Generalisations of the functional equation of the logarithm via del Pezzo surfaces

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Plan

I Introduction

Polylogarithms & Functional identities

Theorem : $\mathbf{HLog}^w = 0$ for $w = 1, \ldots, 6$

II Proof

Del Pezzo surfaces

Hyperlogarithms

III Comparison between $\mathbf{HLog}^2 = \boldsymbol{\mathcal{A}b}$ and $\mathbf{HLog}^3$

IV Approach à la Gelfand-MacPherson

The logarithm

- $\mathbf{L}\mathrm{i}_1(z) = -\mathbf{Log}(1-z)$ $\quad (z \in \mathbb{C})$

- **Integral representation :** $\mathbf{Log}(z) = \int^z \frac{du}{u-0}$

  $\mathbf{L}\mathrm{i}_1(z) = -\int^z \frac{du}{u-1}$

- **Taylor series :** $\mathbf{L}\mathrm{i}_1(z) = \sum_{k=1}^{\infty} \frac{z^k}{k}$

- **Monodromy :** $\mathcal{M}_0 \left(\mathbf{Log} \right) = \mathbf{Log} + 2i\pi$

- **Functional identity :** $\mathbf{Log}(x) - \mathbf{Log}(y) - \mathbf{Log}\left(\frac{x}{y} \right) = 0$

Indoles logarithmorum hac aequatione fundamentali continetur **[Pfaff 1788]**

[The nature of logarithms is contained in this fundamental eq°]

The dilogarithm Li_2

- $\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2}$ $\qquad\qquad \left(|z| < 1 \right)$
- **Integral formulas :**
$$\mathrm{Li}_2(z) = \mathbf{L}_{01}(z) = -\int^z \log(1-u)\,\frac{du}{u-0}$$
$$\mathbf{L}_{10}(z) = \int^z \log(u-0)\,\frac{du}{1-u}$$
- **Monodromy :**
$$\mathcal{M}_1\left(\mathbf{Li}_2\right) = \mathbf{Li}_2 - 2i\pi\,\mathbf{Log}$$
- **Abel's functional identity** $(\mathcal{A}b)$ $\qquad\qquad \left(0 < \boldsymbol{x} < \boldsymbol{y} < 1 \right)$

$$\mathbf{Li}_2(\boldsymbol{x}) - \mathbf{Li}_2(\boldsymbol{y}) - \mathbf{Li}_2\left(\frac{\boldsymbol{x}}{\boldsymbol{y}}\right) - \mathbf{Li}_2\left(\frac{\mathbf{1}-\boldsymbol{y}}{\mathbf{1}-\boldsymbol{x}}\right) + \mathbf{Li}_2\left(\frac{\boldsymbol{x}(\mathbf{1}-\boldsymbol{y})}{\boldsymbol{y}(\mathbf{1}-\boldsymbol{x})}\right) =$$
$$\mathbf{Log}(y)\,\mathbf{Log}\left(\frac{1-y}{1-x}\right) - \frac{\pi^2}{6}$$

The dilogarithm $\mathbf{Li}_2$

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- **Integral formulas :** $\quad \mathbf{Li}_2(z) = \mathbf{L}_{01}(z) = -\int^z \log(1-u)\,\frac{du}{u-0}$

$$\mathbf{L}_{10}(z) = \int^z \log(u-0)\,\frac{du}{1-u}$$

- **Monodromy :** $\quad \mathcal{M}_1\,(\mathbf{Li}_2) = \mathbf{Li}_2 - 2i\pi\,\mathbf{Log}$

- **Abel's functional identity** $(\mathcal{A}\boldsymbol{b})$ $\qquad \left(\ 0 < \boldsymbol{x} < \boldsymbol{y} < 1\ \right)$

$$\mathbf{R}(\boldsymbol{x}) - \mathbf{R}(\boldsymbol{y}) - \mathbf{R}\Big(\frac{\boldsymbol{x}}{\boldsymbol{y}}\Big) - \mathbf{R}\Big(\frac{\mathbf{1}-\boldsymbol{y}}{\mathbf{1}-\boldsymbol{x}}\Big) + \mathbf{R}\Big(\frac{\boldsymbol{x}(\mathbf{1}-\boldsymbol{y})}{\boldsymbol{y}(\mathbf{1}-\boldsymbol{x})}\Big) = 0$$

$$\mathbf{R}(x) = \frac{1}{2}\Big(\mathbf{L}_{01}(x) - \mathbf{L}_{10}(x)\Big) = \mathbf{Li}_2(x) + \frac{1}{2}\mathbf{Log}(x)\mathbf{Log}(1-x) - \frac{\pi^2}{6}$$

The n-th polylogarithm $\mathbf{Li}_n$

- $\mathbf{Li}_n(z) \;=\; \sum_{k=1}^{\infty} \frac{z^k}{k^n}$ $\qquad\qquad \big(\ |z| < 1\ \big)$

- **Integral formula :**

$$\mathbf{Li}_n(z) \;=\; \int^z \mathbf{Li}_{n-1}(u)\frac{du}{u}$$

$$\mathbf{Li}_n{}'(z) \;=\; \mathbf{Li}_{n-1}(z)/z$$

- **Monodromy :** $\quad \mathcal{M}_1\left(\mathbf{Li}_n\right) \;=\; \mathbf{Li}_n - 2i\pi\,\frac{(\mathbf{Log})^{n-1}}{(n-1)!}$

- **Identités fonctionnelles en une variable :**

$$\mathbf{Li}_n\Big(z^r\Big) \;=\; r^{n-1}\sum_{\omega^r=1}\mathbf{Li}_n(\omega\, z) \qquad\qquad \Big(\ |z| < 1\ \Big)$$

$$\mathbf{Li}_n(z) + (-1)^n\mathbf{Li}_n\Big(z^{-1}\Big) \;=\; -\frac{(2i\pi)^n}{n!}\,\mathbf{B}_n\bigg(\frac{\mathrm{Log}\, z}{2i\pi}\bigg) \quad \Big(\ z\in\mathbb{C}\setminus[0,+\infty[\ \Big)$$

The n-th polylogarithm $\mathbf{Li}_n$

- $\mathbf{Li}_n(z) \;=\; \sum_{k=1}^{\infty} \frac{z^k}{k^n}$ $\qquad\qquad \Big(\ |z|< 1\ \Big)$

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- **Monodromy :**

$$\mathcal{M}_1\left(\mathbf{Li}_n\right) \;=\; \mathbf{Li}_n - 2i\pi\,\frac{(\mathbf{Log})^{n-1}}{(n-1)!}$$

- **Functional identities in several variables ($\exists$?) :**

$$\sum_{i\in I} c_i\,\mathbf{Li}_n\big(U_i\big) = \mathbf{Elem}_{<n}$$

$$\Big(\ I\ \text{ finite}\,,\ c_i\in\mathbb{Z}\,,\ U_i\in\mathbb{Q}(x_1,\ldots,x_N)\ \Big)$$

The n-th polylogarithm $\mathbf{L}\mathrm{i}_n$

- $\mathbf{L}\mathrm{i}_n(z) \;=\; \sum_{k=1}^{\infty} \frac{z^k}{k^n}$ $\qquad\qquad \big(\,|z|<1\,\big)$

- **Integral formula :**

$$\mathbf{L}\mathrm{i}_n(z) \;=\; \int^z \mathbf{L}\mathrm{i}_{n-1}(u)\frac{du}{u}$$

$$\mathbf{L}\mathrm{i}_n{}'(z) \;=\; \mathbf{L}\mathrm{i}_{n-1}(z)/z$$

- **Monodromy :** $\quad \boldsymbol{\mathcal{M}}_1\,(\,\mathbf{L}\mathrm{i}_n\,) \;=\; \mathbf{L}\mathrm{i}_n - 2i\pi\,\frac{(\mathbf{Log})^{n-1}}{(n-1)!}$

- **Functional identities in several variables ($\exists$?) :**

$$\sum_{i\in I} c_i\,\mathbf{L}\mathrm{i}_n\big(U_i\big) = \mathbf{Elem}_{<n} \qquad \Longleftrightarrow \qquad \sum_{i\in I} c_i\,\boldsymbol{\mathcal{L}}_n\big(U_i\big) = \mathbf{0}$$

$\Big(\ I\ \text{ finite}\,,\ c_i \in \mathbb{Z}\,,\ U_i \in \mathbb{Q}(x_1,\ldots,x_N)\ \Big)$

Example : $\mathbf{Li}_3$

- $\mathbf{Li}_3(z) \;=\; \sum_{k=1}^{\infty} z^k/k^3 \;=\; \int^z \mathbf{Li}_2(u)\frac{du}{u}$

- **Spence-Kummer identity $\mathcal{SK}$ (1809-1840) :**

$$2\,\mathbf{Li}_3(x) + 2\,\mathbf{Li}_3(y) - \mathbf{Li}_3\Big(\frac{x}{y}\Big) + 2\,\mathbf{Li}_3\Big(\frac{1-x}{1-y}\Big) + 2\,\mathbf{Li}_3\Big(\frac{x(1-y)}{y(1-x)}\Big) - \mathbf{Li}_3\big(xy\big)$$
$$+ 2\,\mathbf{Li}_3\Big(-\frac{x(1-y)}{(1-x)}\Big) + 2\,\mathbf{Li}_3\Big(-\frac{(1-y)}{y(1-x)}\Big) - \mathbf{Li}_3\Big(\frac{x(1-y)^2}{y(1-x)^2}\Big)$$
$$= 2\,\mathbf{Li}_3(1) - \mathrm{Log}\,(y)^2\mathrm{Log}\left(\frac{1-y}{1-x}\right) + \frac{\pi^2}{3}\mathrm{Log}\,(y) + \frac{1}{3}\mathrm{Log}\,(y)^3$$

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$$2\,\mathcal{L}_3(x) + 2\,\mathcal{L}_3(y) - \mathcal{L}_3\Big(\frac{x}{y}\Big) + 2\,\mathcal{L}_3\Big(\frac{1-x}{1-y}\Big) + 2\,\mathcal{L}_3\Big(\frac{x(1-y)}{y(1-x)}\Big) - \mathcal{L}_3\big(xy\big)$$
$$+ 2\,\mathcal{L}_3\Big(-\frac{x(1-y)}{(1-x)}\Big) + 2\,\mathcal{L}_3\Big(-\frac{(1-y)}{y(1-x)}\Big) - \mathcal{L}_3\Big(\frac{x(1-y)^2}{y(1-x)^2}\Big) = \mathbf{0}$$

$$\mathcal{L}_3(z) = \mathbf{Li}_3(z) - \mathbf{Li}_2(z)\,\mathrm{Log}|z| + \tfrac{1}{3}\,\mathbf{Li}_1(z)\big(\mathrm{Log}|z|\big)^2$$

Example : Li_4

- $\mathbf{Li}_4(x) \;=\; \sum_{k=1}^{\infty} x^k/k^4 \qquad \mathcal{L}_4(x) \;=\; \mathbf{Li}_4(x) + \mathbf{Elem}_{<4}(x)$

- **Kummer's functional identity $\mathcal{K}(4)$ (1840) :**

$$\begin{aligned}
&\mathcal{L}_4\Big(-\frac{x^2y\eta}{\zeta}\Big) + \mathcal{L}_4\Big(-\frac{y^2x\zeta}{\eta}\Big) + \mathcal{L}_4\Big(\frac{x^2y}{\eta^2\zeta}\Big) + \mathcal{L}_4\Big(\frac{y^2x}{\zeta^2\eta}\Big)\\
&-6\,\mathcal{L}_4\big(xy\big) - 6\,\mathcal{L}_4\Big(\frac{xy}{\eta\zeta}\Big) - 6\,\mathcal{L}_4\Big(-\frac{xy}{\eta}\Big) - 6\,\mathcal{L}_4\Big(-\frac{xy}{\zeta}\Big)\\
&-3\,\mathcal{L}_4\big(x\eta\big) - 3\,\mathcal{L}_4\big(y\zeta\big) - 3\,\mathcal{L}_4\Big(\frac{x}{\eta}\Big) - 3\,\mathcal{L}_4\Big(\frac{y}{\zeta}\Big)\\
&-3\,\mathcal{L}_4\Big(-\frac{x\eta}{\zeta}\Big) - 3\,\mathcal{L}_4\Big(-\frac{y\zeta}{\eta}\Big) - 3\,\mathcal{L}_4\Big(-\frac{x}{\eta\zeta}\Big) - 3\,\mathcal{L}_4\Big(-\frac{y}{\eta\zeta}\Big)\\
&+6\,\mathcal{L}_4\big(x\big) + 6\,\mathcal{L}_4(y) + 6\,\mathcal{L}_4\Big(-\frac{x}{\zeta}\Big) + 6\,\mathcal{L}_4\Big(-\frac{y}{\eta}\Big) = \mathbf{0}
\end{aligned}$$

$$(\ \zeta = 1 - x,\ \eta = 1 - y\ )$$

- **Abel 1881 (Spence 1809, Hill 1829, Rogers 1907)**

$$\mathbf{R}(x) - \mathbf{R}(y) - \mathbf{R}\left(\frac{x}{y}\right) - \mathbf{R}\left(\frac{1-y}{1-x}\right) + \mathbf{R}\left(\frac{x(1-y)}{y(1-x)}\right) = 0 \qquad (\mathcal{A}b)$$

- **Spence-Kummer :** $\sum_{i=1}^{9} c_i\, \mathcal{L}_3\Big(U_i(x,y)\Big) = 0 \qquad (\mathcal{SK})$

- **Kummer 1840 :** $\sum_i c_i\, \mathcal{L}_n\Big(U_i(x,y)\Big) = 0 \qquad (\ n \leq 5\ ) \qquad (\mathcal{K}_n)$

- $\ldots$

- **Goncharov 1995 :** $\sum_{i=1}^{22} c_i\, \mathcal{L}_3\Big(U_i(a,b,c)\Big) = 0 \qquad (\mathcal{G}on)$

- **Gangl 2003 :** $\sum_i c_i\, \mathcal{L}_n\Big(U_i(x,y)\Big) = 0 \qquad (\ n = 6,7\ ) \qquad (\mathcal{G}an_n)$

- **Charlton, Gangl, Radchenko, Rudenko, Goncharov-Rudenko, ...**

- **Functional identities (FI) of polylogarithms Li_n :**
  - Hyperbolic geometry
  - Web geometry $\quad (n \leq 3)$
  - K-theory of number fields $\quad (n \leq 4)$
  - Theory of periods (MZVs)
  - Particle physics ('*Scattering amplitudes*')
  - Mathematical physics ('*Y-systems*') $\quad (n = 2)$
  - Cluster algebras $\quad (n \leq 4)$
  - Mirror symmetry ('*Scattering diagrams*') $\quad (n = 2)$

- **Main problems :** – find **FI**s for $\mathcal{L}_n$ $\big(\textit{e.g.}\ \exists\ n \geq 8\ ?\ \big)$

  – better understand the polylogarithmic **FI**s

- **Functional identities (FI) of polylogarithms Li_n :**

  - Hyperbolic geometry
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- **Main problems :** – find **FI**s for $\mathcal{L}_n$ (*e.g.* $\exists\, n \geq 8$?)

  – better understanding of polylogarithmic **FI**s

K-theory and polylogarithmic identities

- $\boldsymbol{F}$ = number field $\rightsquigarrow$ $K_n(\boldsymbol{F}) = \begin{cases} n \text{ even} & \checkmark \text{ (Borel)} \\ n \text{ odd} & ? \end{cases}$

- **[Zagier]** $K_3(\boldsymbol{F}) \leftrightsquigarrow B_2(\boldsymbol{F})$ with

$$B_2(\boldsymbol{F}) = \frac{\mathbb{Z}\Big[\, \boldsymbol{F}\backslash\{0,1\} \,\Big]}{\Big\langle\, [x]-[y]-\big[\tfrac{x}{y}\big]-\big[\tfrac{1-y}{1-x}\big]+\big[\tfrac{x(1-y)}{y(1-x)}\big] \;\Big|\; x,y\in\boldsymbol{F}\backslash\{0,1\}\,,\, x\neq y \,\Big\rangle}$$

$$\left(\mathbf{R}(x) - \mathbf{R}(y) - \mathbf{R}\Big(\frac{x}{y}\Big) - \mathbf{R}\Big(\frac{1-y}{1-x}\Big) + \mathbf{R}\Big(\frac{x(1-y)}{y(1-x)}\Big) = 0 \right)$$

- **(Borel's) regulator :**
$$\mathcal{R}_2^B = \boldsymbol{\mathcal{L}}_2 : K_3(\boldsymbol{F}) \longrightarrow \mathbb{R}$$
$$\mathcal{R}_n^B = \boldsymbol{\mathcal{L}}_n : K_{2n-1}(\boldsymbol{F}) \longrightarrow \mathbb{R} \quad (\; n \geq 2 \;)$$

- Understanding FI's of $\boldsymbol{\mathcal{L}}_n$ $\rightarrow$
  - Desc° of $K_{2n-1}(\boldsymbol{F})$ = generators / relat°
  - Applications to "Zagier's Conjecture"

'Scattering amplitudes' and functional identities

- *'Scattering amplitudes'* $\boldsymbol{I} = \int_\Delta \boldsymbol{\Psi}$ (important in HEPP)

$$\boldsymbol{I} = \boldsymbol{I}' + \mathcal{R}$$

$$\boldsymbol{I} = \boldsymbol{I}' + \sum_{i\in I} \mathbf{F}_i(x_i)$$

- **[dDDS]** *'The 2-loop hexagon Wilson loop in $\mathcal{N}=4$ SYM'* (2010)

$\mathcal{R}^{(2)}_{6,WL}$ = 'remainder' given by a 17 pages formula !

[GSVV] $\mathcal{R}^{(2)}_{6,WL} = \sum_{i=1}^{3}\left(L_4(x_i^+, x_i^-) - \frac{1}{2}\mathbf{L}\mathrm{i}_4(v_i)\right) - \frac{1}{8}\left(\sum_{i=1}^{3}\mathbf{L}\mathrm{i}_2(v_i)\right)^2 + \cdots$

- Importance of simplifying $\sum_{i\in I} \mathbf{F}_i(x_i)$ when $\mathbf{F}_i$ = polylogarithms
  $\mathbf{F}_i$ = hyperlogarithms
  $\mathbf{F}_i$ = elliptic polylogs

- Justifies the study of functional indentities $\sum_{j\in J} \mathbf{F}_j(x_j) = \mathrm{cst}$

- $\boldsymbol{Ab}(x,y) = \mathbf{R}(x) - \mathbf{R}(y) - \mathbf{R}\left(\frac{x}{y}\right) - \mathbf{R}\left(\frac{1-y}{1-x}\right) + \mathbf{R}\left(\frac{x(1-y)}{y(1-x)}\right) \equiv 0$

Thm [de Jeu 20] For I finite, $c_i \in \mathbb{Q}$ and $U_i \in \mathbb{Q}[x_1, \ldots, x_m]$:

$\sum_{i\in I} c_i\, \mathbf{R}(U_i) \equiv \mathbf{cst}$ $\Longleftrightarrow$ $\sum_{i\in I} c_i\, \mathbf{R}(U_i)$ is a LC of specialisations of $\boldsymbol{Ab}(X_s, Y_s)$'s with $X_s, Y_s \in \mathbb{Q}[x_1, \ldots, x_m]\ \forall\ s$

- $\mathbf{Log}(x) - \mathbf{Log}(y) - \mathbf{Log}(x/y) = 0$ is the FFI of the log ✓
- $\boldsymbol{\mathcal{A}b}$ $\Longleftrightarrow$ $\big(\boldsymbol{Ab}(x,y) \equiv 0\big)$ is the FFI of the dilog ✓
- $\boldsymbol{\mathcal{G}on}_{22}$ $\Longleftrightarrow$ $\sum_{i=1}^{22} c_i \mathcal{L}_3(U_i) = 0$ is the FFI of the trilog **?**
- $\mathbf{Q}_4$ **[Goncharov-Rudenko]** is the FFI of the tetralog **?**

[Hain MacPherson 1990] *Higher logarithms*

The dilogarithm has properties analogous to those of the logarithm. It has been widely believed, both in the nineteenth century and more recently, that these two functions should be the first two elements of an infinite sequence of higher logarithms which share analogous properties. To date, several sequences of such functions have been proposed, but no function beyond the dilogarithm in any of these sequences is known to possess all the desired properties.

[Griffiths 2002] *The legacy of Abel in algebraic geometry*

...intuitively, we are asking ***whether or not for each n there is an integer*** $\boldsymbol{d(n)}$ ***such that there is a "new"*** $\boldsymbol{d(n)}$***-web of maximum rank one of whose abelian relations is a (the ?) functional equation with*** $\boldsymbol{d(n)}$ ***terms for*** $\mathbf{Li}_n$ ***?*** *Here, 'new" means the general extension of the phenomena above for the logarithm when* $n = 1$*, where* $d(1) = 3$*, for the 5-term identity when* $n = 2$ *and* $d(2) = 5, \ldots$

[Goncharov-Rudenko 2018] *Motivic correlator, cluster algebras ...*

__Conclusion__. *If* $\boldsymbol{n > 3}$, ***the problem of writing explicitly functional equations for*** $\mathbf{Li}_{\boldsymbol{n}}$ ***might not be the "right" problem****. It seems that when n is growing the functional equations become so complicated that one can not write them down on a piece of paper.*

- **Main problems about polylogarithms :**

  – Finding **FI**'s for $\mathcal{L}_n$ $\big($ *e.g.* $\exists\, n \geq 8$? $\big)$

  – Does it exist a sequence $(\boldsymbol{FI}_n)_{n\geq 1}$ of **FI**'s for the polylogs ?

  – Does it exist a (fundamental) **FFI** for $\mathcal{L}_n$ for each $n \geq 1$?

  – Better understand polylogarithmic **FI**'s

- In this talk, considering **hyperlogarithms** $\Big(\supset$ polylogarithms $\Big)$:
  - we describe a series of new hyperlogarithmic identities

$$\mathbf{HLog}^1 \iff \Big(\ \mathbf{Log}(x) - \mathbf{Log}(y) - \mathbf{Log}(x/y) = 0 \ \Big)$$

$$\mathbf{HLog}^2 \iff \Big(\ \mathbf{R}(x) - \mathbf{R}(y) - \mathbf{R}\Big(\frac{x}{y}\Big) - \mathbf{R}\Big(\frac{1-y}{1-x}\Big) + \mathbf{R}\Big(\frac{x(1-y)}{y(1-x)}\Big) = 0 \ \Big)$$

$$\vdots$$

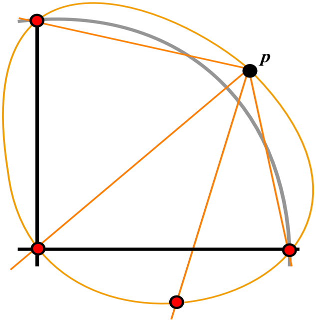
$$\mathbf{HLog}^6 \qquad\qquad \Big(\text{ hyperlogarithmic FI of weight 6 } \Big)$$

  - For $w = 1, \ldots, 6$, one has

$$\mathbf{HLog}^w \quad : \quad \sum_{i=1}^{\kappa} \boldsymbol{AH}_i^w(\boldsymbol{\phi_i}) = 0$$

A geometric view on Abel's identity

- $(\mathcal{A}b)$ $\quad \underset{U_1}{\underset{\|}{\mathrm{R}(x)}} - \underset{U_2}{\underset{\|}{\mathrm{R}(y)}} - \underset{U_3}{\underset{\|}{\mathrm{R}\left(\frac{x}{y}\right)}} - \underset{U_4}{\underset{\|}{\mathrm{R}\left(\frac{1-y}{1-x}\right)}} + \underset{U_5}{\underset{\|}{\mathrm{R}\left(\frac{x(1-y)}{y(1-x)}\right)}} = 0$



Base points of the U_i's :

- $p_1 = [1, 0, 0]$
- $p_2 = [0, 1, 0]$
- $p_3 = [0, 0, 1]$
- $p_4 = [1, 1, 1]$

$\rightsquigarrow$ **Blow-up** $\quad \beta : X_4 = \mathrm{Bl}_{p_1,\ldots,p_4}(\mathbb{P}^2) \longrightarrow \mathbb{P}^2$

A geometric view on Abel's identity

- $(\mathcal{A}b)$ $$\mathbf{R}(\underset{\substack{\| \\ U_1}}{x}) - \mathbf{R}(\underset{\substack{\| \\ U_2}}{y}) - \mathbf{R}\Big(\underset{\substack{\| \\ U_3}}{\frac{x}{y}}\Big) - \mathbf{R}\Big(\underset{\substack{\| \\ U_4}}{\frac{1-y}{1-x}}\Big) + \mathbf{R}\Big(\underset{\substack{\| \\ U_5}}{\frac{x(1-y)}{y(1-x)}}\Big) = 0$$

- **Blow-up** $\beta : X_4 = \mathrm{Bl}_{p_1,\ldots,p_4}(\mathbb{P}^2) \longrightarrow \mathbb{P}^2$

$$\begin{tikzcd}
\mathbf{X}_4 \arrow[d, "\beta"'] \arrow[rr, bend left, "\phi_i"] & & \\
\mathbb{P}^2 \arrow[rr, dashed, "U_i"'] & & \mathbb{P}^1
\end{tikzcd}$$

The $\phi_1, \ldots, \phi_5 : X_4 \longrightarrow \mathbb{P}^1$ are the five fibrations by conics on the quintic del Pezzo surface X_4

- $(\mathcal{A}b) \Longleftrightarrow$ $\exists\,(\epsilon_i)_{i=1}^{5} \in \{\pm 1\}^5$ ($\exists\,!$ up to sign) s. t. $\sum_{i=1}^{5} \epsilon_i \,\mathbf{R}(\phi_i) = 0$

Generalisation to del Pezzo surfaces

- $p_1, \ldots, p_r \in \mathbb{P}^2$: points in general position $\quad (\, r \in \{3, \ldots, 8\} \,)$
- **Blow-up** $\beta_r : X_r = \mathbf{Bl}_{p_1,\ldots,p_r}(\mathbb{P}^2) \longrightarrow \mathbb{P}^2 \quad (\, X_r = \mathbf{dP}_{9-r} \,)$

Prop : 1. *There is a finite number κ_r of conic fibrations* $\quad \phi_1, \ldots, \phi_\kappa : X_r \longrightarrow \mathbb{P}^1$

2. *For any i :* $\boldsymbol{\Sigma_i} = \mathbf{Spectrum}(\phi_i) \subset \mathbb{P}^1$ *has $r-1$ elements*

Def° : *The* **complete antisymmetric hypergarithm of weight $r-2$** : $\quad AH^{r-2}_{\Sigma_i} : \widetilde{\mathbb{P}^1 \setminus \Sigma_i} \longrightarrow \mathbb{C}$

Thm [Castravet-P.] $\exists \, (\epsilon_i)_{i=1}^{\kappa} \in \{ \pm 1 \}^{\kappa}$, $\pm$-unique, such that

$$\left(\mathbf{HLog}^{r-2}\right) \qquad \sum_{i=1}^{\kappa} \epsilon_i \, AH^{r-2}_{\Sigma_i}(\phi_i) = 0$$

$$\boxed{\left(\mathbf{HLog}^{r-2}\right) \qquad \sum_{i=1}^{\kappa} \epsilon_i\, \boldsymbol{AH}_{\boldsymbol{\Sigma_i}}^{r-2}(\phi_{\boldsymbol{i}}) = 0}$$

- One identity $\mathbf{HLog}^{r-2}$ for each del Pezzo $\mathbf{dP}_d = \boldsymbol{X}_r$ $(d = 9 - r)$

[$d = 6$] $\mathbf{dP}_6$ is unique, $\boldsymbol{AH}_{\boldsymbol{\Sigma_i}}^{1} = \mathbf{Log}$ for all i

$$\mathbf{HLog}^1 = \Big(\mathbf{Log}(\,\boldsymbol{x}\,) - \mathbf{Log}(\,\boldsymbol{y}\,) - \mathbf{Log}(\,\boldsymbol{x}/\boldsymbol{y}\,) = \mathbf{0}\Big)$$

[$d = 5$] $\mathbf{dP}_5$ is unique : $\boldsymbol{AH}_{\boldsymbol{\Sigma_i}}^{2} = \frac{1}{2}\Big(\mathbf{L}_{01} - \mathbf{L}_{10}\Big) = \mathbf{R}$ for all i

$$\mathbf{HLog}^2 = \Big(\ \sum_{\boldsymbol{i=1}}^{\mathbf{5}} \boldsymbol{\epsilon_i}\, \mathbf{R}(\,\phi_{\boldsymbol{i}}\,) = \mathbf{0}\ \Big) \qquad (\mathcal{A}\mathbf{b})$$

$$\left(\mathbf{HLog}^{r-2}\right) \qquad \sum_{i=1}^{\kappa} \epsilon_i \, \boldsymbol{AH}^{r-2}_{\boldsymbol{\Sigma_i}}(\phi_i) = 0$$

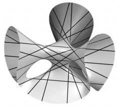
[$d = 4$] $\mathbf{dP}_4$ ∞^2 moduli $\rightsquigarrow$ ∞^2 identities $\mathbf{HLog}^3$

$$\boldsymbol{AH}^3_1(x) + \boldsymbol{AH}^3_2\left(\frac{1}{y}\right) + \boldsymbol{AH}^3_3\left(\frac{y}{x}\right) + \cdots$$

$$\cdots + \boldsymbol{AH}^3_9\left(\frac{y(x-b)}{x(y-a)}\right) + \boldsymbol{AH}^3_{10}\left(\frac{a\,(b-x)}{by-ax}\right) = 0$$

[$d = 3$] $\mathbf{dP}_3$ = cubic surface in $\mathbb{P}^3$ $\rightsquigarrow$ ∞^4 identities $\mathbf{HLog}^4$

$$\sum_{i=1}^{27} \boldsymbol{AH}^4_i(\phi_i) = 0$$



Thm [Castravet-P. 2022]

$\exists\ (\epsilon_i)_{i=1}^{\kappa} \in \{\pm 1\}^{\kappa}$ unique up to sign, such that

$$\left(\mathbf{HLog}^{r-2}\right) \qquad \sum_{i=1}^{\kappa} \epsilon_i \, \boldsymbol{AH}_{\boldsymbol{\Sigma_i}}^{r-2}(\boldsymbol{\phi_i}) = 0$$

$\longrightarrow$ Del Pezzo surfaces

$\longrightarrow$ Hyperlogarithms (aka 'Iterated integrals on $\mathbb{P}^1$')

Del Pezzo surfaces I : properties

- $\mathbf{dP}_d \subset \mathbb{P}^d$ smooth surface, of degree d $\qquad (\, d = 9 - r \,)$
- $\mathbf{dP}_d = \mathbf{X}_r = \mathbf{Bl}_{p_1,\ldots,p_r}(\mathbb{P}^2)$ $\qquad \mathbf{Pic}(\mathbf{dP}_d) = \mathbb{Z}\,\boldsymbol{h} \oplus \big(\bigoplus_{i=1}^r \mathbb{Z}\,\boldsymbol{\ell}_i \big)$
- $-\boldsymbol{K}_{\mathbf{dP}_d} = 3\boldsymbol{h} - \sum_{i=1}^r \boldsymbol{\ell}_i$ ample $\rightsquigarrow$ $\varphi_{|-\boldsymbol{K}|} : \mathbf{dP}_d \hookrightarrow \mathbb{P}^d$ embedding
- $\mathbf{Pic}(\mathbf{dP}_d) \supset \boldsymbol{K}^\perp = \langle \, \boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_r \, \rangle$ $\qquad \boldsymbol{\rho}_i = \boldsymbol{\ell}_i - \boldsymbol{\ell}_{i+1} \quad i \leq r-1$

  $\boldsymbol{\rho}_r = 3\boldsymbol{h} - \sum_{i=1}^3 \boldsymbol{\ell}_i$
- $-(\cdot,\cdot) + \{ \, \boldsymbol{\rho}_i \, \}_{i=1}^r$ $\rightsquigarrow$ Root system $E_r \subset R_r = \boldsymbol{K}^\perp \otimes \mathbb{R}$
- For any root $\boldsymbol{\rho}$: $\qquad s_{\boldsymbol{\rho}} : R_r \longrightarrow R_r$ $\qquad$ (orthog. reflection)

  $$d \longmapsto d + (d, \boldsymbol{\rho})\boldsymbol{\rho}$$
- $\boldsymbol{W}_r = \boldsymbol{W}(E_r) = \langle \, s_{\boldsymbol{\rho}_1}, \ldots, s_{\boldsymbol{\rho}_r} \, \rangle \subset \mathrm{O}(R_r)$: Weyl group of type E_r

Del Pezzo Surfaces I

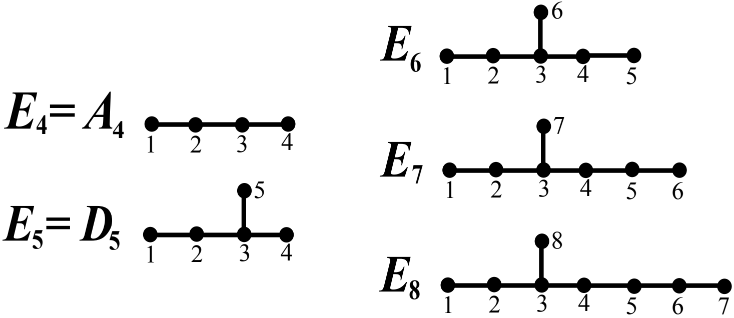


Figure – Dynkin diagram E_r $\big(k$ stands for ρ_k for any $k = 1, \ldots, r\big)$

Lines and conics on $X_r = \mathbf{dP}_d \qquad \left(d = 9 - r \right)$

- Lines $\mathcal{L}_r = \left\{ \ell \in \mathbf{Pic}(X_r) \;\middle|\; (\ell, -K) = 1 \,,\ \ell^2 = -1 \right\}$

$$\begin{array}{l} \mathcal{L}_r \\ \Cup \\ \delta \end{array} \quad \rightsquigarrow \quad |\,\delta\,| = \{\,\delta\,\} \quad \text{with} \quad \mathbb{P}^1 \simeq \delta \subset \mathbf{dP}_d \qquad \deg(\delta) = 1$$

$$\mathcal{L}_r = W_r \cdot \ell_r$$

- Clonics $\mathcal{K}_r = \left\{ \mathfrak{c} \in \mathbf{Pic}(X_r) \;\middle|\; (\mathfrak{c}, -K) = 2 \,,\ \mathfrak{c}^2 = 0 \right\}$

$$\begin{array}{l} \mathcal{K}_r \\ \Cup \\ \mathfrak{c} \end{array} \quad \longrightarrow |\,\mathfrak{c}\,| \simeq \mathbb{P}^1 \rightsquigarrow \quad \text{Conic fibration } \phi_{\mathfrak{c}} : X_r \to \mathbb{P}^1$$

$$\mathcal{K}_r = W_r \cdot (h - \ell_1)$$

r	3	4	5	6	7	8
E_r	$A_2 \times A_1$	A_4	D_5	E_6	E_7	E_8
$W_r = W(E_r)$	$\mathfrak{S}_3 \times \mathfrak{S}_2$	$\mathfrak{S}_5$	$(\mathbf{Z}/2\mathbf{Z})^4 \ltimes \mathfrak{S}_5$	$W(E_6)$	$W(E_7)$	$W(E_8)$
$\omega_r = \lvert W_r \rvert$	12	$5!$	$2^4 \cdot 5!$	$2^7 \cdot 3^4 \cdot 5$	$2^{10} \cdot 3^4 \cdot 5 \cdot 7$	$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$
$l_r = \lvert \mathcal{L}_r \rvert$	6	10	16	27	56	240
$\kappa_r = \lvert \mathcal{K}_r \rvert$	3	5	10	27	126	2160

Example : the lines of dP_2 seen in $\mathbb{P}^2$

- $\mathbf{dP}_2 = \boldsymbol{X}_7 = \mathbf{Bl}_{p_1,\ldots,p_7}(\mathbb{P}^2) \xrightarrow{\beta} \mathbb{P}^2$
- $\ell_i = \beta^{-1}(p_i) \subset \boldsymbol{X}_7 \quad \rightsquigarrow \quad \boldsymbol{\ell}_i \in \mathbf{Pic}(\boldsymbol{X}_7)$

  $\rightsquigarrow \quad \boldsymbol{\ell} = \sum_{i=1}^{7} \boldsymbol{\ell}_i$

Line	Class in $\mathrm{Pic}(X_7)$	Number of such lines	Model in $\mathbf{P}^2$
ℓ_i	$\boldsymbol{\ell}_i$	7	first infinitesimal neighbourhood $p_i^{(1)}$
ℓ_{ij}	$h - \boldsymbol{\ell}_i - \boldsymbol{\ell}_j$	21	line joining p_i to p_j
C_{ij}	$2h - \boldsymbol{\ell} + \boldsymbol{\ell}_i + \boldsymbol{\ell}_j$	21	conic through the p_k's, $k \notin \{i, j\}$
C_i^3	$3h - \boldsymbol{\ell} - \boldsymbol{\ell}_i$	7	cubic through all the p_l's with a node at p_i

TABLE 2. Lines on dP_2 and the corresponding 'curves' in the projective plane

Example : the clonics of dP_2 seen in $\mathbb{P}^2$

Conic class $\mathfrak{c}$	Number of such $\mathfrak{c}$	Linear system $\lvert\mathfrak{C}_{\mathfrak{c}}\rvert$	$\mathfrak{C}_{\mathfrak{c}}^{\mathrm{red}}$
$h - \ell_i$	7	lines through p_i	$\ell_{ij} + \ell_j$
$2h - \sum_{i \in I} \ell_i$	35	conics through the p_i's, $i \in I$	$\ell_{i_1 i_2} + \ell_{i_3 i_4}$ $\ell_{i_3} + C_{i_1 i_2}$
$3h - \boldsymbol{\ell} + \ell_i - \ell_j$	42	cubics through the p_k's for $k \neq i$, with a node at p_j	$\ell_{jk} + C_{ik}$ $\ell_i + C_j^3$
$4h - \boldsymbol{\ell} - \sum_{j \in J} \ell_j$	35	quartics through the p_k's with a node at p_j for $j \in J$	$C_{k_1 k_2} + C_{k_3 k_4}$ $\ell_{j_1 j_2} + C_{j_3}^3$
$5h - 2\boldsymbol{\ell} + \ell_i$	7	quintics through the p_k's with a node at p_k except for $k = i$	$C_{ij} + C_j^3$

TABLE 3. Conic classes on dP_2 and their reducible fibers

Non irreducible conics on $X_r = \mathrm{dP}_d$

- $\boldsymbol{L}_r = \cup_{\ell \in \boldsymbol{\mathcal{L}}_r} \ell \subset \boldsymbol{X}_r \quad \rightsquigarrow \quad \boldsymbol{U}_r = \boldsymbol{X}_r \setminus \boldsymbol{L}_r$

- $\mathcal{K}_r \ni \mathfrak{c} \quad \rightsquigarrow$ Fibration by conics $\phi_{\mathfrak{c}} : \boldsymbol{X}_r \to \mathbb{P}^1$

$$\Sigma_{\mathfrak{c}} = \textbf{Spectrum}(\phi_{\mathfrak{c}}) = \left\{ \sigma \in \mathbb{P}^1 \;\middle|\; \phi_{\mathfrak{c}}^{-1}(\sigma) \text{ not irreducible } \right\}$$
$$= \left\{ \sigma_{\mathfrak{c}}^1 , \ldots , \sigma_{\mathfrak{c}}^{r-2} , \sigma_{\mathfrak{c}}^{r-1} = \infty \right\} \subset \mathbb{P}^1$$

- For $\sigma_{\mathfrak{c}}^i \in \Sigma_{\mathfrak{c}} \;:\; \phi_{\mathfrak{c}}^{-1}(\sigma_{\mathfrak{c}}^i) = L_{\mathfrak{c}}^i + \widetilde{L}_{\mathfrak{c}}^i \qquad \left(L_{\mathfrak{c}}^i , \widetilde{L}_{\mathfrak{c}}^i \in \boldsymbol{\mathcal{L}}_r \right)$

- $\mathcal{H}_{\mathfrak{c}} = \mathbf{H}^0\Big(\mathbb{P}^1 , \Omega^1_{\mathbb{P}^1}(\mathrm{Log}\, \Sigma_{\mathfrak{c}})\Big) = \left\langle \dfrac{dz}{z-\sigma_{\mathfrak{c}}^i} \right\rangle_{i=1}^{r-2} \simeq \mathbb{C}^{r-2}$

$\quad \wr|$

- $\mathbf{H}_{\mathfrak{c}} = \phi_{\mathfrak{c}}^*\big(\mathcal{H}_{\mathfrak{c}}\big) = \left\langle \dfrac{d\phi_{\mathfrak{c}}}{\phi_{\mathfrak{c}} - \sigma_{\mathfrak{c}}^i} \right\rangle_{i=1}^{r-2} \subset \mathbf{H}^0\Big(\boldsymbol{X}_r , \Omega^1_{\boldsymbol{X}_r}(\mathrm{Log}\, \boldsymbol{L}_r)\Big) = \mathbf{H}_{\boldsymbol{X}_r}$

Del Pezzo's webs $\mathcal{W}_{\mathbf{dP}_d}$

- $\mathcal{W}_{\mathbf{dP}_d} = \mathcal{W}\big(\phi_{\mathfrak{c}}\big)_{\mathfrak{c}\in\mathcal{K}_r}$: κ_r-web by conics on $\mathbf{dP}_d$

- **Quest$^{\mathbf{o}}$** : $\exists\ \Big(F_{\mathfrak{c}}(\phi_{\mathfrak{c}})\Big)_{\mathfrak{c}\in\mathcal{K}_r}$ such that $\sum_{\mathfrak{c}\in\mathcal{K}_r} F_{\mathfrak{c}}\big(\phi_{\mathfrak{c}}\big) = 0$

  for some polylogarithms $F_{\mathfrak{c}}$ **?**

Theorem : $\exists\ \big(\epsilon_{\mathfrak{c}}\big)_{\mathfrak{c}\in\mathcal{K}_r} \in \{1\,,\,-1\}^{\mathcal{K}_r}$ $\pm$-unique such that

$$\Big(\mathbf{HLog}^{r-2}\Big) \qquad\qquad \sum_{\mathfrak{c}\in\mathcal{K}_r} \epsilon_{\mathfrak{c}}\, AH_{\mathfrak{c}}^{r-2}\big(\,\phi_{\mathfrak{c}}\,\big) = 0$$

where $\forall\ \mathfrak{c}$: $AH_{\mathfrak{c}}^{r-2}$ = complete antisymmetric hyperlogarithm of weight $r-2$ on $\mathbb{P}^1 \setminus \Sigma_{\mathfrak{c}}$

Iterated integrals

- **Poincaré (1884), Lappo-Danilevski (1928), Chen (1973)**

- $\boldsymbol{Y}$ complex manifold

- $\mathbf{H} = \left\langle \omega_1 \,,\, \ldots \,,\, \omega_m \right\rangle \subset \mathbf{H}^0\Big(\, \boldsymbol{Y}, \Omega^1_{\boldsymbol{Y}} \Big) \;+\; \left[\begin{array}{r} d\omega_i = 0 \\ \omega_i \wedge \omega_j = 0 \end{array} \right]$

- **Ex :** $\phi : \boldsymbol{Y} \to \boldsymbol{C}$ and $\omega_i \in \phi^*\Big(\mathbf{H}^0\big(\, \boldsymbol{C}, \Omega^1_{\boldsymbol{C}}\,\big)\Big)$ $\quad i = 1, \ldots, m$

- Base point $y \in \boldsymbol{Y}$, path $\gamma^x : [0,1] \to \boldsymbol{Y}$ from y to x :
  - $\mathbb{I}_{\omega_i} \;:\; x \longmapsto \int_{\gamma^x} \omega_i \quad \rightsquigarrow \quad \mathbb{I}_{\omega_i} \in \mathcal{O}_y$
  - $\mathbb{I}_{\omega_j\omega_i} \;:\; x \longmapsto \int_{\gamma^x} \omega_j(u) \cdot \mathbb{I}_{\omega_i}(u) \quad \rightsquigarrow \quad \mathbb{I}_{\omega_j\omega_i} \in \mathcal{O}_y$
  - $\mathbb{I}_{\omega_k\omega_j\omega_i} \;:\; x \longmapsto \int_{\gamma^x} \omega_k(u) \cdot \mathbb{I}_{\omega_j\omega_i}(u) \quad \rightsquigarrow \quad \mathbb{I}_{\omega_k\omega_j\omega_i} \in \mathcal{O}_y$

Iterated integrals (polylogarithms)

$$\mathbf{II}^w \,:\, \mathbf{H}^{\otimes w} \longrightarrow \mathcal{O}_y$$

- $\underline{\omega} = \omega_{i_1} \otimes \cdots \otimes \omega_{i_w} \longmapsto \mathbb{I}_{\underline{\omega}} : z \mapsto \int_{\gamma^z} \omega_{i_1}(u) \cdot \mathbb{I}_{\omega_{i_2} \cdots \omega_{i_w}}(u)$

- $\mathbf{II} : \Big(\bigoplus_{w \geq 0} \mathbf{H}^{\otimes w} \,,\, ш \Big) \longrightarrow \mathcal{O}_y$ injective morphism of $\mathbb{C}$-algebras

- $\forall\, \underline{\omega}$: $\mathbb{I}_{\underline{\omega}} \in \mathcal{O}_y \cap \widetilde{\mathcal{O}}(\boldsymbol{Y})$ unipotent monod. $\longrightarrow$ Symbol $\boldsymbol{\mathcal{S}}\Big(\widetilde{\mathbb{I}}_{\underline{\omega}}\Big) = \underline{\omega}$ ✓

- **Ex :** $\boldsymbol{Y} = \mathbb{P}^1 \setminus \Sigma$ with $\Sigma = \{\, 0, 1, \infty \,\}$

$$\mathbf{H} = \Big\langle \tfrac{dz}{z} \,,\, \tfrac{dz}{1-z} \Big\rangle = \mathbf{H}^0\Big(\mathbb{P}^1 \,,\, \Omega^1_{\mathbb{P}^1}(\operatorname{Log} \Sigma) \Big)$$

$$\mathbf{L}\mathrm{i}_n = \mathbf{II}^n\Big(\big(\tfrac{dz}{z}\big)^{\otimes (n-1)} \otimes \big(\tfrac{dz}{1-z}\big) \Big)$$

(**'Polylogarithms'**)

Iterated integrals (hyperlogarithms)

- **Ex :** $\boldsymbol{Y} = \mathbb{P}^1 \setminus \Sigma$ with $\Sigma = \left\{ \sigma^1 , \ldots , \sigma^{r-2} , \sigma^{r-1} = \infty \right\}$

$$\mathbf{H} = \left\langle \tfrac{dz}{z-\sigma^1}, \ldots, \tfrac{dz}{z-\sigma^{r-2}} \right\rangle = \mathbf{H}^0\Big(\mathbb{P}^1 , \Omega^1_{\mathbb{P}^1}(\operatorname{Log}\Sigma) \Big)$$

$$\mathbf{II}^n\Big(\big(\tfrac{dz}{z-\sigma^{i_1}}\big) \otimes \cdots \otimes \big(\tfrac{dz}{z-\sigma^{i_n}}\big) \Big)$$

"Hyperlogarithm"

- **Complete antisymmetric hyperlog of weight $r-2$ on $\mathbb{P}^1 \setminus \boldsymbol{\Sigma}$:**

$$\boldsymbol{AH}^{r-2}_{\Sigma} = \mathbf{II}^n\left(\mathbf{Asym}\left(\left(\frac{dz}{z-\sigma^1}\right) \otimes \cdots \otimes \left(\frac{dz}{z-\sigma^{r-2}}\right)\right) \right)$$

$$= \mathbf{II}^n\left(\tfrac{1}{(r-2)!} \textstyle\sum_{\nu\in\mathfrak{S}_{r-2}} (-1)^\nu \left(\tfrac{dz}{z-\sigma^{\nu(1)}}\right) \otimes \cdots \otimes \left(\tfrac{dz}{z-\sigma^{\nu(r-2)}}\right) \right)$$

- **Ex :** $\boldsymbol{AH}^2_{\{0,1,\infty\}} = \frac{1}{2}\,\mathbf{II}^2\left(\frac{dz}{z} \otimes \frac{dz}{(1-z)} - \frac{dz}{(1-z)} \otimes \frac{dz}{z} \right) = \mathbf{R}$

III Identity $\mathbf{HLog}^{r-2}$: proof(s)

$$\boxed{\Big(\mathbf{HLog}^{r-2}\Big) \;:\; \sum_{\mathfrak{c}\in\mathcal{K}} \epsilon_{\mathfrak{c}}\, \boldsymbol{AH}_{\mathfrak{c}}^{r-2}(\phi_{\mathfrak{c}}) = 0 \quad \text{with} \quad \boldsymbol{AH}_{\mathfrak{c}}^{r-2} = \boldsymbol{AH}_{\Sigma_{\mathfrak{c}}}^{r-2}}$$

- $\phi_{\mathfrak{c}} : \boldsymbol{X}_r \to \mathbb{P}^1 \supset \Sigma_{\mathfrak{c}} = \{\, \sigma_{\mathfrak{c}}^i \,\}_{i=1}^{r-1} \qquad \mathcal{H}_{\mathfrak{c}} = \mathbf{H}^0\Big(\Omega^1_{\mathbb{P}^1}\big(\operatorname{Log}\Sigma_{\mathfrak{c}}\big)\Big)$

- $\phi_{\mathfrak{c}}^*\big(\mathcal{H}_{\mathfrak{c}}\big) = \mathbf{H}_{\mathfrak{c}} \subset \mathbf{H}_{\boldsymbol{X}_r} = \mathbf{H}^0\Big(\Omega^1_{\boldsymbol{X}_r}\big(\operatorname{Log}\boldsymbol{L}_r\big)\Big)$

- $\boldsymbol{AH}_{\mathfrak{c}}^{r-2}(\phi_{\mathfrak{c}}) = \mathbf{II}\left(\left(\frac{d\phi_{\mathfrak{c}}}{\phi_{\mathfrak{c}}-\sigma_{\mathfrak{c}}^1}\right)\wedge\cdots\wedge\left(\frac{d\phi_{\mathfrak{c}}}{\phi_{\mathfrak{c}}-\sigma_{\mathfrak{c}}^{r-2}}\right)\right) \in \mathbf{II}^{r-2}\Big(\wedge^{r-2}\,\mathbf{H}_{\mathfrak{c}}\Big)$

$$\Big\downarrow\ \mathcal{S} \ \text{(symbol)}$$

- $\boldsymbol{\Omega}_{\mathfrak{c}}^{r-2} = \left(\frac{d\phi_{\mathfrak{c}}}{\phi_{\mathfrak{c}}-\sigma_{\mathfrak{c}}^1}\right)\wedge\cdots\wedge\left(\frac{d\phi_{\mathfrak{c}}}{\phi_{\mathfrak{c}}-\sigma_{\mathfrak{c}}^{r-2}}\right) \in \wedge^{r-2}\mathbf{H}_{\mathfrak{c}} \subset \wedge^{r-2}\mathbf{H}_{X_r}$

$$\boxed{\Big(\mathbf{HLog}^{r-2}\Big) \quad \Longleftrightarrow \quad \sum_{\mathfrak{c}} \epsilon_{\mathfrak{c}}\, \boldsymbol{\Omega}_{\mathfrak{c}}^{r-2} = 0 \quad \text{in} \quad \wedge^{r-2}\mathbf{H}_{X_r}}$$

$$\boxed{\textbf{Proofs of :} \quad \mathbf{hlog}^{r-2} = \sum_{\mathfrak{c}} \epsilon_{\mathfrak{c}} \, \mathbf{\Omega}_{\mathfrak{c}}^{r-2} = 0 \quad \textbf{dans } \wedge^{r-2}\mathbf{H}_{X_r}}$$

- $\mathbf{H}_{\boldsymbol{X}_r} = \mathbf{H}^0\Big(\Omega^1_{\boldsymbol{X}_r}\big(\,\mathrm{Log}\,\boldsymbol{L}_r\,\big)\Big) \xrightarrow{\;\oplus_\ell \mathbf{Res}_\ell\;} \mathbb{C}^{\boldsymbol{\mathcal{L}}_r}$ injective

$$\mathbf{\Omega}_{\mathfrak{c}}^{r-2} \in \wedge^{r-2}\mathbf{H}_{X_r} \hookrightarrow \wedge^{r-2}\,\mathbb{C}^{\boldsymbol{\mathcal{L}}_r} \;\curvearrowleft\; \boldsymbol{W}(E_r)$$

[P1] One decomposes $\mathbf{hlog}^{r-2}$ in a natural basis for $\wedge^{r-2}\,\mathbb{C}^{\boldsymbol{\mathcal{L}}_r}$

[P2] $\mathbf{sign}_r \hookrightarrow \oplus_{\mathfrak{c}}\big(\mathbf{H}_{\mathfrak{c}}\big)^{\wedge(r-2)} \longrightarrow \wedge^{r-2}\,\mathbb{C}^{\boldsymbol{\mathcal{L}}_r}$ $\quad \Big\langle\, \mathbf{sign}_r\,,\, \wedge^{r-2}\,\mathbb{C}^{\boldsymbol{\mathcal{L}}_r} \Big\rangle = 0$

$$\mathbf{1} \longmapsto \big(\,\mathbf{\Omega}_{\mathfrak{c}}^{r-2}\,\big)_{\mathfrak{c}} \longmapsto \sum_{\mathfrak{c}} \mathbf{\Omega}_{\mathfrak{c}}^{r-2}$$ (GAP3)

[P3] Explicit descrip° of lines $\longrightarrow$ $\mathbb{Z}^{\boldsymbol{\mathcal{L}}_r} \simeq \mathbb{Z}^{|\boldsymbol{\mathcal{L}}_r|}$

+ linear algebra over $\mathbb{Z}$ $\longrightarrow$ $\sum \epsilon_{\mathfrak{c}} \mathbf{\Omega}_{\mathfrak{c}}^{r-2} = 0$ (Maple)

[P4] Inductive analytic proof (... ?)

IV Comparison between HLog^2 and HLog^3

- $\boxed{\mathbf{HLog}^2}$ $\quad \mathbf{R}(x) - \mathbf{R}(y) - \mathbf{R}\left(\frac{x}{y}\right) - \mathbf{R}\left(\frac{1-y}{1-x}\right) + \mathbf{R}\left(\frac{x(1-y)}{y(1-x)}\right) = 0$

- $\boxed{\mathbf{HLog}^3}$ $\quad \sum_{i=1}^{10} \epsilon_i \, \boldsymbol{AH}^3_{\boldsymbol{\Sigma_i}}(\boldsymbol{\phi_i}) = 0$ with for $\Sigma = \{ b_1, \ldots, b_4 \}$

$$\boldsymbol{AH}^3_\Sigma(x) = \frac{1}{3}\sum_{k=1}^{3}(-1)^{k-1}\mathbf{Log}\left(1-\frac{x}{b_k}\right)\mathbf{R}_{\Sigma\setminus\{b_k\}}(x)$$

$$\boldsymbol{AH}^3_1(\boldsymbol{x}) + \boldsymbol{AH}^3_2\left(\frac{\boldsymbol{1}}{\boldsymbol{y}}\right) + \boldsymbol{AH}^3_3\left(\frac{\boldsymbol{y}}{\boldsymbol{x}}\right) + \boldsymbol{AH}^3_4\left(\frac{\boldsymbol{x-y}}{\boldsymbol{x-1}}\right) + \boldsymbol{AH}^3_5\left(\frac{\boldsymbol{b(a-x)}}{\boldsymbol{ay-bx}}\right)$$

$$+\boldsymbol{AH}^3_6\left(\frac{\boldsymbol{P(x,y)}}{\boldsymbol{(x-1)(y-b)}}\right) + \boldsymbol{AH}^3_7\left(\frac{\boldsymbol{(x-y)(y-b)}}{\boldsymbol{y\,P(x,y)}}\right) + \boldsymbol{AH}^3_8\left(\frac{\boldsymbol{x\,P(x,y)}}{\boldsymbol{(x-y)(x-a)}}\right)$$

$$+\boldsymbol{AH}^3_9\left(\frac{\boldsymbol{y(x-b)}}{\boldsymbol{x(y-a)}}\right) + \boldsymbol{AH}^3_{10}\left(\frac{\boldsymbol{a\,(b-x)}}{\boldsymbol{by-ax}}\right) = \boldsymbol{0}$$

Del Pezzo's webs $\mathcal{W}_{\mathbf{dP}_5}$ and $\mathcal{W}_{\mathbf{dP}_4}$

- $(\mathcal{A}\mathbf{b})$ $\quad \mathbf{R}(\phi_1) - \mathbf{R}(\phi_2) - \mathbf{R}\Big(\,\phi_3\,\Big) - \mathbf{R}\Big(\,\phi_4\,\Big) + \mathbf{R}\bigg(\,\phi_5\,\bigg) = \mathbf{0}$

- For each i : $\mathcal{F}_{\phi_i} =$ foliation by the $\{\,\phi_i = \lambda\,\}$, $\lambda \in \mathbb{P}^1$

- **Web :** $\mathcal{W}_{\mathbf{dP}_5} = \Big(\mathcal{F}_{\phi_1}, \ldots, \mathcal{F}_{\phi_5}\Big)$: 5-tuple of foliations

- $\mathcal{W}_{\mathbf{dP}_5} =$ geometric object $\quad \rightsquigarrow \quad (\,\mathbf{HLog}^2\,) = (\mathcal{A}\mathbf{b})$

- $\mathcal{W}_{\mathbf{dP}_4} = \Big(\mathcal{F}_{\phi_k}\Big)_{\substack{\phi_k\,:\,\mathbf{dP_4} \to \mathbb{P}^1 \\ \text{fib}^\circ \text{ by conics}}} \quad \rightsquigarrow \quad (\,\mathbf{HLog}^3\,)$

Comparison between the webs $\mathcal{W}_{\mathbf{dP}_5}$ and $\mathcal{W}_{\mathbf{dP}_4}$

$\mathcal{W}_{\mathbf{dP}_5}$ and $\mathcal{W}_{\mathbf{dP}_4}$ satisfy similar remarkable properties :

- *formed by the pencils of conics on a del Pezzo surface*
- *non linearizable webs*
- *webs of maximal rank, with all their ARs hyperlogarithmic*
- *one has* $\mathbf{RA}(\mathcal{W}_{\mathbf{dP}_5}) = \mathbf{LogRA}^1(\mathcal{W}_{\mathbf{dP}_5}) \oplus \langle\, \mathbf{HLog}^2 \,\rangle$
  $\mathbf{RA}(\mathcal{W}_{\mathbf{dP}_4}) = \mathbf{LogRA}^1(\mathcal{W}_{\mathbf{dP}_4}) \oplus \mathbf{HLogRA}^2(\mathcal{W}_{\mathbf{dP}_4}) \oplus \langle\, \mathbf{HLog}^3 \,\rangle$

  (decompositions in irreducible $\boldsymbol{W}_r$-representations)
- *caracterized by the matroids of their hexagonal 3-subwebs*
- *are 'canonically algebraizable' (!)*
- *are 'modular webs'*
- *are 'cluster webs'*
- *can be obtained geometrically à la* **[Gelfand-MacPherson]**

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  $\mathbf{RA}(\mathcal{W}_{\mathbf{dP}_4}) = \mathbf{LogRA}^1(\mathcal{W}_{\mathbf{dP}_4}) \oplus \mathbf{HLogRA}^2(\mathcal{W}_{\mathbf{dP}_4}) \oplus \langle\, \mathbf{HLog}^3 \,\rangle$

  (decompositions in irreducible $\boldsymbol{W}_r$-representations)
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Gelfand-MacPherson webs

- $\boldsymbol{G} \supset \boldsymbol{P} \supset \boldsymbol{H}$:
  $\boldsymbol{G}$ = simple Lie group, Dynkin type D, rank r
  $\boldsymbol{P}$ = standard parabolic sub-group (maximal)
  $\boldsymbol{H} \simeq (\mathbb{C}^*)^r$ = Cartan sub-torus

- $\boldsymbol{X} = \boldsymbol{G}/\boldsymbol{P}$: $\boldsymbol{G}$-homogenous projective variety

- $\boldsymbol{X} \subset \mathbb{P}(\boldsymbol{V}_\rho)$:
  $\boldsymbol{V}_\rho$ = rep$^\circ$ of $\boldsymbol{G}$ $\big(\rho : \boldsymbol{G} \hookrightarrow \mathbf{GL}(V_\rho)\big)$ such that
  $\boldsymbol{X} = \boldsymbol{G} \cdot [v_\omega]$ with $v_\omega \in \boldsymbol{V}_\rho$ of highest weight ω
  $\boldsymbol{P} = \mathrm{Stab}_{\boldsymbol{G}}([v_\omega])$ = stabilizer of $[v_\omega] \in \mathbb{P}(\boldsymbol{V}_\rho)$

- $\mathfrak{W}_\rho = \Big\{ \text{weights of } \rho \Big\} \subset \mathfrak{h}^*_{\mathbb{R}} \simeq \mathbb{R}^r$ $\quad \Big(\mathfrak{h} = \mathrm{Lie}(\boldsymbol{H}) \Big)$

  – $\boldsymbol{X} \ni x = \Big[\sum_{w \in \mathfrak{W}} p^w(x) v_w \Big] \in \mathbb{P}(\boldsymbol{V}_\rho)$ **$(p^w(x))_w$ 'generalized Plücker coordinates'**

- **Moment map :** $\mu = \mu_{D,\rho} : \boldsymbol{X} \longrightarrow \mathfrak{h}^*_{\mathbb{R}}\ ,\ x \longmapsto \dfrac{\sum_w |p^w(x)|^2 w}{\sum_w |p^w(x)|^2}$

Gelfand-MacPherson webs

- **Moment map :** $\mu : \boldsymbol{X} \longrightarrow \mathfrak{h}^*_{\mathbb{R}}$, $x \longmapsto \dfrac{\sum_w |p^w(x)|^2 w}{\sum_w |p^w(x)|^2}$

- **Moment polytope :** $\mu(\boldsymbol{X}) = \boldsymbol{\Delta} = \boldsymbol{\Delta}_{D,\rho} = \mathrm{Conv}(\mathfrak{W}_\rho) \subset \mathfrak{h}^*_{\mathbb{R}}$

- One sets $H_+ = \boldsymbol{H}(\mathbb{R}_{>0}) \simeq (\mathbb{R}_{>0})^r$ (split form over $\mathbb{R}$)

Thm [Atiyah, Guillemin-Sternberg, Gelfand-Serganova]

1. For every $x \in \boldsymbol{X}$:

- $\boldsymbol{\Delta_x} = \mu(\overline{\boldsymbol{H}\cdot x})$ is a $\mathfrak{W}_\rho$-sub-polytope of $\boldsymbol{\Delta}$
- μ induces an C^ω-isom of manifolds with corners $\overline{H_+\cdot x} \simeq \boldsymbol{\Delta_x}$

2. For generic x, *i.e.* $x \in \boldsymbol{X}^\circ = \mu^{-1}(\mathring{\boldsymbol{\Delta}})$, one has $\boldsymbol{\Delta_x} = \boldsymbol{\Delta}_{D,\rho}$

Gelfand-MacPherson webs

- F face of Δ : $\boldsymbol{X}_F = \mu^{-1}(F) \subset \boldsymbol{X}$

Prop : 1. $\boldsymbol{X}_F = \boldsymbol{G}_F/\boldsymbol{P}_F$ with $(\boldsymbol{G}_F, \boldsymbol{P}_F)$ of type (D_F, ω_F) and $F \simeq \boldsymbol{\Delta}_{D_F,\omega_F}$

2. One has $\boldsymbol{V}_\rho = \boldsymbol{V}_F \oplus \boldsymbol{V}^F$ as $\boldsymbol{G}_F$-rep° $\rightarrow$ Linear project° $\Pi_F : \boldsymbol{V}_\rho \to \boldsymbol{V}_F$

$$\boldsymbol{X}_F = \boldsymbol{X} \cap \mathbb{P}(\boldsymbol{V}_F) = \Pi_F(\boldsymbol{X}) \subset \mathbb{P}(\boldsymbol{V}_F)$$

3. $\Pi_F(\boldsymbol{X}^\circ) = \boldsymbol{X}_F^\circ$ and $\Pi_F : \boldsymbol{X}^\circ \longrightarrow \boldsymbol{X}_F^\circ$ is a locally trivial fibration in weighted projective spaces

4. $\boldsymbol{H}$-torsor $\nu_{\boldsymbol{H}} : \boldsymbol{X}^\circ \longrightarrow \mathcal{Y}^\circ = \boldsymbol{X}^\circ/\boldsymbol{H}$ $\left(\mathcal{Y}^\circ \subset \mathcal{Y} = \boldsymbol{X}^{ss}/\!/\boldsymbol{H}\right)$

5. $\Pi_F : \boldsymbol{X}^\circ \longrightarrow \boldsymbol{X}_F^\circ$ is $(\boldsymbol{H}, \boldsymbol{H}_F)$-equivariant $(\boldsymbol{H} \twoheadrightarrow \boldsymbol{H}_F)$

Gelfand-MacPherson webs

5. $\mathbb{P}(\boldsymbol{V}_\rho) \supset \boldsymbol{X}^\circ \xrightarrow{\Pi_F} \boldsymbol{X}_F^\circ \subset \mathbb{P}(\boldsymbol{V}_F)$ is $(\boldsymbol{H}, \boldsymbol{H}_{\boldsymbol{F}})$-equivariant

6. $\exists\ \pi_F : \boldsymbol{\mathcal{Y}}^\circ \to \boldsymbol{\mathcal{Y}}_F^\circ = \boldsymbol{X}_F^\circ / \boldsymbol{H}_F$ such that the diagram

$$\begin{tikzcd} \boldsymbol{X}^\circ \arrow[r, "\Pi_F"] \arrow[d, "\nu_{\boldsymbol{H}}"'] & \boldsymbol{X}_F^\circ \arrow[d, "\nu_{\boldsymbol{H}_F}"] \\ \boldsymbol{\mathcal{Y}}^\circ \arrow[r, "\pi_F"'] & \boldsymbol{\mathcal{Y}}_F^\circ \end{tikzcd}$$

be commutative

- **Def°** : **Gelfand-MacPherson webs**

$$\mathcal{W}_{\boldsymbol{X}}^{GM} = \mathcal{W}\Big(\Pi_F \ \Big|\ F \text{ face of codim 1 of } \boldsymbol{\Delta}\Big) \leftarrow \boldsymbol{H}\text{-equivariant}$$

$$\mathcal{W}_{\boldsymbol{\mathcal{Y}}}^{GM} = \mathcal{W}\Big(\pi_F \ \Big|\ F \text{ face of codim 1 of } \boldsymbol{\Delta}\Big) = \Big(\mathcal{W}_{\boldsymbol{X}}^{GM}\Big)/\boldsymbol{H}$$

Gelfand-MacPherson's webs : $G_k(\mathbb{C}^N)$

- $\boldsymbol{G}_k(\mathbb{C}^N) \subset \mathbb{P}(\wedge^k \mathbb{C}^N)$ $\qquad\qquad \Big(\boldsymbol{G} = \mathbf{SL}(\mathbb{C}^N),\ D = \boldsymbol{A_{N-1}},\ \text{etc} \Big)$

- $\mu : \boldsymbol{G}_k(\mathbb{C}^N) \longrightarrow \boldsymbol{\Delta}_k^N = \left\{ (t_i)_{i=1}^N \,\middle|\, \begin{array}{l} 0 \leq t_i \leq 1 \\ \sum_{i=1}^N t_i = k \end{array} \right\}$ **hypersimplex**

- Facets of $\boldsymbol{\Delta}_k^N$ $= \left\{ \begin{array}{lll} \boldsymbol{\Delta}_k^N \cap \{t_i = 0\} = \boldsymbol{\Delta}_k^{N-1} & \leftsquigarrow & \boldsymbol{G_k}(\mathbb{C}^{\boldsymbol{N-1}}) \\ \boldsymbol{\Delta}_k^N \cap \{t_i = 1\} \simeq \boldsymbol{\Delta}_{k-1}^{N-1} & \leftsquigarrow & \boldsymbol{G_{k-1}}(\mathbb{C}^{\boldsymbol{N-1}}) \end{array} \right.$

- For each $i = 1, \ldots, N$, there are two 'face maps' :

$$\boldsymbol{G}^\circ_{\boldsymbol{k-1}}\Big(\mathbb{C}^{\boldsymbol{N-1}}_{\{\boldsymbol{x_i=0}\}}\Big) \xleftarrow{\Pi_{\{t_i=1\}}} \boldsymbol{G}^\circ_{\boldsymbol{k}}(\mathbb{C}^{\boldsymbol{N}}) \xrightarrow{\Pi_{\{t_i=0\}}} \boldsymbol{G}^\circ_{\boldsymbol{k}}\Big(\mathbb{C}^{\boldsymbol{N}}/_{\langle \boldsymbol{e_i}\rangle}\Big)$$

Gelfand-MacPherson's webs : $G_k(\mathbb{C}^N)$

- For each $i = 1, \ldots, N$, there are two 'face maps' :

$$\begin{tikzcd}
\boldsymbol{G}^{\circ}_{\boldsymbol{k-1}}\Big(\mathbb{C}^{\boldsymbol{N-1}}_{\boldsymbol{\{x_i=0\}}}\Big) \arrow[d, "\nu_{i,0}"] & \boldsymbol{G}^{\circ}_{\boldsymbol{k}}(\mathbb{C}^{\boldsymbol{N}}) \arrow[l, "\Pi_{\{t_i=0\}}"'] \arrow[r, "\Pi_{\{t_i=1\}}"] \arrow[d, "\nu"] & \boldsymbol{G}^{\circ}_{\boldsymbol{k}}\Big(\mathbb{C}^{\boldsymbol{N}}/_{\langle \boldsymbol{e_i}\rangle}\Big) \arrow[d, "\nu_{i,1}"] \\
\mathbf{Conf}^{\circ}_{N-1}(\mathbb{P}^{k-2}) & \mathbf{Conf}^{\circ}_{N}(\mathbb{P}^{k-1}) \arrow[l, "\pi_{i,0}"'] \arrow[r, "\pi_{i,1}"] & \mathbf{Conf}^{\circ}_{N-1}(\mathbb{P}^{k-1}) \\
\Big[\mathbf{Proj}_{p_i}(p_k)\Big]_{k\neq i} & {[p_1, \ldots, p_N]} \arrow[l, mapsto] \arrow[r, mapsto] & {[p_1, \ldots, \widehat{p_i}, \ldots, p_N]}
\end{tikzcd}$$

- **GM-web :** $\mathcal{W}^{GM}_{\mathbf{Conf}_N(\mathbb{P}^k)} = \mathcal{W}\begin{pmatrix} N \text{ forgetful maps of a point } + \\ N \text{ maps of proj}^{\circ} \text{ from a point} \end{pmatrix}$

- $k = 2$: $\mathbf{Conf}^{\circ}_N(\mathbb{P}^1) = \mathcal{M}_{0,N}$ $\qquad \mathcal{W}^{GM}_{\mathcal{M}_{0,N}} = \mathcal{W}\Big(\mathcal{M}_{0,N} \xrightarrow{f_i} \mathcal{M}_{0,N-1}\Big)$

$+ N = 5$: $\overline{\mathcal{M}}_{0,5} \simeq \mathbf{dP}_5 = \boldsymbol{X}_4$ $\qquad \mathcal{W}^{GM}_{\mathcal{M}_{0,5}} = \mathcal{W}_{\mathbf{dP}_5} \simeq \mathcal{B} \longleftarrow (\mathcal{A}\mathbf{b})$

Construction of $(\mathcal{A}b)$ by Gelfand and MacPherson

$$
\begin{tikzcd}
\boldsymbol{G}_2^o(\mathbb{R}^5) \arrow[r, "\boldsymbol{F_i}"] \arrow[d, "\nu=\nu_4"'] & \boldsymbol{G}_2^o\left(\mathbb{R}^5/\langle e_i \rangle\right) \arrow[l, bend right, "a_i"'] \arrow[d, "\nu_i=\nu_3"] \\
\boldsymbol{\mathcal{M}}_{0,5}(\mathbb{R}) \arrow[r, "\boldsymbol{f_i}"] & \boldsymbol{\mathcal{M}}_{0,4}(\mathbb{R}) \simeq \mathbb{R} \setminus \{0,1\}
\end{tikzcd}
$$

$\mathbf{P}_1$: 1-st Pontryagin class

$\mathbf{H}^4\Big(\boldsymbol{G}_2(\mathbb{R}^5)\Big) \ni \mathbf{P}_1 = [\,\boldsymbol{\Omega}\,]$

with $\boldsymbol{\Omega} \in \Omega^4\Big(\boldsymbol{G}_2(\mathbb{R}^5)\Big)^{\mathbf{SO}_5(\mathbb{R})}$

- $\int$ along the fibers of the 4-form $\boldsymbol{\Omega}$ $\longrightarrow$
  - $\omega_{0,5} = \nu_*(\boldsymbol{\Omega}) \quad \in \Omega^0(\boldsymbol{\mathcal{M}}_{0,5}(\mathbb{R}))$
  - $\omega_{0,4,i} = (\nu_i)_*(\boldsymbol{\Omega}) \quad \in \Omega^1(\boldsymbol{\mathcal{M}}_{0,4}(\mathbb{R}))$

- At $\nu(\xi)$: $\omega_{0,5} = \int_{\overline{\boldsymbol{H}\cdot\xi}} \boldsymbol{\Omega} \quad +$

  $\overline{\boldsymbol{H}\cdot\xi} \simeq \boldsymbol{\Delta}_2^5$ via $\boldsymbol{G}_2(\mathbb{R}^5) \xrightarrow{\mu} \boldsymbol{\Delta}_2^5$

  $\partial[\,\boldsymbol{\Delta}_2^5\,] = \sum_{i=1}^5 (-1)^i a_{i*}\Big([\,\boldsymbol{\Delta}_{2,i}^4\,]\Big)$

- Stokes for $\int$ along fibers :

$$
d\,\underbrace{\omega_{0,5}}_{=0} = \sum_{i=1}^5 (-1)^i \boldsymbol{f_i}^*\big(\underbrace{\omega_{0,4}}_{=\mathrm{d}\mathbf{R}}\big) \Longrightarrow 0 = \sum_{i=1}^5 (-1)^i \boldsymbol{f_i}^*(\mathrm{d}\mathbf{R}) \quad (\boldsymbol{\mathcal{A}}\mathbf{b})
$$

Cox varieties

- How get $\boldsymbol{G}_2(\mathbb{C}^5)$ from $\boldsymbol{X}_4 = \mathbf{dP}_5 = \overline{\boldsymbol{\mathcal{M}}}_{0,5}$?

It is its Cox variety !

- $\boldsymbol{S}$ = smooth projective variety such that $\mathbf{Pic}_{\mathbb{Z}}(\boldsymbol{S}) = \oplus_{i=0}^{r} \mathbb{Z}\,\boldsymbol{\ell}_i$

$\Big(\boldsymbol{S} = \mathbf{Bl}_{p_1,\ldots,p_r}(\mathbb{P}^2) \quad \ell_0 = \boldsymbol{h} = [\,H\,]$ et $\boldsymbol{\ell}_i = [\,E_i\,] \quad i = 1,\ldots,r \Big)$

- **Def°** : **Cox ring**

$$\mathbf{Cox}(\boldsymbol{S}) = \bigoplus_{n_0,\ldots,n_r \in \mathbb{Z}} \mathbf{H}^0\Big(\boldsymbol{S}, \mathcal{O}_{\boldsymbol{S}}\big(n_0 H + n_1 E_1 + \cdots + n_r E_r\big)\Big)$$

- **Facts** : – $\mathbf{Cox}(\,\mathbb{P}^n\,) = \mathbb{C}\big[x_0,\ldots,x_n\big]$ (homonenous polynomials)
  - $\mathbf{Cox}(\boldsymbol{S}) = \mathbb{C}\big[y_1,\ldots,y_m\big] \iff \boldsymbol{S}$ is toric
  - $\mathbf{Cox}(\boldsymbol{S})$ of finite type $=$ $\boldsymbol{S}$ 'Mori Dream Space' (MMP ✓)

$$\mathbf{Cox}(\boldsymbol{S}) = \frac{\mathbb{C}\big[\Gamma_1,\ldots,\Gamma_m\big]}{\mathcal{J}_S} \longrightarrow \mathbf{A}(\boldsymbol{S}) = \mathrm{Spec}\Big(\mathbf{Cox}(\boldsymbol{S})\Big) \subset \mathbb{A}^m_{\Gamma}$$

Cox varieties of del Pezzo surfaces

- Surface $\boldsymbol{S} \supset \ell$ with $\ell \simeq \mathbb{P}^1$ and $\ell^2 = -1$ $\Longrightarrow$ $\sigma_\ell \in \mathbf{H}^0(\mathcal{O}_{\boldsymbol{X}_r}(\ell)) \setminus \{0\}$ generator of $\mathbf{Cox}(\boldsymbol{S})$

- In $\mathbf{Bl}_{p_1,\ldots,p_9}(\mathbb{P}^2)$: $\exists\ \infty$ of (-1)-lines $\Longrightarrow$ $\mathbf{Cox}\Big(\mathbf{Bl}_{p_1,\ldots,p_9}(\mathbb{P}^2)\Big)$ not of finite type

Thm [Batyrev, Popov] For $r = 3, \ldots, 8$:

$$\mathbf{Cox}\Big(\mathbf{dP}_d = \boldsymbol{X_r} = \mathbf{Bl}_{p_1,\ldots,p_r}(\mathbb{P}^2)\Big) = \mathbb{C}\Big[\sigma_\ell \,\big|\, \ell \in \boldsymbol{\mathcal{L}}_r\Big]/\boldsymbol{\mathcal{J}}_{\mathbf{dP_d}}$$

- $\mathbf{T}_{\mathrm{NS}} = \mathrm{Hom}_{\mathbb{Z}}(\mathbf{Pic}_{\mathbb{Z}}(\boldsymbol{X}_r), \mathbb{C}^*) \circlearrowleft \mathbf{A}(\boldsymbol{X}_r) \rightsquigarrow \boldsymbol{X}_r = \mathbf{A}(\boldsymbol{X}_r) /\!/ \mathbf{T}_{\mathrm{NS}}$

- $\mathbf{A}(\boldsymbol{X}_r) \hookrightarrow \mathbb{C}^{\boldsymbol{\mathcal{L}}_r}$ + $\mathbb{Z}$-graduation on $\mathbf{Cox}(\boldsymbol{X_r})$ induced by $(-K, \cdot)$

$$\longrightarrow \mathbf{P}(\boldsymbol{X}_r) = \underbrace{\mathrm{Proj}\Big(\mathbf{Cox}(\boldsymbol{X_r})\Big)}_{=\big(\mathbf{A}(\boldsymbol{X}_r)-\{0\}\big)/_{\mathbb{C}^*}} \subset \mathbb{P}\Big(\mathbb{C}^{\boldsymbol{\mathcal{L}}_r}\Big) \circlearrowleft \boldsymbol{\mathcal{T}}_{\mathrm{NS}} = \mathbf{T}_{\mathrm{NS}}/_{\mathbb{C}^*}$$

Cox varieties of del Pezzo surfaces

- $\mathbf{P}(\boldsymbol{X}_r) = \operatorname{Proj}\Big(\mathbf{Cox}(\boldsymbol{X}_r)\Big) \subset \mathbb{P}(\mathbb{C}^{\mathcal{L}_r}) \circlearrowright \left.\begin{array}{l} \mathcal{T}_{\mathrm{NS}} = \mathbf{T}_{\mathrm{NS}}/\mathbb{C}^* \\ \boldsymbol{W}(E_r) \end{array}\right\} \rightsquigarrow \boldsymbol{G}(E_r)$

Thm [Batyrev, Popov, Derenthal, Serganova-Skorobogatov]

1. The space $\mathbb{C}^{\mathcal{L}_r}$ is a **minuscule** representation of $\boldsymbol{G}_r = \boldsymbol{G}(E_r)$

2. There is a $(\mathcal{T}_{\mathrm{NS}}, \boldsymbol{H}_r)$-equivariant embedding :

$$\mathbf{P}(\boldsymbol{X}_r) \hookrightarrow \boldsymbol{\mathcal{X}}_r = \boldsymbol{G}_r/\boldsymbol{P}_r \subset \mathbb{P}(\mathbb{C}^{\mathcal{L}_r})$$

3. There is an embedding $f_{SS} : \boldsymbol{X}_r \hookrightarrow \mathcal{Y}_r = \boldsymbol{\mathcal{X}}_r /\!/ \boldsymbol{H}_r$ such that

$$\begin{tikzcd} \mathbf{P}(\boldsymbol{X}_r) \arrow[r, hook] \arrow[d] & \boldsymbol{\mathcal{X}}_r \subset \mathbb{P}(\mathbb{C}^{\mathcal{L}_r}) \arrow[d] \\ \mathbf{dP}_{9-r} = \boldsymbol{X}_r \arrow[r, hook, "f_{SS}"] & \mathcal{Y}_r = \boldsymbol{\mathcal{X}}_r /\!/ \boldsymbol{H}_r \end{tikzcd}$$

is commutative

- $r=4,\ E_r=A_4$: $\quad \mathbf{P}(\boldsymbol{X}_4)=\boldsymbol{G}_2(\mathbb{C}^5)\hookrightarrow \mathbb{P}(\mathbb{C}^{\mathcal{L}_4})\simeq\mathbb{P}^9 \qquad$ (Plücker)

$$\begin{tikzcd}
 & \mathbf{P}(\boldsymbol{X}_4) \arrow[r, equal] \arrow[d] & \boldsymbol{G}_2(\mathbb{C}^5) \arrow[r, dashed, "\Pi_i"] \arrow[d] & \boldsymbol{G}_2(\mathbb{C}^4_i) \arrow[d] \\
\mathbf{dP}_5 = & \boldsymbol{X}_4 \arrow[r, hook, "f_{SS}", "\sim"'] & \boldsymbol{\mathcal{Y}}_4=\overline{\boldsymbol{\mathcal{M}}}_{0,5} \arrow[r, dashed, "\pi_i", "f_i"'] & \overline{\boldsymbol{\mathcal{M}}}_{0,4}\simeq\mathbb{P}^1
\end{tikzcd}$$

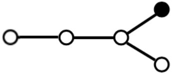
$$\longrightarrow\ \boldsymbol{\mathcal{W}}_{\mathbf{dP}_5}=f_{SS}^*\Big(\boldsymbol{\mathcal{W}}^{GM}_{\boldsymbol{\mathcal{Y}}_4}=\boldsymbol{\mathcal{W}}_{\boldsymbol{\mathcal{M}}_{0,5}}\Big)$$

- For $r\in\{4,\ldots,7\}$ ($r=8$?), one has :

$$\begin{tikzcd}
 & \mathbf{P}(\boldsymbol{X}_r) \arrow[r, hook] \arrow[d] & \boldsymbol{\mathcal{X}}_r=\boldsymbol{G}_r/\boldsymbol{P}_r \arrow[r, dashed, "\Pi_F"] \arrow[d] & \boldsymbol{\mathcal{X}}_F \arrow[d] \\
\mathbf{dP}_{9-r} = & \boldsymbol{X}_r \arrow[r, hook, "f_{SS}"] \arrow[ur, hook, "F_{SS}"] & \boldsymbol{\mathcal{Y}}_r \arrow[r, dashed, "\pi_F"] & \boldsymbol{\mathcal{Y}}_F
\end{tikzcd}$$

$$\boxed{\textbf{Thm}:\quad \boldsymbol{\mathcal{W}}_{\mathbf{dP}_{9-r}}=f_{SS}^*\Big(\boldsymbol{\mathcal{W}}^{GM}_{\boldsymbol{\mathcal{Y}}_r}\Big)=F_{SS}^*\Big(\boldsymbol{\mathcal{W}}^{GM}_{\boldsymbol{G}_r/\boldsymbol{P}_r}\Big)}$$

Example : $\mathcal{W}_{\mathbf{dP}_4}$ (case $r = 5$)

- Type D_5 : 

$$\boldsymbol{G}_5 = \mathrm{Spin}_{10}(\mathbb{C}) \xrightarrow{2:1} \mathrm{SO}_{10}(\mathbb{C})$$

$$\boldsymbol{G}_5/\boldsymbol{P}_5 = \mathbb{S}_{\mathbf{5}} \subset \mathbb{P}(\boldsymbol{S}_{\mathbf{5}}^{+})$$

  – $\mathbb{S}_{\mathbf{5}} \simeq \boldsymbol{OG}_5^{+}(\mathbb{C}^{10}) =$ '*Spinor 10-fold*'

  – $\boldsymbol{S}_{\mathbf{5}}^{+} \simeq \mathbb{C}^{16} =$ '*half-spin representation*'

- $\boldsymbol{\mu} : \mathbb{S}_{\mathbf{5}} \to \boldsymbol{\Delta}_{D_5} =$ **demihypercube** : convexe envelope of the $\frac{1}{2}(\epsilon_1, \ldots, \epsilon_5) \in \frac{1}{2}\{\pm 1\}^5$'s with $\epsilon_1 \cdots \epsilon_5 = 1$

- Facets : $\boldsymbol{F} = \boldsymbol{\Delta}^{\varepsilon}_{D_5, i} = \boldsymbol{\Delta}_{D_5} \cap \{\, t_i = \frac{\varepsilon}{2} \,\} \simeq \boldsymbol{\Delta}_{D_4, \mathbb{Q}^6}$ $\quad \begin{pmatrix} i = 1, \ldots, 5 \\ \varepsilon \in \{\pm 1\} \end{pmatrix}$

- Diagram :

$$\begin{tikzcd}
\mathbf{P}(\boldsymbol{X}_5) \arrow[r, hook] \arrow[d] & \mathbb{S}_5 \arrow[r, dashed, "\Pi_F"] \arrow[d] & \mathbb{Q}^6_F \arrow[d] \\
\mathbf{dP}_4 = \boldsymbol{X}_5 \arrow[r, hook, "f_{SS}"] & \boldsymbol{\mathcal{Y}}_5 \arrow[r, dashed, "\pi_F"] & \boldsymbol{\mathcal{Y}}_F \simeq \mathbb{P}^2
\end{tikzcd}$$

Example : $\mathcal{W}_{\mathbf{dP}_4}$ (case $r=5$)

- Diagram :

$$\begin{tikzcd}
\mathbf{P}(\boldsymbol{X}_5) \arrow[r, hook] \arrow[d] & \mathbb{S}_5 \arrow[r, dashed, "\Pi_F"] \arrow[d] & \mathbb{Q}^6_F \arrow[d] \\
\mathbf{dP}_4 = \boldsymbol{X}_5 \arrow[r, hook, "f_{SS}"] & \boldsymbol{\mathcal{Y}}_5 \arrow[r, dashed, "\pi_F"] & \boldsymbol{\mathcal{Y}}_F \simeq \mathbb{P}^2
\end{tikzcd}$$

- **Gelfand-MacPherson's web of $\mathbb{S}_5$** :

$$\mathcal{W}^{GM}_{\boldsymbol{\mathcal{Y}}_5} = \mathcal{W}\Big(\pi_i^{\varepsilon} : \boldsymbol{\mathcal{Y}}_5 \dashrightarrow \mathbb{P}^2\Big) = \begin{array}{l}\text{10-web of codim 2}\\ \text{on } \boldsymbol{\mathcal{Y}}_5 \simeq_{birat} \mathbb{C}^5\end{array}$$

<u>**Thm**</u> **:** The webs $\mathcal{W}^{GM}_{\boldsymbol{\mathcal{Y}}_4} \sim \mathcal{W}_{\mathbf{dP}_5}$ and $\mathcal{W}^{GM}_{\boldsymbol{\mathcal{Y}}_5}$ are <u>even more</u> <u>similar</u> than $\mathcal{W}_{\mathbf{dP}_5}$ and $\mathcal{W}_{\mathbf{dP}_4} = f_{SS}^*\Big(\mathcal{W}^{GM}_{\boldsymbol{\mathcal{Y}}_5}\Big)$ are !

- $\Big[\mathcal{W}^{GM}_{\boldsymbol{\mathcal{Y}}_4} \to \mathcal{W}^{GM}_{\boldsymbol{\mathcal{Y}}_5} \to \mathcal{W}^{GM}_{\boldsymbol{\mathcal{Y}}_6} \to \cdots\Big]$ 'better than' $\Big[\mathcal{W}_{\mathbf{dP}_5} \to \mathcal{W}_{\mathbf{dP}_4} \to \mathcal{W}_{\mathbf{dP}_3} \to \cdots\Big]$

Questions

- *Can* $\mathbf{HLog}^{r-2} \in \boldsymbol{AR}\big(\mathcal{W}_{\mathbf{dP}_{9-r}}\big)$ *be obtained from an abelian relation* $\mathcal{A}_r$ *of the web* $\mathcal{W}^{GM}_{\mathcal{Y}_r}$ **?** **YES !**

- **Question :** *Can the abelian relation* $\mathcal{A}_r$ *of* $\mathcal{W}^{GM}_{\mathcal{Y}_r}$ *be constructed à la Gelfand-MacPherson from a characteristic class on (a real form of) the homogeneous space* $\mathcal{X}_r = \boldsymbol{G}_r/\boldsymbol{P}_r$ *?*

- $\mathcal{X}_6 = \mathbb{O}_{\mathbb{C}}\mathbf{P}^2 \rightsquigarrow$ $\mathbb{O}\mathbf{P}^2 = F_4/_{\mathrm{Spin}(9)}$, $\mathbb{O}_s\mathbf{P}^2 = E_{6(6)}/_{F_{4(4)}}$ $\qquad \mathbf{H}^8(\mathbb{O}\mathbf{P}^2) = \mathbb{R}\big[\,\Omega^8_{\mathbb{O}}\big]$ (split $\mathbb{R}$-form)

- **Projective differential geometry of surfaces :**
  For any 'sufficiently generic' Surface $\boldsymbol{S} \subset \mathbb{P}^d$

  d = 3 : [Moutard, Darboux - 1880] 27-web on $\boldsymbol{S} \subset \mathbb{P}^3$

  d = 4 : [Darboux 1880] 10-web on $\boldsymbol{S} \subset \mathbb{P}^4$

  d = 5 : [C. Segre -1921] 5-web on $\boldsymbol{S} \subset \mathbb{P}^5$

$\longrightarrow$ Plenty of questions !

Many other questions...

- **Applications** – $\mathbf{HLog}^1 = \Big(\mathbf{Log}(x) - \mathbf{Log}(y) - \mathbf{Log}\big(\frac{x}{y}\big) = 0 \Big)$ ✓

  – $\mathbf{HLog}^2 = \Big(\mathbf{R}(x) - \mathbf{R}(y) - \mathbf{R}\big(\frac{x}{y}\big) - \mathbf{R}\big(\frac{1-y}{1-x}\big) + \mathbf{R}\Big(\frac{x(1-y)}{y(1-x)}\Big) = 0 \Big)$ ✓

  – $\mathbf{HLog}^3 = \Big(\sum_{i=1}^{10} \boldsymbol{AH}_i^3\big(U_i(x,y)\big) = 0 \Big)$?

- **Construction** of $\mathbf{HLog}^3$ à la Gelfand-MacPherson ?

- **Interpretation** of $\mathbf{HLog}^3$ in terms of the SC of $\mathbf{dP}_4$?

- **Versions** Unival. $\mathbf{HLog}^3_{\mathrm{univ}}$? Quantum $\mathbf{HLog}^3_q$? Motivic $\mathbf{HLog}^3_{\mathrm{mot}}$?

- **Singular/real del Pezzo surfaces** ?

- **Blow-ups** $\mathbf{Bl}_{p_1,\ldots,p_r}(\mathbb{P}^2)$ with $r \geq 9$: $\sum_{\mathfrak{c}\in\mathcal{K}} \boldsymbol{AH}_{\mathfrak{c}}^{r-2}(\varphi_{\mathfrak{c}}) = 0$?