_{g,\lambda}\mathrm{v}, \quad \forall\,\lambda \in \mathbb{S}^1,\\ \phi_{g,1} M_1 &= \mathrm{Im}\,\phi_{g,1}\mathrm{c} \oplus \mathrm{Ker}\,\phi_{g,1}\mathrm{v}.\end{aligned}$$

Let us argue with $\phi_{g,1}$ for example. Recall that $M = \bigoplus_\ell M_\ell$ and $M_1 = \bigoplus_\ell M_{1,\ell}$, with $M_\ell \in \mathsf{pHM}(X, w-1+\ell)$ and $M_{1,\ell-1} \in \mathsf{pHM}(X, w+\ell-1)$, and that $\phi_{g,1}\mathrm{c}$ is a morphism $\phi_{g,1} M_\ell \to \phi_{g,1} M_{1,\ell-1}$. It is strictly compatible with the weight filtration on these spaces, which is nothing but $\mathrm{M}_{w+\ell-1+\bullet}(\mathrm{N}_g)$, hence with the monodromy filtration of N_g. The same argument holds for v. It is thus enough to prove

$$\mathrm{gr}_j^{\mathrm{M}} \phi_{g,1} M_{1,\ell-1} = \mathrm{Im}\,\mathrm{gr}_j^{\mathrm{M}} \phi_{g,1}\mathrm{c} \oplus \mathrm{Ker}\,\mathrm{gr}_j^{\mathrm{M}} \phi_{g,1}\mathrm{v}.$$

We can therefore apply $(10)_{<d}$ to the quiver

$$(\mathrm{gr}_j^{\mathrm{M}} \phi_{g,1} M_\ell, \mathrm{gr}_j^{\mathrm{M}} \phi_{g,1} M_{1,\ell-1}, \mathrm{gr}_j^{\mathrm{M}} \phi_{g,1}\mathrm{c}, \mathrm{gr}_j^{\mathrm{M}} \phi_{g,1}\mathrm{v}),$$

with central weight $w + \ell - 1 + j$. □

14.2.27. The category of polarizable $\mathfrak{sl}_2^{\mathrm{k}}$-Hodge modules

In the presence of k commuting nilpotent endomorphisms, we can extend the definition of the category $\mathsf{p}\mathfrak{sl}_2\mathsf{HM}(X,w)$ of $\mathfrak{sl}_2$-Hodge modules to that of the category $\mathsf{p}\mathfrak{sl}_2^k\mathsf{HM}(X,w)$ of $\mathfrak{sl}_2^{\mathrm{k}}$-Hodge modules. The objects of $\mathsf{p}\mathfrak{sl}_2^k\mathsf{HM}(X,w)$ are $\mathbb{Z}^k$-graded polarizable Hodge modules $M=\bigoplus_{\boldsymbol{\ell}\in\mathbb{Z}^k}M_{\boldsymbol{\ell}}$ such that

- for each $\boldsymbol{\ell}$, $M_{\boldsymbol{\ell}}$ is an object in $\mathsf{pHM}\big(X,w+\sum_i\ell_i\big)$,
- M is endowed with actions ρ_i of $\mathfrak{sl}_2$ $(i=1,\dots,k)$ such that, for each i, $\mathrm{H}_i=\ell_i\,\mathrm{Id}$ on $M_{\boldsymbol{\ell}}$ and $\mathrm{Y}_i:M_{\boldsymbol{\ell}}\to M_{\boldsymbol{\ell}-2\mathbf{1}_i}(-1)$, $\mathrm{X}_i:M_{\boldsymbol{\ell}}\to M_{\boldsymbol{\ell}+2\mathbf{1}_i}(1)$ satisfy the isomorphism property for an $\mathfrak{sl}_2$-Hodge module, so that there is a Lefschetz k-decomposition (argue as in Exercise 3.9),
- M can be endowed with a *polarization* S, that is, a (multi) graded morphism $\mathrm{S}:M\to M^*(-w)$ (i.e., S sends $M_{\boldsymbol{\ell}}$ to $M^*_{\boldsymbol{\ell}}(-w)=(M_{-\boldsymbol{\ell}})^*(-w)$), such that each $\mathrm{Y}_i,\mathrm{X}_i$ is skew-adjoint with respect to S (i.e., S is a morphism $(M,\mathrm{Y})\to(M,\mathrm{Y})^*(-w)$) and that, for every $\boldsymbol{\ell}=(\ell_1,\dots,\ell_k)$ with non-negative components, the induced morphism (see Section 3.4.c)

$$\mathrm{X}_1^{*\ell_1}\cdots\mathrm{X}_k^{*\ell_k}\circ\mathrm{S}:M_{-\boldsymbol{\ell}}\longrightarrow(M_{-\boldsymbol{\ell}})^*(-w-\boldsymbol{\ell})$$

induces a polarization of the object $\mathrm{P}_{-\boldsymbol{\ell}}M_{-\boldsymbol{\ell}}:=\bigcap_{i=1}^k\operatorname{Ker}\mathrm{X}_i^{\ell_i+1}$ of $\mathsf{HM}(X,w-\boldsymbol{\ell})$. (One can also use the Y_i's or use alternatively Y_i's and X_j's with an obvious modification of the twists and the signs, e.g. $(-\mathrm{Y}_1)^{*\ell_1}\cdots(-\mathrm{Y}_k)^{*\ell_k}\circ\mathrm{S}$ should induce a polarization on $\mathrm{P}_{\boldsymbol{\ell}}M_{\boldsymbol{\ell}}$.)

Morphisms should be compatible with the $\mathfrak{sl}_2^{\mathrm{k}}$-structure, hence k-graded of k-degree zero. The category is abelian, and any morphism is strict and strictly $\mathbb{R}$-specializable (this is proved as 14.2.17(7)).

14.2.28. Lemma. *Let $(M,\mathrm{X},\mathrm{Y},\mathrm{H})$ be an object of the category $\mathsf{p}\mathfrak{sl}_2^k\mathsf{HM}(X,w)$ and let g be a germ of holomorphic function. Then, for every $\lambda\in\mathbb{S}^1$, the graded nearby cycle object $\big(\mathrm{gr}^{\mathrm{M}}_{\bullet}\psi_{g,\lambda}M,(\mathrm{gr}^{\mathrm{M}}_{\bullet}\psi_{g,\lambda}\mathrm{Y},\mathrm{N}_g)\big)$ is an object of $\mathsf{p}\mathfrak{sl}_2^{k+1}\mathsf{HM}\big(X,w-1\big)$ and for each $\boldsymbol{\ell}\in\mathbb{Z}^k$ and $\ell\in\mathbb{Z}$, $\mathrm{P}_{\boldsymbol{\ell}}\mathrm{gr}^{\mathrm{M}}_{\ell}\psi_{g,\lambda}M_{\boldsymbol{j}}=\mathrm{gr}^{\mathrm{M}}_{\ell}\psi_{g,\lambda}\mathrm{P}_{\boldsymbol{\ell}}M_{\boldsymbol{j}}$, where $\mathrm{P}_{\boldsymbol{\ell}}$ denotes the multi-primitive part. A similar statement holds with $\phi_{g,1}$ and $\mathsf{p}\mathfrak{sl}_2^{k+1}\mathsf{HM}\big(X,w\big)$.*

Proof. The lemma is a direct consequence of the strict compatibility of $\psi_{t,\lambda}\mathrm{X}_i,\psi_{t,\lambda}\mathrm{Y}_i$ with the monodromy filtration $\mathrm{M}(\mathrm{N}_g)$, as follows from 14.2.17(9) applied to the morphisms $\mathrm{X}_i,\mathrm{Y}_i$. □

14.2.29. Lemma. *The category $\mathsf{p}\mathfrak{sl}_2^k\mathsf{HM}(X,w)$ has an inductive definition as in Definition 14.2.2. Furthermore, Properties 14.2.17(5)–(7) hold for this category.*

Proof. This directly follows from the commutativity of $\mathrm{P}_{\boldsymbol{\ell}}$ and $\mathrm{gr}^{\mathrm{M}}_{\ell}\psi_{g,\lambda}$ and $\mathrm{gr}^{\mathrm{M}}_{\ell}\phi_{g,1}$ shown in Lemma 14.2.28. □

14.3. Introduction to the direct image theorem

The theory of polarizable Hodge modules was developed in order to give an analytic proof, relying on Hodge theory, of the decomposition theorem of the pushforward by a projective morphism of the intersection complex attached to a local system underlying a polarizable variation of Hodge structure. Two questions arise in this context:

- to relate polarizable variations of Hodge structure on a smooth analytic Zariski open subset of a complex analytic set with a polarizable Hodge module on a complex manifold containing this analytic set as a closed analytic subset (the structure theorem),
- to prove the Hodge-Saito (i.e., direct image) theorem for the pushforward by a projective morphism of a polarizable Hodge module.

Recall Definition 12.7.28 for the pushforward functor in the category $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$, and the corresponding definition of the pushforward of a pre-polarization S. In particular, we consider the pushforward ${}_{\mathrm{T}}f_*^{(\bullet)}\widetilde{\mathcal{T}}$ as a graded object in $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(Y)$. The Hodge-Saito theorem describes the behaviour by projective pushforward of an object of $\mathsf{pHM}(X,w)$. The case of the constant map $X \to \mathrm{pt}$ and of the Hodge module ${}_{\mathrm{H}}\mathcal{O}_X$ corresponds to the results of Section 2.4.

The proof of the Hodge-Saito theorem is obtained by reducing to the case of a constant map, by using the nearby cycle functor and its compatibility with pushforward. In the case of the constant map, one can reduce to the case where the Hodge module is a polarizable variation of Hodge structure on the complement of a normal crossing divisor in a complex manifold by using Hironaka's theorem on resolution of singularities, and the decomposition theorem already proved (by induction) for the resolution morphism. One can use a Lefschetz pencil to apply an inductive process, after having blown up the base locus of the pencil. In such a way, one is reduced to the case of the constant map on a smooth projective curve, where one can apply the Hodge-Saito theorem 7.4.19.

Another approach in the case of a constant map would make full use of the higher dimensional analogues of the results proved in Chapter 6 for polarized variations of Hodge structure, but this would need to include in the inductive process the structure theorem for polarizable Hodge modules in the normal crossing case.

The Hodge-Saito theorem enables us to give a proof of a simple case of the structure theorem, namely, that a variation of Hodge structure of weight w on a complex manifold X is a polarizable Hodge module of weight $w + \dim X$. It is indeed difficult to check the behaviour along an arbitrary holomorphic function g (e.g. strict $\mathbb{R}$-specializability), but the case where the function is a monomial can be reduced to the case where the function is a product of coordinates, and in that case Example 12.7.24 provides the result by induction on the dimension. The pushforward theorem 12.7.32 enables us to obtain the result for an arbitrary holomorphic function, according to Hironaka's resolution of singularities of holomorphic functions.

***14.3.1. Theorem* (Hodge-Saito theorem).** *Let $f : X \to Y$ be a projective morphism between complex analytic manifolds and let M be a polarizable Hodge module of weight w on X. Let $\mathcal{L}$ be an ample line bundle on X and let $\mathrm{X}_{\mathcal{L}} = (2\pi \mathrm{i})\mathrm{L}_{\mathcal{L}}$ be the corresponding Lefschetz operator. Then $\big({}_{\mathrm{T}}f_*^{(\bullet)}M, \mathrm{X}_{\mathcal{L}}\big)$ is an object of $\mathsf{p}\mathfrak{sl}_2\mathsf{HM}(Y, w)$.*

A special case of the Hodge-Saito theorem is the case where f is a closed embedding, which is a consequence of Kashiwara's equivalence 14.2.9.

Let us make explicit the statement of Theorem 14.3.1. Let us choose a polarization S on $M = (\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}'', \mathfrak{s})$. It induces an isomorphism $\widetilde{\mathcal{M}}'' \simeq \widetilde{\mathcal{M}}'(w)$ and we can assume that M corresponds to a Hodge-Hermitian pair $(\widetilde{\mathcal{M}}, \mathcal{S})$, i.e., $M = (\widetilde{\mathcal{M}}, \widetilde{\mathcal{M}}(w), \mathcal{S})$ with polarization $\mathrm{S} = (\mathrm{Id}, \mathrm{Id})$.

***14.3.2. Theorem* (Reformulation of the Hodge-Saito theorem)**

For f and $(\widetilde{\mathcal{M}}, \mathcal{S})$ as above, the following properties hold.

(1) *${}_{\mathrm{D}}f_*\widetilde{\mathcal{M}}$, regarded as an object of $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\widetilde{\mathcal{D}}_Y)$, is strict, that is, for every k, ${}_{\mathrm{D}}f_*^{(k)}\widetilde{\mathcal{M}}$ is a strict $\widetilde{\mathcal{D}}_Y$-module (see Proposition 8.8.23). Moreover, ${}_{\mathrm{D}}f_*^{(k)}\widetilde{\mathcal{M}}$ is S-decomposable.*

(2) *Each ${}_{\mathrm{T}}f_*^{(k)}M$ is a polarizable Hodge module of weight $w + k$ on Y.*

(3) (Relative hard Lefschetz theorem) *For each $k \geqslant 0$, the Lefschetz operator $\mathrm{X}_{\mathcal{L}}$ induces isomorphisms in $\mathsf{pHM}(Y, w + k)$:*

$$\mathrm{X}_{\mathcal{L}}^k : {}_{\mathrm{T}}f_*^{(-k)}M \xrightarrow{\ \sim\ } {}_{\mathrm{T}}f_*^{(k)}M(k),$$

so that $\big({}_{\mathrm{T}}f_*^{(\bullet)}M, \mathrm{X}_{\mathcal{L}}\big)$ is an object of $\mathsf{p}\mathfrak{sl}_2\mathsf{HM}(Y, w)$, that is, a graded Hodge-Lefschetz Hermitian pair with central weight w.

(4) *The object $({}_{\mathrm{T}}f_*^{(\bullet)}\widetilde{\mathcal{M}}, {}_{\mathrm{T}}f_*^{(\bullet)}\mathcal{S}), \mathrm{X}_{\mathcal{L}})$ (see Section 12.4.a for ${}_{\mathrm{T}}f_*\mathcal{S}$) is an $\mathfrak{sl}_2$-Hodge-Hermitian pair.*

One of the most notable consequences of the Hodge-Saito theorem is the decomposition theorem.

***14.3.3. Theorem* (Decomposition theorem).** *Let $f : X \to Y$ be a projective morphism of complex manifolds. Let $\widetilde{\mathcal{M}}$ be a $\widetilde{\mathcal{D}}_X$-module underlying a polarizable Hodge module. Then the complex ${}_{\mathrm{D}}f_*\widetilde{\mathcal{M}}$ in $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\widetilde{\mathcal{D}}_Y)$ decomposes (in a non-canonical way) as $\bigoplus_k {}_{\mathrm{D}}f_*^{(k)}\widetilde{\mathcal{M}}[-k]$. Similarly, if $\mathcal{M} = \widetilde{\mathcal{M}}/(z-1)\widetilde{\mathcal{M}}$ is the underlying $\mathcal{D}_X$-module, then there exists a (non canonical) decomposition ${}_{\mathrm{D}}f_*\mathcal{M} \simeq \bigoplus_k {}_{\mathrm{D}}f_*^{(k)}\mathcal{M}[-k]$ in $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_Y)$.*

***Proof*.** This is a direct consequence of Deligne's criterion 3.3.8 for a spectral sequence to degenerate at E_2. We apply this theorem to ${}_{\mathrm{D}}f_*\widetilde{\mathcal{M}}$ as an object of $\mathsf{D}^{\mathrm{b}}(\widetilde{\mathcal{D}}_Y)$, by using the Hard Lefschetz theorem furnished by the Hodge-Saito theorem. □

***14.3.4. Sketch of the proof of Theorem* 14.3.1.** That holonomicity is preserved by proper pushforward is recalled in Remark 8.8.31. We will now focus on the other properties defining a polarizable Hodge module. The proof of Theorem 14.3.1 is done by induction on the pair

$$(n, m) = (\dim \operatorname{Supp} M, \dim \operatorname{Supp} {}_{\mathrm{T}}f_*M)$$

ordered lexicographically. Note that the pairs occurring satisfy $0 \leqslant m \leqslant n$.

(a) In the case where $n = 0$, the assertion of Theorem 14.3.1 is easily obtained: we can assume that M is supported on a point x_o, hence is equal the pushforward by $\iota : \{x_o\} \hookrightarrow X$ of a polarizable Hodge structure, and ${}_{\mathrm{T}}f_*M$ is equal to this Hodge structure.

(b) In the case where $\dim X = 1$ with X smooth, and $m = 0$, it is straightforward to reduce to the case where X is also connected, so that f factorizes as $X \to \mathrm{pt} \hookrightarrow Y$. As already remarked for the case of a closed embedding, we are left with considering the case of the constant map $a_X : X \to \mathrm{pt}$ from a compact Riemann surface, which has been treated in Chapter 7 (see Corollary 7.4.14 and the Hodge-Saito theorem 7.4.19 in dimension 1, i.e., the Hodge-Zucker theorem 6.11.1).

Both (a) and (b) provide the property $[(14.3.1)_{(\leqslant 1,0)}$ with $\mathrm{Supp}\, M$ smooth$]$.

(c) $(14.3.1)_{(n,m)} \implies (14.3.1)_{(n+1,m+1)}$ is proved in Section 14.4. In such a case, the behaviour of f_*M with respect to nearby and vanishing cycles for a function g on the base is controlled by the behaviour of M with respect to nearby and vanishing cycles for the function $g \circ f$ on the source, plus a good behaviour of these by the pushforward ${}_{\mathrm{T}}f_*$ relying on 12.7.32. The main point is provided by Proposition 14.4.2.

(d) $(14.3.1)_{(\leqslant n-1,0)}$ & $[(14.3.1)_{(\leqslant 1,0)}$ with $\mathrm{Supp}\, M$ smooth$] \implies (14.3.1)_{(n,0)}$ for $n \geqslant 1$ is proved in Section 14.5 by using the method of Lefschetz pencils. In this case, f is the constant map and we factor it through a map to $\mathbb{P}^1$ (up to taking a blowing-up along the axis of the pencil). If such a blow-up is not needed, i.e., a factorization of f exists, the proof relies on the analysis of the corresponding Leray spectral sequence. The general case follows the same strategy.

Conclusion. Let us check that the statements (a)–(d) lead to the proof of Theorem 14.3.1.

Given a pair $(n, m) \in \mathbb{N}^2$ with $m \leqslant n$, let us assume that the theorem is proved for every pair $(n', m') < (n, m)$. If $m \geqslant 1$, (c) gives the theorem by induction since $(n-1, m-1) < (n, m)$. We can thus assume that $m = 0$. By (a), it is enough to consider the case $n \geqslant 1$. Then (d), together with (a) and (b), reduces the proof to that of $(14.3.1)_{(n-1,0)}$, which is also true by induction. ☐

14.4. Behaviour of the Hodge module properties by projective pushforward

In this section we fix n and we assume that $(14.3.1)_{(n',m')}$ holds for any $n' \leqslant n$ and any $m' \leqslant n'$. We aim at proving that $(14.3.1)_{(n+1,m+1)}$ holds for any $m \leqslant n$.

Let $f : X \to Y$ be a projective morphism between complex manifolds, let h be a holomorphic function on Y and set $g = h \circ f : X \to \mathbb{C}$. Let $\mathcal{L}$ be a relatively ample line bundle on X. In other words, we choose a relative embedding

(14.4.1)
$$\begin{array}{ccc} X & \hookrightarrow & Y \times \mathbb{P}^N \\ & {\scriptstyle f}\searrow & \downarrow \\ & & Y \xrightarrow{\ h\ } \widetilde{\mathbb{C}} \end{array} \qquad g = h \circ f : X \to \widetilde{\mathbb{C}}$$

so that $\mathcal{L}$ comes by pullback from an ample line bundle on $\mathbb{P}^N$. We aim at proving that the properties 14.2.2(1) and (2) relative to the given g are preserved (in some sense) under pushforward by f under weak assumptions on (M,S), and a support condition that allows the application of the induction hypothesis $(14.3.1)_{(n,m)}$.

14.4.2. Proposition. *Let $\widetilde{\mathcal{T}}=(\widetilde{\mathcal{M}}',\widetilde{\mathcal{M}}'',\mathfrak{s})$ be a an object of $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)_{\mathrm{hol}}$ and let S be a pre-polarization of $\widetilde{\mathcal{T}}$ of weight w. We assume*

(a) $\dim(\operatorname{Supp}\widetilde{\mathcal{T}}\cap g^{-1}(0))\leqslant n$,

(b) $(\widetilde{\mathcal{T}},\mathrm{S})$ *satisfies* $14.2.2(1)_g$ *and* $(2)_g$. *In other words, we assume that the objects* $(\mathrm{gr}^{\mathrm{M}}_{\bullet}\psi_{g,\lambda}\widetilde{\mathcal{T}},\mathrm{grN},\mathrm{gr}_{\bullet}\psi_{g,\lambda}\mathrm{S})$ *and* $(\mathrm{gr}^{\mathrm{M}}_{\bullet}\phi_{g,1}\widetilde{\mathcal{T}},\mathrm{grN},\mathrm{gr}_{\bullet}\phi_{g,1}\mathrm{S})$ *are respectively polarized* $\mathfrak{sl}_2$*-Hodge triples with central weight* $w-1$ *and* w *(*grN *of type* Y *in both cases and denoted* Y_g*).*

Then, if Theorem 14.3.1 *holds for pairs* (n',m') *with* $n'\leqslant n$*, the following holds.*

(1) ${}_{\mathrm{T}}f_*^{(k)}\widetilde{\mathcal{T}}$ *is strictly* $\mathbb{R}$*-specializable and S-decomposable along* (h) *for every* $k\in\mathbb{Z}$.

(2) $\big(\bigoplus_{k,\ell}\mathrm{gr}^{\mathrm{M}}_{\ell}\psi_{h,\lambda}({}_{\mathrm{T}}f_*^{(k)}\widetilde{\mathcal{T}}),(\mathrm{X}_{\mathcal{L}},\mathrm{Y}_g),\mathrm{gr}^{\mathrm{M}}_{\ell}\psi_{h,\lambda}({}_{\mathrm{T}}f_*^{(k)}\mathrm{S})\big)$ *is a polarized bi-*$\mathfrak{sl}_2$ *Hodge triple with central weight* $w-1$.

(3) $\big(\bigoplus_{k,\ell}\mathrm{gr}^{\mathrm{M}}_{\ell}\phi_{h,1}({}_{\mathrm{T}}f_*^{(k)}\widetilde{\mathcal{T}}),(\mathrm{X}_{\mathcal{L}},\mathrm{Y}_g),\mathrm{gr}^{\mathrm{M}}_{\ell}\phi_{h,1}({}_{\mathrm{T}}f_*^{(k)}\mathrm{S})\big)$ *is a polarized bi-*$\mathfrak{sl}_2$ *Hodge triple with central weight* w.

Before giving the proof of this proposition, we will introduce the technical tools that are needed for it.

14.4.a. bi-$\mathfrak{sl}_2$ Hodge modules

14.4.3. Proposition. *The conclusions of Propositions* 3.2.26 *and* 3.2.27 *remain valid for polarizable bi-*$\mathfrak{sl}_2$ *Hodge modules.*

Proof.

(1) Let us start with Proposition 3.2.27. Let $\big((M_{\boldsymbol{j}\in\mathbb{Z}^2},\rho_1,\rho_2\big)$ be an object of $\mathsf{p}\mathfrak{sl}_2^2\mathsf{HM}(X,w)$ with a polarization S. We assume that it comes equipped with a bi-graded differential $d:M_{\bullet}\to M_{\bullet-(1,1)}(-1)$ which commutes with Y_1 and Y_2 and is self-adjoint with respect to S. In particular, d is strict and strictly specializable and we have, for any germ g of holomorphic function, any $\lambda\in\mathbb{S}^1$ and any $\ell\geqslant 0$,

$$\mathrm{P}_{\ell}\psi_{g,\lambda}(\operatorname{Ker}d/\operatorname{Im}d)=\operatorname{Ker}(\mathrm{P}_{\ell}\psi_{g,\lambda}d)/\operatorname{Im}(\mathrm{P}_{\ell}\psi_{g,\lambda}d)$$

(see Corollary 14.2.23). By induction on the dimension of the support, we can assert that $\big(\mathrm{P}_{\ell}\psi_{g,\lambda}(\operatorname{Ker}d/\operatorname{Im}d),\mathrm{P}_{\ell}\psi_{g,\lambda}\rho_1,\mathrm{P}_{\ell}\psi_{g,\lambda}\rho_2\big)$ is an object of $\mathsf{p}\mathfrak{sl}_2^2\mathsf{HM}(X,w-1+\ell)$ with polarization $\mathrm{P}_{\ell}\psi_{g,\lambda}\mathrm{S}$, and we conclude with Lemma 14.2.29. The case where the dimension of the support is zero is obtained from Proposition 3.2.27.

(2) The analogue of Proposition 3.2.26 is proved similarly. □

***14.4.4. Corollary* (Degeneration of a spectral sequence).** *Let $(\widetilde{\mathcal{T}}^{\bullet},d)$ be a bounded complex in $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$, with $d:\widetilde{\mathcal{T}}^j\to\widetilde{\mathcal{T}}^{j+1}(1)$ and $d\circ d=0$. Let us assume that it is equipped with the following data:*

(a) *a morphism of complexes* $\mathrm{S}:(\widetilde{\mathfrak{T}}^\bullet,d)\to(\widetilde{\mathfrak{T}}^\bullet,d)^*(-w)$ *which is* $(-1)^w$*-Hermitian, that is, for every* k*, a morphism* $\mathrm{S}:\widetilde{\mathfrak{T}}^k\to(\widetilde{\mathfrak{T}}^{-k})^*(-w)$ *which is compatible with* d *and* d^**, and such that* $\mathrm{S}^*=(-1)^w\mathrm{S}$*,*

(b) *a morphism* $\mathrm{X}':(\widetilde{\mathfrak{T}}^\bullet,d)\to(\widetilde{\mathfrak{T}}^{\bullet+2}(1),d)$ *which is self-adjoint with respect to* S*,*

(c) *a morphism* $\mathrm{N}:(\widetilde{\mathfrak{T}}^\bullet,d)\to(\widetilde{\mathfrak{T}}^\bullet(-1),d)$ *which is nilpotent, commutes with* X'*, and self-adjoint with respect to* S*, with monodromy filtration of* $\mathrm{M}_\bullet(\mathrm{N})$*.*

Let us consider the spectral sequence associated to the filtered complex $(\mathrm{M}_{-\ell}\widetilde{\mathfrak{T}}^\bullet,d)$ *with* $E_1^{\ell,j-\ell}=H^j\mathrm{gr}^{\mathrm{M}}_{-\ell}\widetilde{\mathfrak{T}}^\bullet$*. We set* $\mathrm{Y}=\mathrm{grN}$*. We assume that*

$$\bigoplus_{j,\ell}\Big(E_1^{\ell,j-\ell}=H^j(\mathrm{gr}^{\mathrm{M}}_{-\ell}\widetilde{\mathfrak{T}}^\bullet),(H^j\mathrm{gr}^{\mathrm{M}}_{-\ell}\mathrm{X}',H^j\mathrm{Y}),H^j\mathrm{gr}^{\mathrm{M}}_{-\ell}\mathrm{S}\Big)$$

is a polarized object of $\mathsf{p}\mathfrak{sl}_2^2\mathsf{HM}(X,w)$*. Then*

(1) *the spectral sequence degenerates at* E_2*,*

(2) *the filtration* $W_\ell H^j(\widetilde{\mathfrak{T}}^\bullet):=\mathrm{image}[H^j(\mathrm{M}_\ell\widetilde{\mathfrak{T}}^\bullet)\to H^j(\widetilde{\mathfrak{T}}^\bullet)]$ *is equal to the monodromy filtration* $\mathrm{M}_\bullet H^j(\widetilde{\mathfrak{T}}^\bullet)$ *associated to* $H^j\mathrm{N}:H^j(\widetilde{\mathfrak{T}}^\bullet)\to H^j(\widetilde{\mathfrak{T}}^\bullet)$*,*

(3) *the object*

$$\bigoplus_{j,\ell}\Big(\mathrm{gr}^{\mathrm{M}}_{-\ell}H^j(\widetilde{\mathfrak{T}}^\bullet),(\mathrm{gr}^{\mathrm{M}}_{-\ell}H^j\mathrm{X}',\mathrm{gr}H^j\mathrm{N}),\mathrm{gr}^{\mathrm{M}}_{-\ell}H^j\mathrm{S}\Big)$$

is a polarized object of $\mathsf{p}\mathfrak{sl}_2^2\mathsf{HM}(X,w)$*.*

Proof. Let us first make clear the statement. Since d and X' commute with N, d and X' are compatible with the monodromy filtration $\mathrm{M}_\bullet(\mathrm{N})$, hence for each ℓ we have a graded complex $(\mathrm{gr}^{\mathrm{M}}_{-\ell}\widetilde{\mathfrak{T}}^\bullet,d)$, and X' induces for every ℓ a morphism $\mathrm{gr}^{\mathrm{M}}_{-\ell}\mathrm{X}':(\mathrm{gr}^{\mathrm{M}}_{-\ell}\widetilde{\mathfrak{T}}^\bullet,d)\to(\mathrm{gr}^{\mathrm{M}}_{-\ell}\widetilde{\mathfrak{T}}^{\bullet+2}(1),d)$, and thus a morphism $H^j\mathrm{gr}^{\mathrm{M}}_{-\ell}\mathrm{X}':E_1^{\ell,j-\ell}\to E_1^{\ell,j+2-\ell}(1)$. Similarly, $H^j\mathrm{Y}$ is a morphism $E_1^{\ell,j-\ell}\to E_1^{\ell+2,j-\ell-2}(-1)$. We consider the bi-grading such that $E_1^{\ell,j-\ell}$ is in bi-degree (j,ℓ).

The differential $d_1:H^j(\mathrm{gr}^{\mathrm{M}}_{-\ell}\widetilde{\mathfrak{T}}^\bullet)\to H^{j+1}(\mathrm{gr}^{\mathrm{M}}_{-\ell-1}\widetilde{\mathfrak{T}}^\bullet)(1)$ is a morphism in $\mathsf{HM}(X,w+j-\ell)$ that commutes with $H^j\mathrm{gr}^{\mathrm{M}}_{-\ell}\mathrm{X}'$ and $H^j\mathrm{Y}$. We will check below that d_1 is self-adjoint with respect to $H^j\mathrm{gr}^{\mathrm{M}}_{-\ell}\mathrm{S}$. From the analogue of Proposition 3.2.27 (see Proposition 14.4.3), we deduce that $\bigoplus_{j,\ell}E_2^{\ell,j-\ell}$ is part of an object of $\mathsf{p}\mathfrak{sl}_2^2\mathsf{HM}(X,w)$. Now, one shows inductively that, for $r\geqslant 2$, $d_r:E_2^{\ell,j-\ell}\to E_2^{\ell+r,j-\ell-r+1}$ is a morphism of pure Hodge modules, the source having weight $w+j-\ell$ and the target $w+j-\ell-r+1<w+j-\ell$ and thus, by applying 14.2.17(2), that $d_r=0$. This proves 14.4.4(1).

In order to prove (2), we notice that, due to the degeneration property above, we have an identification

$$\mathrm{gr}^W_\ell H^j(\widetilde{\mathfrak{T}}^\bullet)\simeq E_2^{\ell,j-\ell},$$

and the action of grN on the left-hand side is that induced by $H^j\mathrm{Y}$ on the right-hand side. By the $\mathfrak{sl}_2$ property of E_2 relative to $H^j\mathrm{Y}$, we deduce that grN satisfies the Lefschetz property on $\mathrm{gr}^W_\bullet H^j(\widetilde{\mathfrak{T}}^\bullet)$. In other words, (2) holds.

Lastly, due to the above identification, (3) amounts to the bi-$\mathfrak{sl}_2$ Hodge property of E_2. ∎

Proof that d_1 ***is self-adjoint.*** We regard $\mathrm{gr}^{\mathrm{M}}_{-\ell}\mathrm{S}$ as a morphism $\mathrm{gr}^{\mathrm{M}}_{-\ell}M^j \to (\mathrm{gr}^{\mathrm{M}}_{\ell}M^{-j})^*$. It is compatible with d and d^* on these complexes, since N commutes with d. Then, $H^j\mathrm{gr}^{\mathrm{M}}_{-\ell}\mathrm{S}$ is a morphism $H^j\mathrm{gr}^{\mathrm{M}}_{-\ell}M^\bullet \to (H^{-j}\mathrm{gr}^{\mathrm{M}}_{\ell}M^{-\bullet})^*$. Since d_1 is obtained by a standard formula from d on the filtered complex, the equality $\mathrm{S}\circ d = d^*\circ\mathrm{S}$ implies $H^j\mathrm{gr}^{\mathrm{M}}_{-\ell}\mathrm{S}\circ d_1 = (d_1)^*\circ H^j\mathrm{gr}^{\mathrm{M}}_{-\ell}\mathrm{S}$. □

14.4.b. Proof of Proposition 14.4.2 and of 14.3.4(c)

***Proof of Proposition* 14.4.2.** One of the points to understand is the way to pass from properties of ${}_{\mathrm{T}}f_*^{(k)}\mathrm{gr}^{\mathrm{M}}_{-\ell}\psi_{g,\lambda}\widetilde{\mathcal{T}}$ to properties of $\mathrm{gr}^{\mathrm{M}}_{-\ell}\psi_{h,\lambda}({}_{\mathrm{T}}f_*^{(k)}\widetilde{\mathcal{T}})$, and similarly with $\phi_{g,1}$. Although we know that $\psi_{t,\lambda}({}_{\mathrm{T}}f_*^{(k)}\widetilde{\mathcal{T}})$ is isomorphic to ${}_{\mathrm{T}}f_*^{(k)}\psi_{g,\lambda}\widetilde{\mathcal{T}}$ if the latter is strict, according to 12.7.32, we have to check the strictness property. Moreover, we are left with the question of passing from ${}_{\mathrm{T}}f_*^{(k)}\mathrm{gr}^{\mathrm{M}}_{-\ell}$ to $\mathrm{gr}^{\mathrm{M}}_{-\ell}{}_{\mathrm{T}}f_*^{(k)}$. Here, we do not have a commutation property, but we will use Corollary 14.4.4 to analyze the corresponding spectral sequence. At this point, the existence of a polarization is essential. The S-decomposability is not obvious either, and the polarization also plays an essential role for proving it.

Since we assume that Theorem 14.3.1 holds for objects in $\mathsf{pHM}_{\leqslant n}(X)$ and since $\dim(\mathrm{Supp}\,\widetilde{\mathcal{T}}\cap g^{-1}(0))\leqslant n$, we deduce that, for every $\lambda\in\mathbb{S}^1$,

$$\Big(\bigoplus_{k,\ell}{}_{\mathrm{T}}f_*^{(k)}\mathrm{gr}^{\mathrm{M}}_{-\ell}\psi_{g,\lambda}\widetilde{\mathcal{T}},(\mathrm{X}_{\mathcal{L}},{}_{\mathrm{T}}f_*^{(k)}\mathrm{grN}),{}_{\mathrm{T}}f_*^{(k)}\mathrm{gr}^{\mathrm{M}}_{-\ell}\psi_{g,\lambda}\mathrm{S}\Big)$$

is a polarized object of $\mathsf{p\mathfrak{sl}_2^2HM}(Y,w-1)$ if we keep here the grading convention used in Corollary 14.4.4. This corollary implies that

$$\Big(\bigoplus_{k,\ell}\mathrm{gr}^{\mathrm{M}}_{-\ell}\,{}_{\mathrm{T}}f_*^{(k)}\psi_{g,\lambda}\widetilde{\mathcal{T}},(\mathrm{X}_{\mathcal{L}},\mathrm{gr}\,{}_{\mathrm{T}}f_*^{(k)}\mathrm{N}),\mathrm{gr}^{\mathrm{M}}_{-\ell}\,{}_{\mathrm{T}}f_*^{(k)}\psi_{g,\lambda}\mathrm{S}\Big)$$

is a polarized object of $\mathsf{p\mathfrak{sl}_2^2HM}(Y,w-1)$. In particular, each $\mathrm{gr}^{\mathrm{M}}_{-\ell}\,{}_{\mathrm{T}}f_*^{(k)}\psi_{g,\lambda}\widetilde{\mathcal{T}}$ is strict, and therefore so is ${}_{\mathrm{T}}f_*^{(k)}\psi_{g,\lambda}\widetilde{\mathcal{T}}$. We argue similarly for $\phi_{g,1}$.

We can now apply Corollary 9.8.9 to conclude that ${}_{\mathrm{T}}f_*^{(k)}\widetilde{\mathcal{T}}$ is strictly $\mathbb{R}$-specializable along (g) for every k. We also conclude from 12.7.32 that

$$(\psi_{h,\lambda}\,{}_{\mathrm{T}}f_*^{(k)}\widetilde{\mathcal{T}},\mathrm{N})={}_{\mathrm{T}}f_*^{(k)}(\psi_{g,\lambda}\widetilde{\mathcal{T}},\mathrm{N}),\qquad(\phi_{h,1}\,{}_{\mathrm{T}}f_*^{(k)}\widetilde{\mathcal{T}},\mathrm{N})={}_{\mathrm{T}}f_*^{(k)}(\phi_{g,1}\widetilde{\mathcal{T}},\mathrm{N}).$$

We have thus proved that

$$\Big(\bigoplus_{k,\ell}\mathrm{gr}^{\mathrm{M}}_{-\ell}\psi_{h,\lambda}\,{}_{\mathrm{T}}f_*^{(k)}\widetilde{\mathcal{T}},(\mathrm{X}_{\mathcal{L}},\mathrm{grN}),\mathrm{gr}^{\mathrm{M}}_{-\ell}\psi_{h,\lambda}\,{}_{\mathrm{T}}f_*^{(k)}\mathrm{S}\Big)$$

is a polarized object of $\mathsf{p\mathfrak{sl}_2^2HM}(Y,w-1)$, and a corresponding assertion for $\phi_{h,1}$. □

Proof of* 14.3.4(c)*, i.e., $(14.3.1)_{(n,m)} \implies (14.3.1)_{(n+1,m+1)}$. Let $f:X\to Y$ be a projective morphism and let (M,S) be a polarized object of $\mathsf{pHM}_S(X,w)$, where S is an irreducible analytic subset of X of dimension $n+1$. We can assume that (M,S) is represented as a Hodge-Hermitian pair $(\widetilde{\mathcal{M}},\mathcal{S})$ of weight w, and we will omit $\mathrm{S}=(\mathrm{Id},\mathrm{Id})$ in the notation. Assume that $f(S)$ has dimension $m+1$ and that $(14.3.1)_{(n,m)}$ holds. Since Theorem 14.3.1 is a local statement on Y, we can work in an open neighbourhood of a point $y_o\in f(S)$, that we can take as small as needed.

By the S-decomposability of $(\widetilde{\mathcal{M}},\mathcal{S})$ on X, we can therefore assume that S and $f(S)$ are irreducible when restricted to a fundamental basis of neighborhoods of $f^{-1}(y_o)$ and y_o respectively.

Let h be a holomorphic function on some $\mathrm{nb}(y_o)$ and set $g=h\circ f$. We distinguish two cases. We note that strictness of ${}_{\mathrm{T}}f_*^{(k)}M$ on $\mathrm{nb}(y_o)$ is obtained by choosing any h as in Case (1) below.

(1) $h^{-1}(0)\cap f(S)$ has codimension 1 in $f(S)$. Then $g^{-1}(0)\cap S$ has codimension 1 in S. We can thus apply Proposition 14.4.2. It follows that each ${}_{\mathrm{T}}f_*^{(k)}M$ is strict and satisfies $14.2.2(1)_h$ and $(2)_h$.

(2) The function h vanish identically on the closed irreducible subset $f(S)\cap\mathrm{nb}(y_o)$ of $\mathrm{nb}(y_o)$. We now omit referring to $\mathrm{nb}(y_o)$. We denote by

$$\iota_g : X \hookrightarrow X\times\mathbb{C}_t \quad\text{and}\quad \iota : X\times\{0\}\hookrightarrow X\times\mathbb{C}_t$$

the respective graph and trivial inclusions, and similarly on Y. The only property to be checked relative to h is that ${}_{\mathrm{D}}f_*^{(k)}\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along (h), that is, $14.2.2(1)_h$: indeed, in such a case, Proposition 12.7.15 implies $\phi_{g,1}({}_{\mathrm{D}}f_*^{(k)}\widetilde{\mathcal{M}})={}_{\mathrm{D}}f_*^{(k)}\widetilde{\mathcal{M}}$ and $\psi_{g,\lambda}({}_{\mathrm{D}}f_*^{(k)}\widetilde{\mathcal{M}})=0$ for any $\lambda\in\mathbb{S}^1$, so $14.2.2(2)_h$ is trivially satisfied. Since ${}_{\mathrm{D}}f_*^{(k)}\widetilde{\mathcal{M}}$ is strict, ${}_{\mathrm{D}}\iota_*({}_{\mathrm{D}}f_*^{(k)}\widetilde{\mathcal{M}})$ is strictly $\mathbb{R}$-specializable along (t) and it is enough to prove

$${}_{\mathrm{D}}\iota_{h*}({}_{\mathrm{D}}f_*^{(k)}\widetilde{\mathcal{M}})={}_{\mathrm{D}}\iota_*({}_{\mathrm{D}}f_*^{(k)}\widetilde{\mathcal{M}})\quad\forall\,k.$$

The left-hand term is equal to ${}_{\mathrm{D}}f_*^{(k)}\widetilde{\mathcal{M}}_g$, if we still denote by f the map $f\times\mathrm{Id}_{\mathbb{C}}$. Similarly the right-hand term is equal to ${}_{\mathrm{D}}f_*^{(k)}{}_{\mathrm{D}}\iota_*\widetilde{\mathcal{M}}$, with obvious abuse of notation. Since $g\equiv 0$ on S and $\widetilde{\mathcal{M}}$ is assumed to be strictly $\mathbb{R}$-specializable along (g), we have $\widetilde{\mathcal{M}}_g={}_{\mathrm{D}}\iota_*\widetilde{\mathcal{M}}$, hence the desired assertion. ∎

14.5. End of the proof of the Hodge-Saito theorem

Recall that we wish to prove

(d)

$$(14.3.1)_{(\leqslant n-1,0)}\ \&\ [(14.3.1)_{(1,0)}\text{ with }\operatorname{Supp}M\text{ smooth}]\implies(14.3.1)_{(n,0)}\text{ for }n\geqslant 1.$$

We thus fix $n\geqslant 1$ in this section and assume that both properties of the left term hold true. It follows then from the results of Section 14.4 that $(14.3.1)_{(\leqslant n,m)}$ is true for any $m\geqslant 1$. As already noticed in the case 14.3.4(b), we only have to consider the case of the constant map $a_X : X\to\mathrm{pt}$.

Let (M,S) be a polarized Hodge module of weight w on a smooth complex projective variety X and let $\mathcal{L}$ be an ample line bundle on X. We can assume that $\mathrm{S}=(\mathrm{Id},\mathrm{Id})$ and consider the Hodge-Hermitian pair $(\widetilde{\mathcal{M}},\mathcal{S})$ for M as in Proposition 14.2.25. We can also assume that M has pure support Z, which is an irreducible closed n-dimensional algebraic subset of X $(n\geqslant 1)$. It is not restrictive to assume that $\mathcal{L}$ is very ample, so that, by Kashiwara's equivalence (Proposition 14.2.9), we can further assume that $X=\mathbb{P}^N$ and $\mathcal{L}=\mathcal{O}_{\mathbb{P}^N}(1)$.

14.5.a. The case where X maps to a curve

In order to emphasize the main steps, we start with the simpler case where we assume that there exists a morphism f from X to a curve C which is non-constant on the irreducible pure support $Z = \operatorname{Supp} M$, that we decompose as in (14.4.1) with $Y = C$. We use the corresponding notations for the ample line bundles $\mathcal{L}$ on $\mathbb{P}^N$ and $\mathcal{L}'$ on C. We decompose the constant map a_X on the projective manifold X of dimension n as $X \xrightarrow{f} C \xrightarrow{a_C} \mathrm{pt}$ and we consider the Leray spectral sequence for this decomposition (see Corollary 12.7.38).

Our induction hypothesis implies that Theorem 14.3.1 holds for both maps f and a_C: indeed, $(14.3.1)_{(\leqslant n,1)}$ holds true, and thus $\big({}_{\mathrm{T}}f_*^{(\bullet)}(M,\mathrm{S}),\mathrm{X}_{\mathcal{L}}\big)$ is a polarized $\mathfrak{sl}_2$-Hodge module with central weight w; furthermore, by $(14.3.1)_{(1,0)}$, the push-forward $\big({}_{\mathrm{T}}a_{C*}^{(\bullet)}({}_{\mathrm{T}}f_*^{(\bullet)}(M,\mathrm{S})),\mathrm{X}_{\mathcal{L}},\mathrm{X}_{\mathcal{L}'}\big)$ by the constant map a_C on the curve C is a polarized bi-$\mathfrak{sl}_2$ Hodge structure with central weight w. We are thus led to analyzing the Leray spectral sequence in order to get that $\big({}_{\mathrm{T}}a_{X*}^{(\bullet)}(M,\mathrm{S}),(\mathrm{X}_{\mathcal{L}}+\mathrm{X}_{\mathcal{L}'})\big)$ is a polarized $\mathfrak{sl}_2$-Hodge structure.

According to Corollary 12.7.38, there exists a spectral sequence in $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(\mathrm{pt})$ whose E_2 term is $E_2^{p,q} = {}_{\mathrm{T}}a_{C*}^{(p)}({}_{\mathrm{T}}f_*^{(q)}M)$. Since $\dim C = 1$, we have $E_2^{p,q} = 0$ unless $p = -1, 0, 1$. By our induction hypothesis, we can apply the decomposition theorem 14.3.3 to f and (M,S), and the spectral sequence degenerates at E_2.

Furthermore, our induction hypothesis implies that $((E_2^{\bullet,\bullet},\mathrm{S}),\mathrm{X}_{\widetilde{\mathcal{L}}},\mathrm{X}_{\widetilde{\mathcal{L}}'})$ is a polarized $\mathfrak{sl}_2^2$-Hodge structure with central weight w. We set $E_2^k = \bigoplus_{p+q=k} E_2^{p,q}$. We apply Proposition 3.2.26 to deduce a polarized $\mathfrak{sl}_2$-Hodge structure $((E_2^{\bullet},\mathrm{S}),\mathrm{X}_{\widetilde{\mathcal{L}}}+\mathrm{X}_{\widetilde{\mathcal{L}}'})$ with central weight w. It follows that $\big({}_{\mathrm{T}}a_{X*}^{(\bullet)}M,(\mathrm{X}_{\mathcal{L}}+\mathrm{X}_{\mathcal{L}'})\big)$ has a filtration $\mathrm{Ler}^\bullet$ (the Leray filtration attached to the spectral sequence) whose graded term is an $\mathfrak{sl}_2$-Hodge structure polarized by the pre-polarization induced from ${}_{\mathrm{T}}a_{X*}^{(\bullet)}\mathrm{S}$. From this property one deduces at once that $\big({}_{\mathrm{T}}a_{X*}^{(\bullet)}M,(\mathrm{X}_{\mathcal{L}}+\mathrm{X}_{\mathcal{L}'})\big)$ is an $\mathfrak{sl}_2$-Hodge structure of central weight w and that S induces a pre-polarization of it. However, at this step, we cannot assert that S is a polarization (i.e., that the positivity property holds), since it is only a successive extension of polarizations.

In order to overcome this difficulty, we will make use of the criterion provided by Theorem 3.2.20, which relies on the weak Lefschetz property. Since we have at our disposal a pre-polarization, we will work with the Hodge Hermitian pair $(\widetilde{\mathcal{M}},\mathcal{S},w)$ attached to (M,S) (see Proposition 14.2.25).

The operator $\mathrm{X}_{\mathcal{L}}+\mathrm{X}_{\mathcal{L}'}$ is the Lefschetz operator attached to the ample line bundle $\mathcal{L}\boxtimes\mathcal{L}'$ and, up to multiplying $\mathrm{X}_{\mathcal{L}\boxtimes\mathcal{L}'}$ by some positive integer, we can assume that it is very ample. It defines an embedding $X \hookrightarrow \mathbb{P}^{N'}$ and its restriction to X takes the form $\widetilde{\mathcal{O}}_X(H)$ for a general hyperplane of $\mathbb{P}^{N'}$ that we can assume to be non-characteristic with respect $\widetilde{\mathcal{M}}$. Since, by Definition $14.2.2(1)_g$ for any local equation g of H, $\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along H, it follows from Proposition 9.5.2 that H is strictly non-characteristic with respect to $\widetilde{\mathcal{M}}$.

For each $k \in \mathbb{Z}$, we consider the X-$\mathfrak{sl}_2$-Hodge quiver with center $w-1$, with

$$\begin{aligned}(H_\bullet, \mathrm{X}) &= ({}_{\mathrm{T}}a_{X*}^{(\bullet)}(\widetilde{\mathcal{M}}, \mathcal{S}, w), \mathrm{X}_{\mathcal{L}\boxtimes\mathcal{L}'}),\\ (G_\bullet, \mathrm{X}) &= ({}_{\mathrm{T}}a_{H*}^{(\bullet+1)}(\widetilde{\mathcal{M}}_H, \mathcal{S}_H, w-1), \mathrm{X}_{\mathcal{L}\boxtimes\mathcal{L}'}),\\ &\mathrm{can} = \mathrm{restr}_H, \quad \mathrm{var} = \mathrm{Gys}_H\,.\end{aligned}$$

Our induction hypothesis yields that $(G_\bullet, \mathrm{X})$ is a polarized $\mathfrak{sl}_2$-Hodge-Hermitian pair of weight $w-1$. Furthermore, $\mathrm{Gys}_H : {}_{\mathrm{D}}a_{H*}^{(k)}\widetilde{\mathcal{M}}_H \to {}_{\mathrm{D}}a_{X*}^{(k+1)}\widetilde{\mathcal{M}}(1)$, being induced by a morphism of Hodge structures, is a strict morphism for each k. We can therefore apply the criterion of Proposition 11.2.28 to deduce that the X-$\mathfrak{sl}_2$-Hodge quiver $((H_\bullet, \mathrm{X}), (G_\bullet, \mathrm{X}), \mathrm{can}, \mathrm{var})$ satisfies the weak Lefschetz property. According to Theorem 3.2.20, we are left with proving the positivity of $\mathrm{P}({}_{\mathrm{T}}a_{X*}^{(0)}\mathrm{S})$ on $\mathrm{P}({}_{\mathrm{T}}a_{X*}^{(0)}M)$ in order to deduce the desired positivity property for the pre-polarization of $\big({}_{\mathrm{T}}a_{X*}^{(\bullet)}M, (\mathrm{X}_{\mathcal{L}} + \mathrm{X}_{\mathcal{L}'})\big)$.

14.5.1. Lemma. *The pure Hodge structure* $\mathrm{P}({}_{\mathrm{T}}a_{X*}^{(0)}M)$ *is a Hodge sub-structure of* ${}_{\mathrm{T}}a_{C*}^{(0)}\big(\mathrm{P}(f_*^{(0)}M_H)\big)$ *and the pre-polarization* $\mathrm{P}({}_{\mathrm{T}}a_{X*}^{(0)}\mathrm{S})$ *is induced by* ${}_{\mathrm{T}}a_{C*}^{(0)}\big(\mathrm{P}(f_*^{(0)}\mathrm{S}_H)\big)$.

The proof of (d) is achieved with this lemma, according to Exercise 2.12. □

***Proof of Lemma* 14.5.1.** We can assume that S and S_H are of the form (Id, Id), so that we only need to show the first part.

We first check that $\mathrm{P}({}_{\mathrm{T}}a_{X*}^{(0)}M) \subset \mathrm{Ler}^0({}_{\mathrm{T}}a_{X*}^{(0)}M)$. Due to the weak Lefschetz property, we have $\mathrm{P}({}_{\mathrm{T}}a_{X*}^{(0)}M) = \mathrm{Ker}[\mathrm{restr}_H : {}_{\mathrm{T}}a_{X*}^{(0)}M \to {}_{\mathrm{T}}a_{H*}^{(1)}M_H]$ (see Remark 3.1.14(2)). Since the Leray filtration has only three terms $0 \subset \mathrm{Ler}^1 \subset \mathrm{Ler}^0 \subset \mathrm{Ler}^{-1}$, we are reduced to showing that $\mathrm{gr}_{\mathrm{Ler}}^{-1}\mathrm{restr}_H$ is injective. This is the restriction morphism ${}_{\mathrm{T}}a_{C*}^{(-1)}({}_{\mathrm{T}}f_*^{(1)}M) \to {}_{\mathrm{T}}a_{C*}^{(-1)}({}_{\mathrm{T}}f_*^{(2)}M_H)$ induced by the restriction morphism ${}_{\mathrm{T}}f_*^{(1)}M \to {}_{\mathrm{T}}f_*^{(2)}M_H$ relative to f: the latter is the connecting morphism in the long exact sequence in $\mathsf{pHM}(C)$ obtained as in (11.2.19) by applying ${}_{\mathrm{T}}f_*$ to the exact sequence (12.7.27 ∗). Since any morphism in $\mathsf{pHM}(C)$ is strict, we can apply the criterion for the weak Lefschetz property in Proposition 11.2.28 and deduce that this morphism is an isomorphism, hence in particular the desired injectivity.

We then claim that it is enough to check that $\mathrm{P}({}_{\mathrm{T}}a_{X*}^{(0)}M)$ does not intersect $\mathrm{Ler}^1({}_{\mathrm{T}}a_{X*}^{(0)}M)$, hence injects into $\mathrm{gr}_{\mathrm{Ler}}^0({}_{\mathrm{T}}a_{X*}^{(0)}M) = {}_{\mathrm{T}}a_{C*}^{(0)}({}_{\mathrm{T}}f_*^{(0)}M)$. Indeed, having proved this, we note that the action of $\mathrm{X}_{\mathcal{L}'}$ on this space is zero since ${}_{\mathrm{T}}a_{C*}^{(2)}(\bullet) = 0$. Therefore,

$$\begin{aligned}\mathrm{P}({}_{\mathrm{T}}a_{X*}^{(0)}M) &\subset \mathrm{Ker}\big[(\mathrm{X}_{\mathcal{L}} + \mathrm{X}_{\mathcal{L}'}) : \mathrm{gr}_{\mathrm{Ler}}^0({}_{\mathrm{T}}a_{X*}^{(0)}M) \longrightarrow \mathrm{gr}_{\mathrm{Ler}}^0({}_{\mathrm{T}}a_{X*}^{(2)}M)\big]\\ &= \mathrm{Ker}\big[{}_{\mathrm{T}}a_{C*}^{(0)}(\mathrm{X}_{\mathcal{L}}) : {}_{\mathrm{T}}a_{C*}^{(0)}({}_{\mathrm{T}}f_*^{(0)}M) \longrightarrow {}_{\mathrm{T}}a_{C*}^{(0)}({}_{\mathrm{T}}f_*^{(2)}M)\big].\end{aligned}$$

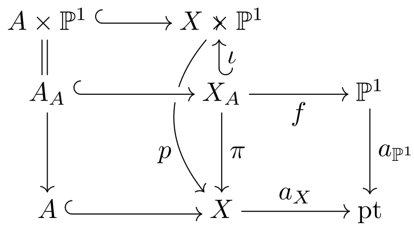
Due to the Lefschetz decomposition of ${}_{\mathrm{T}}f_*^{(\bullet)}M$ with respect to $\mathrm{X}_{\mathcal{L}}$, ${}_{\mathrm{T}}f_*^{(0)}M$ decomposes as $\mathrm{P}({}_{\mathrm{T}}f_*^{(0)}M) \oplus \mathrm{X}_{\mathcal{L}}({}_{\mathrm{T}}f_*^{(-2)}M)$, and $\mathrm{X}_{\mathcal{L}} : \mathrm{X}_{\mathcal{L}}({}_{\mathrm{T}}f_*^{(-2)}M) \to {}_{\mathrm{T}}f_*^{(2)}M$ is an isomorphism. Then ${}_{\mathrm{T}}a_{C*}^{(0)}(\mathrm{X}_{\mathcal{L}}) : {}_{\mathrm{T}}a_{C*}^{(0)}\big(\mathrm{X}_{\mathcal{L}}({}_{\mathrm{T}}f_*^{(-2)}M)\big) \to {}_{\mathrm{T}}a_{C*}^{(0)}({}_{\mathrm{T}}f_*^{(2)}M)$ is also an isomorphism, hence $\mathrm{P}({}_{\mathrm{T}}a_{X*}^{(0)}M)$ does not intersect its source, that is, $\mathrm{P}({}_{\mathrm{T}}a_{X*}^{(0)}M) \subset {}_{\mathrm{T}}a_{C*}^{(0)}\big(\mathrm{P}({}_{\mathrm{T}}f_*^{(0)}M)\big)$, which is the desired inclusion.

For the claim, we have $\mathrm{Ler}^1({}_{\mathrm{T}}a_{X*}^{(0)}M) = \mathrm{gr}^1_{\mathrm{Ler}}({}_{\mathrm{T}}a_{X*}^{(0)}M) = {}_{\mathrm{T}}a_{C*}^{(1)}({}_{\mathrm{T}}f_*^{(-1)}M)$, and the action of $\mathrm{X}_{\mathcal{L}} + \mathrm{X}_{\mathcal{L}'}$ reduces to that of $\mathrm{X}_{\mathcal{L}}$. By the Lefschetz decomposition of ${}_{\mathrm{T}}f_*^{(\bullet)}M$ with respect to $\mathrm{X}_{\mathcal{L}}$, the morphism $\mathrm{X}_{\mathcal{L}} : {}_{\mathrm{T}}f_*^{(-1)}M \to {}_{\mathrm{T}}f_*^{(1)}M$ is an isomorphism, hence so is the morphism ${}_{\mathrm{T}}a_{C*}^{(1)}(\mathrm{X}_{\mathcal{L}}) : {}_{\mathrm{T}}a_{C*}^{(1)}({}_{\mathrm{T}}f_*^{(-1)}M) \to {}_{\mathrm{T}}a_{C*}^{(1)}({}_{\mathrm{T}}f_*^{(1)}M)$. It follows that $\mathrm{Ler}^1({}_{\mathrm{T}}a_{X*}^{(0)}M) \cap \mathrm{P}({}_{\mathrm{T}}a_{X*}^{(0)}M) = 0$. □

14.5.b. The general case. In general however, we do not have such a decomposition $X \xrightarrow{f} C \xrightarrow{a_C} \mathrm{pt}$ of the constant map as in Section 14.5.a, and the usual trick is to consider a Lefschetz pencil instead, a procedure that introduces a supplementary complication due to the base locus of the pencil, that we can choose as generic as we want nevertheless.

Let us choose a pencil of hyperplanes in $X = \mathbb{P}^N$ with axis $A \simeq \mathbb{P}^{N-2}$. It defines a map $X \smallsetminus A \to \mathbb{P}^1$, whose graph is contained in $(X \smallsetminus A) \times \mathbb{P}^1$. Let X_A be the closure of this graph in $X \times \mathbb{P}^1$ with projection π to X, and let A_A be the pullback $\pi^{-1}(A)$. By definition, X_A is the blow-up space of X along the axis A of the pencil, and A_A is a smooth divisor in it. We have the following commutative diagram:

(14.5.2)



The restriction of π to any fiber $f^{-1}(t)$ is an isomorphism onto the corresponding hyperplane in X and, conversely, the pullback by π of this hyperplane is the union of $f^{-1}(t)$ and $A_A = A \times \mathbb{P}^1$, whose intersection $f^{-1}(t) \cap A_A = A \times \{t\}$ is transversal. Similarly, the pullback $\pi^{-1}Z$ of the support Z of M consists of the union of the strict transform S_A of Z by π, i.e., the blow-up space of S along the ideal $\mathcal{I}_A\mathcal{O}_S$, and $(A \cap Z) \times \mathbb{P}^1$.

We set $\mathcal{L}' = \mathcal{O}_{\mathbb{P}^1}(1)$, and we consider the ample line bundle $\mathcal{L} \otimes \mathcal{L}'$ on $X \times \mathbb{P}^1$. We will simply denote by X, X' the Lefschetz operators $\mathrm{X}_{\mathcal{L}}, \mathrm{X}_{\mathcal{L}'}$, so that $\mathrm{X} + \mathrm{X}'$ is the Lefschetz operator that is to be considered on $X \times \mathbb{P}^1$.

We consider the pullback $({}_{\mathrm{T}}\pi^*M, \mathrm{S})$. Although we cannot assert, at this stage of the theory, that it is a polarized Hodge module, we will prove that it enjoys a similar behaviour along the divisors $(f - t)$ when t varies in $\mathbb{P}^1$. This will enable us, by decomposing a_{X_A} as $a_{\mathbb{P}^1} \circ f$, to obtain for ${}_{\mathrm{T}}a_{X_A*}({}_{\mathrm{T}}\pi^*M, \mathrm{S})$ the same results as in the simple case 14.5.a.

On the other hand, we consider the decomposition of a_{X_A} as $a_X \circ \pi$. We will show (with the induction hypothesis at hand) that the pushforward ${}_{\mathrm{T}}\pi_*({}_{\mathrm{T}}\pi^*M)$ decomposes as the direct sum of its cohomology objects, and that M is a direct summand of ${}_{\mathrm{T}}\pi_*^{(0)}({}_{\mathrm{T}}\pi^*M)$. It follows that $\big(\bigoplus_k {}_{\mathrm{T}}a_{X*}^{(k)}M, \mathrm{X}\big)$ is a direct summand of $\big(\bigoplus_k {}_{\mathrm{T}}a_{X_A*}^{(k)}({}_{\mathrm{T}}\pi^*M), \mathrm{X} + \mathrm{X}'\big)$. Then, according to the previous step, the result follows from stability of polarizable $\mathfrak{sl}_2$-Hodge modules by direct summand (see Lemma 5.2.8 and Exercise 2.12(1), as already used in 14.2.8).

The detailed proof will take various steps.

Step 1. We define ${}_{\mathrm{T}}\pi^*$ as the composition ${}_{\mathrm{T}}\iota^* \circ {}_{\mathrm{T}}p^*$. This first step aims at showing that, under a non-characteristic condition,

- *the pullback* ${}_{\mathrm{T}}\pi^*(M, \mathrm{S})$ *is well-defined, is strict and satisfies* 14.2.2(1)$_{f-t}$ *and* (2)$_{f-t}$*, for every* $t \in \mathbb{P}^1$.

The smooth pullback ${}_{\mathrm{T}}p^*M$ is well-defined as an object of $\widetilde{\mathcal{D}}$-$\mathsf{Triples}(X \times \mathbb{P}^1)$ (see Section 12.7.12). In order to define ${}_{\mathrm{T}}\iota^*({}_{\mathrm{T}}p^*M)$, we will prove strict $\mathbb{R}$-specializability of ${}_{\mathrm{T}}p^*M$ along the graph $\iota(X_A)$. Note however that we do not know that the pullback ${}_{\mathrm{T}}p^*M$ satisfies Hodge properties along *every* germ of holomorphic function on $X \times \mathbb{P}^1$. Non-characteristic properties obtained by choosing the axis of the pencil generic enough will help us to overcome this difficulty.

More precisely, let us choose the pencil generic enough so that the axis A of the pencil is *non-characteristic* with respect to $\widetilde{\mathcal{M}}$ (see Section 8.8.d). If the characteristic variety of $\widetilde{\mathcal{M}}$ is contained in $\Lambda \times \mathbb{C}_z$ with Λ Lagrangian in T^*X, there exists a complex stratification of the support of $\widetilde{\mathcal{M}}$ by locally closed sub-manifolds Z_i^o with analytic closure Z_i, such that $\Lambda \subset \bigsqcup_i T^*_{Z_i^o}X$. Then A is chosen to be transversal to every stratum S_i^o. In particular, since the axis A has codimension two, it does not intersect any zero- and one-dimensional stratum. Moreover, for every i, the blow-up Z_{iA} of Z_i contains $(A \cap Z_i) \times \mathbb{P}^1$. This implies that $Z_{iA} = \pi^{-1}(Z_i)$.

14.5.3. Lemma.

(1) *The inclusion* $\iota : X_A \hookrightarrow X \times \mathbb{P}^1$ *is strictly non-characteristic with respect to* ${}_{\mathrm{D}}p^*\widetilde{\mathcal{M}}$.

(2) *We have* ${}_{\mathrm{D}}\pi^*\widetilde{\mathcal{M}} = {}_{\mathrm{D}}\pi^{*(0)}\widetilde{\mathcal{M}}$.

(3) *The* $\widetilde{\mathcal{D}}_{X_A}$*-module* ${}_{\mathrm{D}}\pi^*\widetilde{\mathcal{M}}$ *is holonomic, strict and strictly* $\mathbb{R}$*-specializable along each divisor* $(f - t)$.

According to this lemma, the pullback functor ${}_{\mathrm{T}}\iota^*$ is defined as in Section 12.7.22.

Proof.

(1) We first prove the non-characteristic property. We postpone the proof of strict $\mathbb{R}$-specializability after the proof of (3). Since p is a projection, the characteristic variety of ${}_{\mathrm{D}}p^*\widetilde{\mathcal{M}}$ is contained in the union of the sets $T^*_{S_i \times \mathbb{P}^1}(X \times \mathbb{P}^1) \times \mathbb{C}_z$.

- Away from $A_A = A \times \mathbb{P}^1$, ι is the graph inclusion of a map to $\mathbb{P}^1$ and, in a local setting, we are reduced to proving the claim for the inclusion $\iota : U = U \times \{0\} \hookrightarrow U \times \mathbb{C}$ and the projection $p : U \times \mathbb{C} \to \mathbb{C}$, where the claim is obviously true.
- Let us now consider the neighbourhood of a point of $A_A = A \times \mathbb{P}^1$ in $X \times \mathbb{P}^1$. Since A is non-characteristic with respect to each S_i, so is A_A with respect to each $S_i \times \mathbb{P}^1$ — and therefore so is X_A near any point of A_A, since in such a point the space $T^*_{X_A}(X \times \mathbb{P}^1)$ is contained in $T^*_{A_A}(X \times \mathbb{P}^1)$. The non-characteristic property is then also true along A_A.

(2) We now prove that $L^k {}_{\mathrm{D}}\pi^*\widetilde{\mathcal{M}} = 0$ for $k \neq 0$. Since X_A is of codimension 1 in $X \times \mathbb{P}^1$, this amounts to the property that ${}_{\mathrm{D}}p^*\widetilde{\mathcal{M}}$ has no local section supported on X_A. Notice that ${}_{\mathrm{D}}p^*\widetilde{\mathcal{M}}$ is strict, since $\widetilde{\mathcal{O}}_{X\times\mathbb{P}^1}$ is $\widetilde{\mathcal{O}}_X$-flat. Any coherent $\widetilde{\mathcal{D}}_{X\times\mathbb{P}^1}$-submodule of ${}_{\mathrm{D}}p^*\widetilde{\mathcal{M}}$ is then also strict, and it is supported on X_A if and only if the associated $\mathcal{D}_{X\times\mathbb{P}^1}$-module is so. But such a coherent $\mathcal{D}_{X\times\mathbb{P}^1}$-module is a submodule of ${}_{\mathrm{D}}\pi^*\mathcal{M}$, hence is holonomic with characteristic variety contained in $\Lambda \times T^*_{\mathbb{P}^1}\mathbb{P}^1$. This cannot be the characteristic variety of a holonomic $\mathcal{D}_{X\times\mathbb{P}^1}$-module with support on X_A.

(3) Note that, as a consequence of Theorem 8.8.16, the characteristic variety of ${}_{\mathrm{D}}\pi^*\widetilde{\mathcal{M}} = {}_{\mathrm{D}}\iota^*{}_{\mathrm{D}}p^*\widetilde{\mathcal{M}}$ is contained in the union of sets $(T^*_{S_{iA}}X_A) \times \mathbb{C}_z$. Hence ${}_{\mathrm{D}}\pi^*\widetilde{\mathcal{M}}$ is holonomic.

We also claim that, for every $t \in \mathbb{P}^1$, the inclusion $A \times \{t\} \hookrightarrow X_A$ is non-characteristic with respect to ${}_{\mathrm{D}}\pi^*\widetilde{\mathcal{M}}$. Indeed, by the choice of A, for every S_i as above, the intersection of $T^*_{A\times\{t\}}(X \times \mathbb{P}^1)$ with $T^*_{S_i\times\mathbb{P}^1}(X \times \mathbb{P}^1)$ is contained in the zero-section of $T^*(X \times \mathbb{P}^1)$. As we have $T^*_{A\times\{t\}}(X \times \mathbb{P}^1) = (T^*\iota)^{-1}(T^*_{A\times\{t\}}X_A)$, it follows that $T^*_{A\times\{t\}}X_A \cap T^*_{S_{iA}}X_A \subset T^*_{X_A}X_A$.

This implies that, for every $t \in \mathbb{P}^1$, the inclusion $f^{-1}(t) \hookrightarrow X_A$ is non-characteristic for ${}_{\mathrm{D}}\pi^*\widetilde{\mathcal{M}}$ near any point $(x_o, t) \in A \times \{t\}$ since $A \times \{t\}$ is contained in $f^{-1}(t)$.

Let us fix a point $x_o \in A$ and let $g = 0$ be a local equation of the hyperplane $f = t$ of X near x_o. We will prove strict $\mathbb{R}$-specializability of ${}_{\mathrm{D}}\pi^*\widetilde{\mathcal{M}}$ along $(f - t)$ and we will identify $\psi_{f-t}({}_{\mathrm{T}}\pi^*M)$ near $(x_o, t) \in A_A$ with $\psi_g M$.

Since f is smooth, we can locally consider good V-filtrations along $(f-t)$ in order to compute $\psi_{f-t}({}_{\mathrm{D}}\pi^*\widetilde{\mathcal{M}})$. Arguing as in the beginning of the proof of Proposition 9.5.2, one obtains that ${}_{\mathrm{D}}\pi^*\widetilde{\mathcal{M}}$ is specializable along $f = t$ and that there exists a good V-filtration for which $\mathrm{gr}^V_{-1}({}_{\mathrm{D}}\pi^*\widetilde{\mathcal{M}}) = {}_{\mathrm{D}}\iota^*_{f^{-1}(t)}({}_{\mathrm{D}}\pi^*\widetilde{\mathcal{M}})$. The latter module is equal to ${}_{\mathrm{D}}\iota^*_{g^{-1}(0)}\widetilde{\mathcal{M}}$, which itself is equal to $\psi_{g,1}\widetilde{\mathcal{M}}$, as $\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along (g) according to $14.2.2(1)_g$; it follows that ${}_{\mathrm{D}}\pi^*\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along $(f - t)$, hence strictly non-characteristic along $(f - t)$ (see Proposition 9.5.2(2)).

(1) Let us end the proof of the first statement. The strict $\mathbb{R}$-specializability property for ${}_{\mathrm{D}}p^*\widetilde{\mathcal{M}}$ amounts to strictness of ${}_{\mathrm{D}}\pi^*\widetilde{\mathcal{M}}$. A local section of ${}_{\mathrm{D}}\pi^*\widetilde{\mathcal{M}}$ which is of z-torsion is supported on $A \times \mathbb{P}^1$ since $\widetilde{\mathcal{M}}$ is strict. It is thus a local section of the coherent submodule of ${}_{\mathrm{D}}\pi^*\widetilde{\mathcal{M}}$ supported on the divisor $(f - t)$, for every t. Since ${}_{\mathrm{D}}\pi^*\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along $(f - t)$ by (3), this submodule is strict, according to Exercise 9.21. □

***End of the proof of Step* 1.** A similar argument is used to identify the sesquilinear pairings. The identification of the pre-polarizations S is straightforward, as they all both equal to (Id, Id).

Using the identification above near the axis, and the properties assumed for (M, S) on and out of the axis, we get all properties asserted for ${}_{\mathrm{T}}\pi^*(M, \mathrm{S})$ with respect to $(f - t)$ for any t. This concludes the first step. □

Step 2. Let us set $(M_A, \mathrm{S}_A) = {}_{\mathrm{T}}\pi^*(M, \mathrm{S})$ that we consider as a pre-polarized object of $\widetilde{\mathcal{D}}$-$\mathsf{Triples}(X_A)$ since we do not yet know that it is a polarized Hodge module of weight w. Nevertheless, we aim at showing that, for the constant map $a_{X_A}: X_A \to \mathrm{pt}$ and the object (M_A, S_A),

- $\big({}_{\mathrm{T}}a^{(\bullet)}_{X_A *}(M_A, \mathrm{S}_A), \mathrm{X}+\mathrm{X}'\big)$ *is a polarized* $\mathfrak{sl}_2$*-Hodge structure of weight* w.

The support of M_A is $\pi^{-1}Z$, which is equal to the blow-up Z_A of Z as we have seen above, and the fibers of $f_{|Z_A}$ all have dimension $n-1$ ($n = \dim Z$). According to Step 1 and to Assumption $(14.3.1)_{(n-1,0)}$, the assumptions of Proposition 14.4.2 are satisfied by the pre-polarized triple (M_A, S_A), and the conclusion of this proposition yields that $\big({}_{\mathrm{T}}f^{(\bullet)}_{*}\big((M_A, \mathrm{S}_A), \mathrm{X}\big)$ is a polarized $\mathfrak{sl}_2$-Hodge module of weight w. From this point, the arguments developed for the simple case of Section 14.5.a apply with no change to the present situation, and they yield the desired assertion.

Step 3. We now prove that

- *the pushforward* ${}_{\mathrm{T}}\pi^{(0)}_{*}(M_A)$ *decomposes as a direct sum in* $\widetilde{\mathcal{D}}$-$\mathsf{Triples}(X)$*, one summand being* M.

Let us first check that this is a local statement on X. If such a decomposition exists locally, then ${}_{\mathrm{T}}\pi^{(0)}_{*}(M_A) = M \oplus M_1$ locally, with M_1 supported on A. We need to prove that this decomposition is unique, in order to glue it along X (along A in fact, since π is an isomorphism away from A). Let g be a local equation for the hyperplane $f = t$ near a point $x_o \in A$. We claim that ${}_{\mathrm{D}}\pi^{(0)}_{*}(\widetilde{\mathcal{M}}_A)$ is strictly $\mathbb{R}$-specializable along (g). Indeed, we have seen in Step 1 that $\widetilde{\mathcal{M}}_A$ is strictly $\mathbb{R}$-specializable along $(f-t)$ and we have identified locally $\psi_{f-t}(\widetilde{\mathcal{M}}_A)$ with $\psi_g\widetilde{\mathcal{M}}$ (and we have a strict non-characteristic property, so that $\phi_{f-t,1}(\widetilde{\mathcal{M}}_A)$ is zero). We have also used that $\pi : \{f = t\} \to \{g = 0\}$ is an isomorphism. By the pushforward theorem 9.8.8 or 10.5.4, we conclude that ${}_{\mathrm{D}}\pi^{(0)}_{*}(\widetilde{\mathcal{M}}_A)$ is strictly $\mathbb{R}$-specializable along (g). Since $\widetilde{\mathcal{M}}$ has pure support S, if ${}_{\mathrm{D}}\pi^{(0)}_{*}(\widetilde{\mathcal{M}}_A)$ decomposes locally as $\widetilde{\mathcal{M}}\oplus\widetilde{\mathcal{M}}_1$ with $\widetilde{\mathcal{M}}_1$ supported in A, hence in $\{g = 0\}$, we can apply Proposition 9.7.2 to conclude that there does not exist any non-zero morphism $\widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}_1$ and $\widetilde{\mathcal{M}}_1 \to \widetilde{\mathcal{M}}$, and thus the local decomposition of $\widetilde{\mathcal{M}}$ is unique. Similarly, according to Lemma 12.3.10, any sesquilinear pairing between $\mathcal{M}$ and $\mathcal{M}_1$ is zero, hence ${}_{\mathrm{D},\overline{\mathrm{D}}}\pi^{(0)}_{*}({}_{\mathrm{D},\overline{\mathrm{D}}}\pi^{*}\mathfrak{s})$ decomposes uniquely as $\mathfrak{s} \oplus \mathfrak{s}_1$.

Let us then consider the local statement near (x_o, t_o), that we can assume to belong to $A \times \mathbb{P}^1$, as π is an isomorphism outside of A. Let g be a local equation of a hyperplane containing A.

We claim that $\widetilde{\mathcal{M}}_A$ is strictly non-characteristic along both components of $g \circ \pi = 0$ and their intersection. The components consist of

- the germ at x_o of the hyperplane $f = t_o$ containing A, for which the assertion has been proved in Step 1,
- the germ at (x_o, t_o) of $A \times \mathbb{P}^1$; by considering the left square in (14.5.2), the assertion follows from the property that $\widetilde{\mathcal{M}}$ is strictly non-characteristic along A, since $A_A \to A$ is smooth;
- the germ at (x_o, t_o) of $A \times \{t_o\}$, for which we apply the same argument as the previous one.

We can therefore apply the results of Section 12.7.23 together with Remark 14.2.3(4). They show that (M_A, S_A) satisfies $14.2.2(1)_{g\circ\pi}$ and $(2)_{g\circ\pi}$.

Arguing as in Proposition 14.4.2 (this is permissible due to the inductive hypothesis $(14.3.1)_{(\leqslant(n-1),0)}$, as the fibers of $\pi : S_A \to S$ have dimension $\leqslant n-1$), we conclude that $\big(\bigoplus_k {}_{\mathrm{T}}\pi_*^{(k)}(M_A), \mathrm{X}'\big)$ is strict and satisfies $14.2.2(1)_g$ and $(2)_g$ in the sense of Lemma 14.2.29. We can cover A by finitely many open sets where the previous argument applies.

Let us set $M_0 := {}_{\mathrm{T}}\pi_*^{(0)}(M_A)$. We note that, as $\mathrm{X}'^2 = 0$, $M_0 = \mathrm{P}'_0{}_{\mathrm{T}}\pi_*^{(0)}(M_A)$ is strict and satisfies $14.2.2(1)_g$ and $(2)_g$. By applying 14.2.17(10) to the quiver $(\psi_{g,1}M_0, \phi_{g,1}M_0, \mathrm{c}, \mathrm{v})$ and arguing as in the proof of $((10)_{<d} \Rightarrow (1)_d)$, we find that M_0 is S-decomposable along (g). We will identify M with a direct summand of it.

Let us set $M_0 = (\widetilde{\mathcal{M}}_0, \mathcal{S}_0)$. It decomposes therefore as $M_1 \oplus M_2$, with M_2 supported on $g^{-1}(0)$ and M_1 being a middle extension along (g). By Proposition 8.7.30, there is an adjunction morphism $\widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}_0$. This morphism is an isomorphism away from A, hence from $g^{-1}(0)$, and is injective, as $\widetilde{\mathcal{M}}$ has no coherent submodule supported on $g^{-1}(0)$. Its image is thus contained in $\widetilde{\mathcal{M}}_1$.

At this point, we cannot assert that the image is equal to $\widetilde{\mathcal{M}}_1$, since the middle extension property 9.7.2(2) of $\widetilde{\mathcal{M}}_1$ only implies the vanishing of some quotient modules, and not all of them a priori. Nevertheless, the morphism $\mathcal{M} \to \mathcal{M}_1$ between the underlying $\mathcal{D}_X$-modules is an isomorphism (since no restriction occurs in 9.7.2(2) for $\mathcal{D}_X$-modules). It follows then from Proposition 12.3.8 applied to any germ of hyperplane containing A that $\mathcal{S} = \mathcal{S}_1$. It also follows that the cokernel of $\widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}_1$ is of z-torsion.

We thus have a monomorphism of Hermitian pairs $M \to M_1$. It is strictly $\mathbb{R}$-specializable along (g), since the associated nearby and vanishing cycle morphisms are morphisms in $\mathsf{pHLM}(X, w-1)$ or $\mathsf{pHLM}(X, w)$. Therefore, this morphism is strict, according to Corollary 10.7.6. The cokernel, being strict and of z-torsion, must then vanish, and $M \simeq M_1$, as wanted.

Step 4. As X' vanishes on M, we conclude from Step 3 that $({}_{\mathrm{T}}a_{X,*}M, \mathrm{N})$ is a direct summand of $({}_{\mathrm{T}}a_{X_A*}M_A, \mathrm{X}+\mathrm{X}')$. From Step 2 and **[Del68]** we have a (non canonical) decomposition ${}_{\mathrm{T}}a_{X_A*}M_A \simeq \bigoplus_k {}_{\mathrm{T}}a_{X_A*}^{(k)}(M_A)[-k]$. Therefore, this decomposition can be chosen to induce a decomposition ${}_{\mathrm{T}}a_{X,*}M \simeq \bigoplus_k {}_{\mathrm{T}}a_{X,*}^{(k)}(M)[-k]$. In particular, $\big(\bigoplus_k {}_{\mathrm{T}}a_{X,*}^{(k)}(M), \mathrm{X}\big)$ is an $\mathfrak{sl}_2$-Hodge structure with central weight w, being a direct summand of the $\mathfrak{sl}_2$-Hodge structure $\big(\bigoplus_k {}_{\mathrm{T}}a_{X_A*}^{(k)}(M_A), \mathrm{X}+\mathrm{X}'\big)$ with central weight w.

Step 5. It remains to show the polarization property. By the result of Step 2, $\big(\bigoplus_k {}_{\mathrm{T}}a_{X_A*}^{(k)}(M_A), \mathrm{X}+\mathrm{X}'\big)$ is polarized by $(\mathrm{Id}, \mathrm{Id})$, which induces the desired prepolarization on $\big(\bigoplus_k {}_{\mathrm{T}}a_{X,*}^{(k)}(M), \mathrm{X}\big)$. That the latter is a polarization is a particular case of Proposition 3.4.18(2). This concludes the proof of 14.3.4(d), hence that of the Hodge-Saito theorem 14.3.1. $\square$

14.6. Variations of Hodge structure are Hodge modules

The first non trivial example of a polarizable Hodge module is given by a polarizable variation of Hodge structure. The following theorem is a partial converse of Proposition 14.2.10.

14.6.1. Theorem. *Let X be a complex manifold of dimension n and let ${}_{\mathrm{H}}H$ be a smooth Hodge triple of weight w, that is, a polarizable variation of pure Hodge structure of weight $w-n$ (see Definition 5.4.7). Then ${}_{\mathrm{H}}H$ is a polarizable Hodge module of weight w.*

14.6.2. Example. The two basic examples ${}_{\mathrm{H}}\widetilde{\mathcal{O}}_X$ and ${}_{\mathrm{H}}\widetilde{\omega}_X$ of 12.7.9 are the left and the right Hodge module representatives of the trivial variation of Hodge structure of weight 0. They belong to $\mathsf{pHM}(n)$.

From Theorem 14.6.1 and the Hodge-Saito theorem 14.3.1, we deduce:

14.6.3. Corollary. *Let $(\mathcal{H},\nabla)$ be vector bundle with connection on X underlying a variation of polarizable Hodge structure of weight w. Then its direct image (in the category of $\widetilde{\mathcal{D}}$-modules) by a projective morphism $f: X\to Y$ decomposes non-canonically in $\mathsf{D}^{\mathrm{b}}(\mathcal{D}_Y)$*

$$ {}_{\mathrm{D}}f_*(\mathcal{H},\nabla)\simeq\bigoplus_k {}_{\mathrm{D}}f_*^{(k)}(\mathcal{H},\nabla), $$

and each ${}_{\mathrm{D}}f_*^{(k)}(\mathcal{H},\nabla)$ underlies a polarizable Hodge module of weight $w+\dim X+k$. □

***Proof of Theorem* 14.6.1.** This assertion is not trivially satisfied since one has to check in an iterative way that nearby cycles and vanishing cycles along *any* germ of holomorphic function are polarizable Hodge modules. We assume that the polarization is $\mathrm{S}=(\mathrm{Id},\mathrm{Id})$, i.e., we realize ${}_{\mathrm{H}}H$ as a Hermitian pair $(\widetilde{\mathcal{H}},\mathcal{S})$.

We first note that 14.2.2$(1)_g$ and $(2)_g$ hold for $({}_{\mathrm{H}}H,(\mathrm{Id},\mathrm{Id}))$ if g is a local coordinate on X. According to Remark 14.2.3(3), these properties also hold when g is a power of a local coordinate on X. As a consequence, the assertion of the theorem holds if $\dim X=1$.

If $\dim X\geqslant 2$, the proof is by induction on $\dim X$. We thus assume that the theorem holds for $\dim X<n$ $(n\geqslant 2)$, and we assume $\dim X=n$. We wish to prove that, for any germ of holomorphic function g on X, 14.2.2$(1)_g$ and $(2)_g$ hold for $({}_{\mathrm{H}}H,(\mathrm{Id},\mathrm{Id}))$.

Step 1: reduction to the case where $D:=(g)$ is a normal crossing divisor

We assume that 14.2.2$(1)_g$ and $(2)_g$ hold for $({}_{\mathrm{H}}H,(\mathrm{Id},\mathrm{Id}))$ if g defines a normal crossing divisor in X. Let us then take any germ g on X centered at $x\in X$. We simply denote by X the germ (X,x) and by D the germ of the reduced divisor defined by g. Let $f: X'\to X$ be a projective modification which is an isomorphism $X'\smallsetminus f^{-1}(D)\to X\smallsetminus D$ such that $g':=g\circ f$ defines a normal crossing divisor D' in X'.

The pullback $({}_{\mathrm{H}}H',(\mathrm{Id},\mathrm{Id})):={}_{\mathrm{T}}f^*({}_{\mathrm{H}}H,(\mathrm{Id},\mathrm{Id}))$ is also a polarized variation of pure Hodge structure of weight $w-n$ (see 12.7.13) and is strict as an object of $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X')$. Furthermore, by our assumption, 14.2.2$(1)_{g'}$ and $(2)_{g'}$ hold for $({}_{\mathrm{H}}H',(\mathrm{Id},\mathrm{Id}))$. It follows from Proposition 14.4.2 that $({}_{\mathrm{T}}f_*^{(0)}{}_{\mathrm{H}}H',(\mathrm{Id},\mathrm{Id}))$ satisfies 14.2.2$(1)_g$ and $(2)_g$, that it is S-decomposable along (g) as an object of $\widetilde{\mathcal{D}}\text{-Triples}(X)$ and Corollary 10.7.8 yields that it is strict.

Let us denote by $({}_{\mathrm{H}}H'_0, (\mathrm{Id}, \mathrm{Id}))$ the component of $({}_{\mathrm{T}}f_*^{(0)}{}_{\mathrm{H}}H', (\mathrm{Id}, \mathrm{Id}))$ with pure support X. It also satisfies 14.2.2(1)$_g$ and (2)$_g$, is strict, and is a middle extension along (g). It corresponds to a coherently filtered $\mathcal{D}_X$-module $(\mathcal{H}'_0, F_\bullet\mathcal{H}'_0)$. We will show that $({}_{\mathrm{H}}H'_0, (\mathrm{Id}, \mathrm{Id}))$ is isomorphic to $({}_{\mathrm{H}}H, (\mathrm{Id}, \mathrm{Id}))$, concluding thereby the first step.

We start with identifying the $\widetilde{\mathcal{D}}_X$-module components. Composing the adjunction morphism $\widetilde{\mathcal{H}} \to {}_{\mathrm{D}}f_*^{(0)}\widetilde{\mathcal{H}}'$ of Proposition 8.7.30 with the projection (coming from the S-decomposition) ${}_{\mathrm{D}}f_*^{(0)}\widetilde{\mathcal{H}}' \to \widetilde{\mathcal{H}}'_0$ yields a morphism $\widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}'_0$ which is an isomorphism on the complement of D. Since $\widetilde{\mathcal{H}}$ is $\widetilde{\mathcal{O}}_X$-locally free, this morphism is injective. On the other hand, $\mathcal{H}'_0$ is a middle extension along (g) (Example 11.5.3 and Remark 11.5.5). Therefore, $\mathcal{H} \to \mathcal{H}'_0$ is an isomorphism (see Exercise 9.35(1)).

What about the Hodge filtrations? We know that the morphism $F_p\mathcal{H} \to F_p\mathcal{H}'_0 =: F'_p\mathcal{H}$ is injective and is an isomorphism on $X \smallsetminus D$, so $F'_p\mathcal{H}/F_p\mathcal{H}$ is supported in D. On the other hand, $\mathcal{H}/F_p\mathcal{H}$ is $\mathcal{O}_X$-locally free, being a successive extension of $\mathcal{O}_X$-locally free modules $\mathrm{gr}_q^F\mathcal{H}$. Since we have an inclusion $F'_p\mathcal{H}/F_p\mathcal{H} \hookrightarrow \mathcal{H}/F_p\mathcal{H}$, it follows that $F'_p\mathcal{H}/F_p\mathcal{H} = 0$, that is, $F'_p\mathcal{H} = F_p\mathcal{H}$, as desired.

What about the sesquilinear pairing $\mathcal{S}$ on $\mathcal{H}$ and $\mathcal{S}'$ on $\mathcal{H}'_0 \simeq \mathcal{H}$? Both take values in $\mathcal{C}_X^\infty$ (Lemma 12.3.6) and coincide on $X \smallsetminus D$, hence they coincide. □

Step 2: reduction to the case where (g) ***is a reduced normal crossing divisor.*** According to Step 1, we can assume that g is a monomial $x_1^{r_1} \cdots x_\ell^{r_\ell}$ in a local coordinate system $(x_1, \dots, x_n)$. We still denote by X the corresponding local coordinate chart. There exists a multi-cyclic ramified covering $f : X' \to X$ such that $h := g \circ f$ is a power of a product of local coordinates $(x'_1 \cdots x'_\ell)^r$. Set $h' = x'_1 \cdots x'_\ell$ and let us assume that 14.2.2(1)$_{h'}$ and (2)$_{h'}$ hold for $({}_{\mathrm{H}}H', (\mathrm{Id}, \mathrm{Id})) := {}_{\mathrm{T}}f^*({}_{\mathrm{H}}H, (\mathrm{Id}, \mathrm{Id}))$. Then 14.2.2(1)$_h$ and (2)$_h$ hold for $({}_{\mathrm{H}}H', (\mathrm{Id}, \mathrm{Id})))$, according to Remark 14.2.3(3). We wish to prove that 14.2.2(1)$_g$ and (2)$_g$ hold for $({}_{\mathrm{H}}H, (\mathrm{Id}, \mathrm{Id}))$. We argue in a way similar to that of Step 1 and take the same notation. In particular, $({}_{\mathrm{H}}H'_0, (\mathrm{Id}, \mathrm{Id}))$ is the component of ${}_{\mathrm{T}}f_*^{(0)}({}_{\mathrm{H}}H', (\mathrm{Id}, \mathrm{Id})))$ with pure support X, and 14.2.2(1)$_g$ and (2)$_g$ hold for it.

According to 12.7.31, $({}_{\mathrm{H}}H, (\mathrm{Id}, \mathrm{Id}))$ is a direct summand of $({}_{\mathrm{H}}H'_0, (\mathrm{Id}, \mathrm{Id}))$. Since 14.2.2(1)$_g$ and (2)$_g$ are stable by direct summand in $\widetilde{\mathcal{D}}\text{-Triples}(X)$, as follows from 14.2.8, they hold for $({}_{\mathrm{H}}H, (\mathrm{Id}, \mathrm{Id}))$. □

Step 3: case where (g) ***is a reduced normal crossing divisor.*** Assume now that g is a product of distinct coordinates of a local coordinate system. We are thus in the setting of Example 12.7.24. We then know that ${}_{\mathrm{H}}H$ is strictly $\mathbb{R}$-specializable and a middle extension along (g), so we only need to check 14.2.2(2a)$_g$, according to Remark 14.2.3(4).

We are therefore led to showing that the right-hand side in (12.7.24 $*$) is a polarized Hodge module of weight $w+\ell-1$ ($\ell \geqslant 0$), where each ι_I occurring there is the inclusion of a codimension $(\ell+1)$ submanifold in X. By induction on $\dim X$, each variation ${}_{\mathrm{T}}\iota_I^*({}_{\mathrm{H}}H, \mathrm{S})$ for $J = I^{\mathrm{c}} \in \mathcal{J}_{\ell+1}$ is a polarized Hodge module of weight $w-(\ell+1)$, since its support has dimension $n-(\ell+1)$. Hence, by Kashiwara's equivalence of Proposition

14.2.9, $({}_{\mathrm{T}}\iota_{I*}({}_{\mathrm{T}}\iota_{I\mathrm{H}}^{*}H), (\mathrm{Id}, \mathrm{Id}))$ is a polarized Hodge module of weight $w-(\ell+1)$, and $({}_{\mathrm{T}}\iota_{I*}({}_{\mathrm{T}}\iota_{I\mathrm{H}}^{*}H), (\mathrm{Id}, \mathrm{Id}))(-\ell)$ is a polarized Hodge module of weight $w+\ell-1$. □

14.7. Some properties of the category of W-filtered Hodge modules

In this section, we consider the category $\mathsf{WHM}(X)$ of W-filtered Hodge modules introduced in Section 14.2.12. In Sections 14.7.a and 14.7.b, we prove regularity and strict holonomicity of the holonomic $\widetilde{\mathcal{D}}_X$-modules underlying an object of $\mathsf{WHM}(X)$. In particular, these properties hold true for polarizable Hodge modules. In Section 14.7.f, relying on the Hodge-Saito theorem 14.3.1, we partially extend it in the sense that we prove stability of WHM by projective pushforward and we analyze the behavior of nearby and vanishing cycles by such a pushforward.

14.7.a. Regularity

14.7.1. Theorem. *Let $(M, W_\bullet M)$ be an object of $\mathsf{WHM}(X)$. Then the underlying $\mathcal{D}_X$-modules $\mathcal{M}', \mathcal{M}''$ of M are regular holonomic.*

Proof. According to Proposition 10.7.13 we can use Definition 10.7.12 for the notion of regularity. Furthermore, it is enough to prove the theorem for each graded object $\mathrm{gr}_\ell^W M$, which is pure polarizable. We argue by induction on the dimension of the support. Since the definition of $\mathsf{pHM}(X)$ and that of regularity are inductive, we are left with checking the property 10.7.12(1), which follows from Corollary 10.7.10, as by definition $\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}''$ are strict. □

14.7.b. Strict holonomicity. We refer to Section 8.8.g for the notion of strict holonomicity and to Proposition 8.8.38 for various consequences.

14.7.2. Theorem. *Let $(M, W_\bullet M)$ be an object of $\mathsf{WHM}(X)$. Then the underlying $\widetilde{\mathcal{D}}_X$-modules $\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}''$ are strictly holonomic.*

Proof. As for Theorem 14.7.1, we can assume that M is pure polarizable. Then, in view of the inductive definition of $\mathsf{pHM}(X)$, the result is a direct consequence of Theorem 10.7.14, the case where the support of M is punctual being clear. □

14.7.c. Nondegeneracy. Let $(M, W_\bullet M)$ be an object of $\mathsf{WHM}(X)$. Each component $\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}''$ is strictly holonomic by Theorem 14.7.2, and the underlying $\mathcal{D}_X$-modules are regular holonomic by Theorem 14.7.1. In order to apply the duality functor of Section 13.4.c, we only need to check that the underlying object of $\mathcal{D}\text{-}\mathsf{Triples}(X)$ is nondegenerate (see Definition 13.4.3). That the dual object is also an object of $\mathsf{WHM}(X)$ is a stronger property that will be addressed in Chapter 16.

14.7.3. Lemma. *The underlying triple $\mathcal{T} = (\mathcal{M}', \mathcal{M}'', \mathfrak{s})$ of an object M of $\mathsf{WHM}(X)$ is nondegenerate.*

Proof. By induction on the length of the filtration $W_\bullet M$, we only need to consider the case of an object of $\mathsf{pHM}(X, w)$ having pure support on a closed irreducible subvariety Z of X. Since the question is local, we can assume that there exists a holomorphic function $g : X \to \mathbb{C}$ which is not identically zero on Z, such that M is a middle extension along H and its restriction to $X \smallsetminus g^{-1}(0)$ is the pushforward by $\iota : Z \smallsetminus g^{-1}(0) \hookrightarrow X \smallsetminus g^{-1}(0)$ of a polarizable variation of Hodge structure of some weight w. It is immediate to check that a polarizable variation of Hodge structure is nondegenerate when considered as a triple, and Proposition 13.4.5 implies that its pushforward by ι is nondegenerate. Then Corollary 13.4.4 yields the conclusion. □

14.7.d. Localization and dual localization

14.7.4. Proposition. *Let H be a smooth hypersurface of X. If M is an object of $\mathsf{pHM}(X)$, then both $M[*H]$ and $M[!H]$ underlie objects of $\mathsf{WHM}(X)$.*

Proof. The assertion is local, so we can assume that $X = H \times \Delta_t$. We can also assume that M has strict support a closed irreducible analytic subset $Z \subset X$. If $Z \subset H$, the assertion is trivial. Since M is a minimal extension along H, we have exact sequences in $\widetilde{\mathcal{D}}$-$\mathsf{Triples}(X)$:

$$0 \longrightarrow M \longrightarrow M[*H] \longrightarrow \operatorname{Coker} \operatorname{var}_t \longrightarrow 0$$
$$0 \longrightarrow \operatorname{Ker} \operatorname{can}_t \longrightarrow M[!H] \longrightarrow M \longrightarrow 0,$$

and we have

$$\operatorname{Coker} \operatorname{var}_t \simeq \operatorname{Coker}[\mathrm{N} : \psi_t M \longrightarrow \psi_t M(-1)]$$
$$\operatorname{Ker} \operatorname{can}_t \simeq \operatorname{Ker}[\mathrm{N} : \psi_t M \longrightarrow \psi_t M(-1)].$$

We can then apply Lemma 3.3.7. □

14.7.e. Non-characteristic restriction

14.7.5. Proposition. *Let $(M, W_\bullet M)$ be an object of $\mathsf{WHM}(X)$ and let Y be a closed submanifold of X which is non-characteristic with respect to each $\widetilde{\mathcal{D}}_X$-module component $\widetilde{\mathcal{M}}$ of M. Then it is strictly non-characteristic with respect to each $\widetilde{\mathcal{D}}_X$-module component of $\mathrm{gr}^W_\ell M$ for every $\ell \in \mathbb{Z}$, and the restriction $({}_{\mathrm{T}}\iota^*_Y M, {}_{\mathrm{T}}\iota^*_Y W_\bullet M)$ as defined by 12.7.22, is an object of $\mathsf{WHM}(Y)$.*

Proof. The question is local on X. We argue by induction on the codimension of Y. If H is a smooth hypersurface containing Y, then H is non-characteristic with respect to each $\widetilde{\mathcal{D}}_X$-module component $\widetilde{\mathcal{M}}$ of M, hence to each $\mathrm{gr}^W_\ell \widetilde{\mathcal{M}}$. As $\mathrm{gr}^W_\ell \widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along H, it follows that H is strictly non-characteristic with respect to each $\mathrm{gr}^W_\ell \widetilde{\mathcal{M}}$ (see Proposition 9.5.2(2)). Furthermore, ${}_{\mathrm{T}}\iota^*_H \mathrm{gr}^W_\ell M = {}_{\mathrm{T}}\iota^{*(0)}_H \mathrm{gr}^W_\ell M$.

Setting $H = \{t = 0\}$, we have ${}_{\mathrm{T}}\iota_{H*}({}_{\mathrm{T}}\iota^*_H \mathrm{gr}^W_\ell M) = \psi_{t,1} \mathrm{gr}^W_\ell M$, which is an object of $\mathsf{pHM}(H)$, so by induction Y is strictly non-characteristic with respect to each $\widetilde{\mathcal{D}}_X$-module component of ${}_{\mathrm{T}}\iota^{*(0)}_H \mathrm{gr}^W_\ell M$. By Remark 8.8.19(2), Y is strictly non-characteristic with respect to each $W_\ell \widetilde{\mathcal{M}}$. Hence, ${}_{\mathrm{T}}\iota^*_Y W_\ell M$ has cohomology in degree zero at most for each ℓ, so that $({}_{\mathrm{T}}\iota_Y^* W_\ell M)_\ell$ defines a filtration of ${}_{\mathrm{T}}\iota_Y^*(M, W_\bullet M)$ in $\mathsf{WHM}(X)$. □

14.7.f. Stability by projective pushforward

14.7.6. Theorem. *Let $f : X \to Y$ be a projective morphism and let M be an object of $\mathsf{WHM}(X)$. Then, for each $k \in \mathbb{Z}$, the pushforward ${}_{\mathrm{T}}f_*^{(k)}M$, together with the shifted image filtration $W[k]_\bullet({}_{\mathrm{T}}f_*^{(k)}M)$, is an object of $\mathsf{WHM}(X)$. Furthermore, the spectral sequence attached to $W_\bullet$ degenerates at E_2.*

Proof. The term $E_1^{-\ell,k+\ell}$ of the spectral sequence of Corollary 12.7.37 is an object of $\mathsf{pHM}(Y, k+\ell)$, according to the Hodge-Saito theorem 14.3.1, and the differential d_1 is a morphism in $\mathsf{pHM}(Y, k+\ell)$, so that $E_2^{-\ell,k+\ell}$ is also an object of $\mathsf{pHM}(Y, k+\ell)$, due to the abelianity of this category (Theorem 14.2.17(4)). Then, for $r \geqslant 2$, $d_r = 0$ since the weight of $E_r^{-\ell+r,k+\ell-r+1}$ is $k+\ell-r+1 < k+\ell$ (Theorem 14.2.17(2)). □

14.7.g. Semi-simplicity

Any polarizable Hodge module of weight w is semi-simple in the category $\mathsf{pHM}(X, w)$ (Corollary 14.2.20). If X is a projective complex manifold, semi-simplicity also holds for the underlying holonomic $\widetilde{\mathcal{D}}_X$-module, that is, the analogue of Theorem 4.3.3 holds for polarizable Hodge modules.

***14.7.7. Theorem* (Semi-simplicity).** *Assume X is projective. Let M be a polarized Hodge module of weight w (so that $\widetilde{\mathcal{M}}' \simeq \widetilde{\mathcal{M}}''$ by means of a polarization). Then the underlying $\mathcal{D}_X$-module $\mathcal{M}$ is semi-simple.*

Proof. By the S-decomposition theorem (Corollary 14.2.20), we can assume that M has pure support an irreducible variety $Z \subset X$. If $\dim Z = 0$, the result is clear by Definition 14.2.2(0). If $\dim Z \geqslant 1$, the restriction of (M, S) to a suitable smooth Zariski dense open subset Z^o of Z is a polarized variation of Hodge structure of weight $w - \dim Z$ (Proposition 14.2.10).

We claim that $\mathcal{M}$ has no submodule and no quotient module supported on $Z \smallsetminus Z^o$. Indeed, let g be any local holomorphic function at $x \in Z \smallsetminus Z^o$. As $\widetilde{\mathcal{M}}$ is a middle extension along (g), so is $\mathcal{M}$ (Exercise 9.35(4)), so that $\mathcal{M}$ does not have any submodule supported on $g^{-1}(0)$. As g is arbitrary, this proves the claim.

Since $\mathcal{M}$ is regular holonomic (Theorem 14.7.1), we can apply the criterion of Corollary 13.2.11, and we are led with proving that the underlying local system on Z^o is semi-simple.

- If $\dim Z^o = 1$, the underlying local system is semi-simple (Corollary 6.4.2).
- If $\dim Z^o \geqslant 2$, we fix a projective embedding of Z. The Zariski-Lefschetz theorem [**HL85**, Th. 1.1.3(ii)] implies that, for a generic hyperplane H, the inclusion $H \cap Z^o \hookrightarrow Z^o$ induces a *surjective* morphism of fundamental groups. By induction and according to Remark 4.3.2(2), we conclude that the local system underlying $M_{|Z^o}$ is semi-simple. □

When both X and Y are projective, we can combine Theorems 14.3.3, 14.3.1 and 14.7.7 to obtain:

14.7.8. Corollary. *Let $f : X \to Y$ be a morphism between projective complex manifolds and let $\mathcal{M}$ be a semi-simple holonomic $\mathcal{D}_X$-module underlying a polarizable Hodge module. Then ${}_{\mathrm{D}}f_*\mathcal{M}$ decomposes non-canonically as $\bigoplus_k {}_{\mathrm{D}}f_*^{(k)}\mathcal{M}[-k]$, and each ${}_{\mathrm{D}}f_*^{(k)}\mathcal{M}$ is itself a semi-simple holonomic $\mathcal{D}_Y$-module.* □

14.8. Exercises

Exercise 14.1. Show that if the conditions in Definition 14.2.2 hold for a function g, they hold for g^r for any $r \in \mathbb{N}^*$. [*Hint*: Use the example of Section 9.9.a.] Conclude that, if $n = 1$, Definition 14.2.2 reduces to Definition 7.4.7.

14.9. Comments

The relation between Hodge theory and the theory of nearby or vanishing cycles in dimension bigger than one starts with the work of Steenbrink [**Ste76, Ste77**]. It concerns 1-parameter families of projective varieties, regarded as proper functions from a complex manifold to a disc. A canonical Hodge structure is constructed on the cohomology of the nearby fiber of a singular fiber of the family by means of replacing the special fiber with a divisor with normal crossings and by computing the nearby or vanishing cohomology in terms of a logarithmic de Rham complex, in order to apply Deligne's method in [**Del71b**]. This gives a geometric construction of Schmid's limit mixed Hodge structure in the case of a variation of geometric origin. The need of passing from the assumption of unipotent monodromy, as used in the work of Schmid [**Sch73**] to the assumption of quasi-unipotent monodromy is justified by this geometric setting. This leads Steenbrink [**Ste77**] to developing the notion of logarithmic de Rham complex in the setting of V-manifolds. Steenbrink also obtains, as a consequence of this construction, the local invariant cycle theorem and the Clemens-Schmid exact sequence. We can regard this work as the localization of Hodge theory in the analytic neighbourhood of a projective variety.

The work of Varchenko [**Var82**] and others on asymptotic Hodge theory has localized even more Hodge theory. This work is concerned with an isolated singularity of a germ of holomorphic function and it constructs a Hodge-Lefschetz structure on the space of vanishing cycles of this function, by taking advantage that the vanishing cycles are supported at the isolated singularity, which is trivially a projective variety. The construction of Varchenko has been later analyzed in terms of $\widetilde{\mathcal{D}}$-modules by Pham [**Pha83**], Saito [**Sai83b, Sai83a, Sai84, Sai85**] and Scherk-Steenbrink [**SS85**]. It is then natural to consider the cohomology of the vanishing cycle sheaf of a holomorphic function on a complex manifold whose critical locus is projective, but possibly not the special fiber of the function, and to ask for a mixed Hodge structure on it.

The theory of polarizable Hodge modules, as developed by Saito in [**Sai88**], emphasizes the local aspect of Hodge theory, by constructing a category defined by local properties in a way similar, but much more complicated, to the definition of a the category of variations of Hodge structure. It can then answer the question above. This idea has proved very efficient, eventually allowing to use the formalism of Grothendieck's six operations in Hodge theory. Many standard cohomological results, like the Clemens-Schmid exact sequence and the local invariant cycle theorem, can be read in this functorial way. The proofs given in this chapter follow those of [**Sai88, Sai90**] by adapting them to the setting of triples.

The definition of complex Hodge modules as developed here, not relying on a $\mathbb{Q}$-structure and on the notion of a perverse sheaf, is inspired by the extension of the notion of polarizable Hodge module to twistor theory, as envisioned by Simpson [**Sim97**], and achieved by Sabbah [**Sab05**] and Mochizuki [**Moc07, Moc15**], although the way the sesquilinear pairing is used on both theories is not exactly the same. We refer to the comments of Chapter 12 for the idea of using sesquilinear pairings in the framework of holonomic $\widetilde{\mathcal{D}}$-modules.