The Dynamical André-Oort conjecture for rational maps

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Backgrounds

Complex Dynamics, introduced by Fatou and Julia in the 1900s, study the iteration of rational maps on the Riemann sphere $\mathbb{P}^1(\mathbb{C})$, topics includes e.g. Fatou set and Julia set ($=$ the closure of repelling periodic points), local dynamics, bifurcation theory...

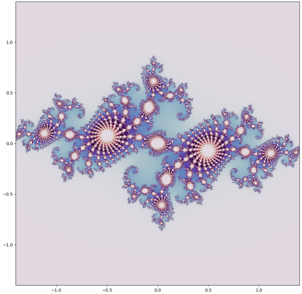

Figure 1: The Julia set of $z^2 + (-0.744 + 0.148i)$

Backgrounds

Arithmeric Dynamics, started in 1980s, is a relatively new area study the arithmetic and algebraic properties of dynamics of rational maps on $\mathbb{P}^1$, motivated by the anology between Arithmetic Dynamics and Diophantine Geometry.

Arithmetic Dynamics	Arithmetic of Elliptic Curves
Rational map	Elliptic curve
rational/integral orbit	rational/integral point
Preperiodic point	Torsion point
PCF map	Elliptic curve with complex multiplication

Here a PCF map is a rational map such that its critical orbits are finite.

The parameter space $\mathrm{Rat}_d(\mathbb{C})$

Let $d \geq 2$. The parameter space of all degree d rational maps on the Riemann sphere, denoted by Rat_d, can be seen as a Zariski open subset $\mathrm{Rat}_d \subset \mathbb{P}^{2d+1}$, via the expression

$$f(z) = \frac{a_d z^d + \cdots + a_0}{b_d z^d + \cdots + b_0}.$$

The group $\mathrm{PGL}(2, \mathbb{C})$ acts on Rat_d via conjugacy: Let f be a rational map and let $\sigma \in \mathrm{PGL}(2, \mathbb{C})$, the conjugacy $\sigma f \sigma^{-1}$ has the same dynamical behavior with f up to the coordinate change induced by σ.

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Dynamical André-Oort conjecture

The following conjecture proposed by Baker and DeMarco in 2011 concerning the distribution of PCF maps, is a bridge between Complex Dynamics and Arithmetic Dynamics.

Conjecture (DAO for curves)

Let $(f_t)_{t \in X}$ be a non-isotrivial algebraic family of rational maps with degree $d \geq 2$, parametrized by an algebraic curve Λ over $\mathbb{C}$. Then the following are equivalent:

(i) *There are infinitly many $t \in \Lambda$ such that f_t is PCF;*

(ii) *Any two marked critical points are dynamically related.*

Dynamical André-Oort conjecture

1.The meaning of non-isotrivial: Maps in Λ are not all conjugated by elements in $\mathrm{PGL}(2,\mathbb{C})$.

2. The meaning of dynamically related critical points: We view a family of rational maps $(f_t)_{t\in X}$ as a single rational map f defined over the function field $k := \mathbb{C}(\Lambda)$. Two critical points $c_1 \in \mathbb{P}^1_k$ and $c_2 \in \mathbb{P}^1_k$ are called dynamically related if they satisfy an algebraic relation, i.e. there exists a irreducible algebraic curve $V \subset \mathbb{P}^1_k \times \mathbb{P}^1_k$ such that $(c_1, c_2) \in V$ and V is preperiodic under the product map $f \times f$ on $\mathbb{P}^1_k \times \mathbb{P}^1_k$.

A typical example of algebraic relation: $f_t^m(c_1(t)) = f_t^n(c_2(t))$ for some $m, n > 0$. Here we can choose $V \subset \mathbb{P}^1_k \times \mathbb{P}^1_k$ to be the irreducible component of the algebraic curve $f^m(z_1) = f^n(z_2)$ contaning (c_1, c_2).

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A Bogomolov type generalization of the DAO conjecture

Let Λ be an algebraic curve over $\overline{\mathbb{Q}}$ and let $(f_t)_{t\in\Lambda}$ be a non-isotrivial algebraic family of rational maps over $\overline{\mathbb{Q}}$. The critical height $h_{\mathrm{crit}} : \Lambda(\overline{\mathbb{Q}}) \to \mathbb{R}$ is given by

$$t \in \Lambda(\overline{\mathbb{Q}}) \mapsto \widehat{h}_{f_t}(\mathcal{C}_{f_t}),$$

where $\mathcal{C}_{f_t}$ is the critical locus of f_t and $\widehat{h}_{f_t}$ is the canonical height of f_t. It is clear that a parameter $t \in \Lambda(\overline{\mathbb{Q}})$ is PCF if and only if $h_{\mathrm{crit}}(t) = 0$.

Theorem ($\Rightarrow$ DAO for curves, Ji-Xie)

Let Λ be an algebraic curve over $\overline{\mathbb{Q}}$ and let $(f_t)_{t\in\Lambda}$ be a non-isotrivial algebraic family of rational maps over $\overline{\mathbb{Q}}$. Then the following are equivalent:

(i) *There are infinitely many $t \in \Lambda$ such that f_t is PCF;*

(ii) *Any two marked critical points are dynamically related;*

(iii) *For every $\varepsilon > 0$, the set $\{t \in \Lambda(\overline{\mathbb{Q}}) \mid h_{crit}(t) < \varepsilon\}$ is infinite.*

Motivations

Complex Analytic part:
PCF maps play a fundamental role both in complex dynamics.

1. Globally, a rational map is expanding on the Riemann sphere (it is a branched covering of degree d), but looking at the local behavior of f at a critical point, it is stronly contracting. For PCF maps the critical orbit is simple, so the dyanmics can be well understood.

2. Every hyperbolic component with connected Julia set contains a unique PCF hyperbolic map.

3. PCF maps are equidistributed with respect to the bifurcation measure, which is a natural maesure supported on the strong bifurcation locus.

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Arithmetic part:
PCF maps are also special from the arithmetic point of view.

1. A consequence of Thurston's rigidity theorem shows that PCF maps are defined over $\overline{\mathbb{Q}}$ in the moduli space of rational maps of fixed degree, except for the well-understood one parameter family of flexible Lattès maps.

2. Moreover PCF maps are Zariski dense, and form a set of bounded Weil heightafter excluding the flexible Lattès family.

There is an analogy between PCF points and CM points (on Shimura Varieties), and the DAO conjecture can be seen as an analogy of the André-Oort Conjecture in arithmetic geometry.

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Previous results

Baker-DeMarco proved the Conjecture for families of polynomials parameterized by the affine line with coefficients that were polynomial in t.

Since the fundamental work of Baker-DeMarco, plenty of works are devoted to prove special cases of the Conjecture. Most progress are made in the setting that the families are given by polynomials, studied by many authors Ghioca, Hsia, Tucker, Krieger, Nguyen, Ye, Favre, Gauthier. Among theses results, a remarkable work of Favre and Gauthier confirmed this Conjecture for families of polynomials.

In the case that the family is not given by polynomials, DeMarco-Wang-Ye proved the Conjecture for some dynamical meaningful algebraic curves in the moduli space of quadratic rational maps.

Sketch of the proof

We view a family of rational maps f_t over Λ as a single map f on $\Lambda \times \mathbb{P}^1$.

$$f : \Lambda \times \mathbb{P}^1 \to \Lambda \times \mathbb{P}^1 \tag{1}$$

$$(t, z) \mapsto (t, f_t(z)). \tag{2}$$

A marked point P is a morphism $P : \Lambda \to \mathbb{P}^1$.

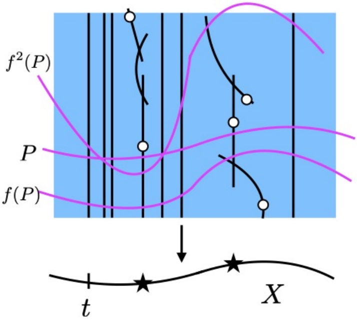

Figure 2: the graph of P and its images by f, picture by DeMarco.

Sketch of the proof

The direction that (ii) implies (i) was proved by DeMarco. We only need to show that (i) implies (ii). Assume there are infinitly many PCF parameters in Λ. Since PCF maps are defined over $\overline{\mathbb{Q}}$, the family itself is defined over $\overline{\mathbb{Q}}$. Assume for the sake of contradiction that there are two marked critical points c_1 and c_2 that are not dynamically related.

Step 1: Construct bifurcation measures μ_{c_i} on Λ with respect to a critical point $c_i(t)$. It is a natural measure with Hölder continuous potential satisfying the condition

$$\mu_{f(c_i)} = d\mu_{c_i}.$$

Dujardin-Favre showed that the bifurcation measure μ_{c_i} is non-zero if and only if c_i is not preperiodic.

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Step 2: The equidistribution theorem for small points of Yuan and Zhang tells us that the infinite number of PCF parameters are equidistributed with respect to the bifurcation measure μ_{c_i}. As a consequence for every non-preperiodic c_i, μ_{c_i} are proportional, i.e.

$$\mu_{c_i} = c\mu_{c_j}$$

for some $c > 0$.

The application of various equidistribution theorems is one of the most successful ideas in arithmetic dynamics, which backs to the works of Ullmo and Zhang, in where they solved the Bogomolov Conjecture.

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Sketch of the proof

Step 3: We introduce new strategy in the next three steps. Our strategy, different from previous proofs, is to work on general points with respect to

$$\mu_{\text{bif}} := \sum_{1 \le i \le 2d-2} \mu_{c_i}.$$

We prove that for μ_{bif}-a.e. point $t \in X$, f_t satisfies a list of conditions. Then we show a contradiction from these conditions.
First, three non-uniform hyperbolic conditions:
(1) t is marked Collet-Eckmann: $|df_t^n(c_i(t))| \ge \lambda^n$ for some $\lambda > 1$.
(2) t is Parametric Collet-Eckmann: $|\frac{\partial f_t^n(c_i(t))}{\partial t}| \ge \lambda_0^n$ for some $\lambda_0 > 1$.
(3) t is polynomial recurrence: $\text{dist}(f_t^n(c_i(t)), \text{Crit}) \ge n^{-s}$ for some $s > 0$.

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Sketch of the proof

Step 3 continue: We define $\mathrm{Corr}(\mathbb{P}^1)_*^{f_t}$ be the set of $f_t \times f_t$-invariant Zariski closed subsets $\Gamma_t \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ of pure dimension 1 such that the maps of projections to coordinates are finite (there are at most countably many of them).
We prove μ_{bif}-a.e. point $t \in X$ satisfies
(4) the Frequently Separated condition $\mathrm{FS}(\Gamma_t)$ for every $\Gamma_t \in \mathrm{Corr}(\mathbb{P}^1)_*^{f_t}$, which means that in most of the time $n \geq 0$ (i.e. with large density), the distance between $(f_t^n(c_1(t)), f_t^n(c_2(t)))$ and Γ_t is larger than a fixed positive constant.

The proof uses pluripotential theory and arithmetic intersection theory.

Sketch of the proof

Step 4: Let c_1 be a marked critical point. We prove similarity between the bifurcation measure μ_{bif} on the parameter space and the maximal entropy measure μ_{f_t} on the phase space, for t satisfying (1), (2) and (3).

This can be thought as a generalization of Tan Lei's work, in where she got such a similarity at PCF points in the boundary of the Mandelbrot set.

To prove this result, we study the distortions for non-injective maps and we use the binding argument in complex dynamics.

Sketch of the proof

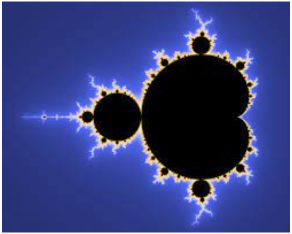

Figure 3: The boundary of the Mandelbrot set, i.e. the bifurcation locus of $z^2 + t$.

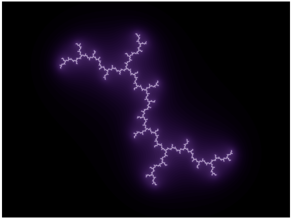

Figure 4: The Julia set of $z^2 + i$.

Sketch of the proof

Step 5: Using the above similarity for c_1 and c_2, we can get a symmetry of μ_{f_t}, i.e. biholomorphic map $\sigma : U \to \sigma(U)$ on a disk U such that
(1) $U \cap J(f_t) \neq \emptyset$;
(2) $\sigma_*(\mu_{f_t})$ and μ_{f_t} are propotional.

Lemma (Ji-Xie)

Assume f_t is not exceptional. Then the biholomorphic symmetry σ comes from an algebraic correspondence, i.e. there exists $\Gamma_t \in Corr(\mathbb{P}^1)_*^{f_t}$ such that the graph of σ is contained in Γ_t.

Finally, how to get a contradiction? We show that the consitions (1)+(2)+(3), together with the above lemma implies the opposite of (4). This can not happen, since (1), (2), (3) and (4) are both generic conditions with respect to μ_{bif}.

An application of DAO for curves

For every $f \in \mathrm{Rat}_d$ and $n \geq 1$, f^n has exactly $d^n + 1$ fixed points counted with multiplicity. The multiplier of a f^n-fixed point x is the differential $df^n(x) \in \mathbb{C}$. Using elementary symmetric polynomials, their multipliers define a point $S_n(f) \in \mathbb{C}^{d^n+1}$. The multiplier spectrum of f is the sequence $S_n(f), n \geq 1$.

Theorem (McMullen's quasi-finiteness theorem)

Except for the flexible Lattès family, the multiplier spectrum determines the conjugacy class of rational maps up to finitely many choices.

Using DAO for curves, we improve McMullen's theorem.

Theorem (Generically injectivity, Ji-Xie)

The choice is unique on a Zariski open subset of the parameter space.

Final remark

The following conjecture is still open.

Conjecture (DAO conjecture for arbitrary varieties, Baker-DeMarco)

Let $(f_t)_{t\in\Lambda}$ be a non-isotrivial algebraic family of rational maps parametrized by a quasiprojective variety Λ over $\mathbb{C}$ of dimension $N \geq 1$. Then the following are equivalent:

(i) *The parameter $t \in \Lambda$ such that f_t is PCF are Zariski dense;*

(ii) *Any $N+1$ marked critical points are dynamically related.*

Thanks for your attention!