Extending morphisms of torsors for finite flat group schemes

Addendum to "A Purity Theorem for Torsors"

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March 12, 2023

In this note, we give a positive answer to a question left open in [2]. Let us recall the context of the mentioned work. For a scheme S and a finite flat S-group scheme $\pi\colon G \to S$, denote by $\mathrm{Tors}(S, G)$ the category of fppf G-torsors over S. The main purpose of [2] was to provide a proof of the following fact, which was previously stated (without proof) in [4].

Theorem 1 ([4], Lemme 2; [2], Theorem 3.1)**.** *Let S be a regular scheme, $U \subseteq S$ an open subscheme, $Z = S \setminus U$ its closed complement and suppose that the codimension of Z in S is at least 2. Let $\pi\colon G \to S$ be a finite flat S-group scheme and denote by $\pi_U\colon G_U \to U$ its restriction to U. Then, the restriction functor:*

$$\mathrm{Tors}(S, G) \longrightarrow \mathrm{Tors}(U, G_U)$$

is an equivalence of categories.

This result is analogous to the purity theorem for finite étale coverings (cf. [1, §X.3]), originally due to Zariski and Nagata as "purity of the branch locus". In that context, it is investigated in [2] what remains true after relaxing the assumption on the codimension of Z in S. It turns out that for U any dense open subscheme of S, the restriction functor from the category of finite étale coverings of S to that of finite étale coverings of U is still fully faithful. In fact, this holds even more generally for S just a normal scheme and it is due to the following result, proved in [2] as an application of Zariski's main theorem.

Lemma 2 ([2], Proposition 1.9)**.** *Let S be a locally Noetherian scheme, $U \subseteq S$ a dense open subscheme, X and Y two finite flat S-schemes; set $X_U \coloneqq X \times_S U$ and $Y_U \coloneqq Y \times_S U$. Suppose that X is normal. Then, writing* Hom_S *and* Hom_U *for the homomorphisms of schemes respectively over S and over U, the restriction map:*

$$\mathrm{Hom}_S(X, Y) \longrightarrow \mathrm{Hom}_U(X_U, Y_U)$$

is bijective.

In analogy with the case of finite étale coverings, it is then natural to ask whether, for U any dense open subscheme of S, the functor of Theorem 1 remains fully faithful. Using the same Lemma 2, we can give a positive answer to this question, again only requiring S to be normal.

Theorem 3. *Let S be a normal scheme, $U \subseteq S$ a dense open subscheme. Let $\pi \colon G \to S$ be a finite flat S-group scheme and denote by $\pi_U \colon G_U \to U$ its restriction to U. Then, the restriction functor:*

$$\mathrm{Tors}(S, G) \longrightarrow \mathrm{Tors}(U, G_U)$$

is fully faithful.

Proof. Let $X, Y \in \mathrm{Tors}(S, G)$ and consider the following fppf sheaf of sets on the category of S-schemes:

$$\begin{aligned} \underline{\mathrm{Hom}}_G(X, Y) \colon \mathrm{Sch}_{/S} &\longrightarrow \mathrm{Sets} \\ (T \to S) &\longmapsto \mathrm{Hom}_{\mathrm{Tors}(T, G_T)}(X_T, Y_T), \end{aligned}$$

where we denote $G_T \coloneqq G \times_S T$, $X_T \coloneqq X \times_S T$ and $Y_T \coloneqq Y \times_S T$. Let $V \to S$ be a faithfully flat and finitely presented covering trivialising both X and Y. Then, $\underline{\mathrm{Hom}}_G(X, Y)$ restricted to V is isomorphic to G_V. Thus, by a similar argument as in [3, Theorem III.4.3(a)] (for the representability of G-torsors) and by faithfully flat descent, we have that $\underline{\mathrm{Hom}}_G(X, Y)$ is represented by some finite flat S-scheme $Z \to S$. Therefore, by Lemma 2, the restriction map:

$$\mathrm{Hom}_{\mathrm{Tors}(S,G)}(X, Y) = \mathrm{Hom}_S(S, Z) \longrightarrow \mathrm{Hom}_U(U, Z_U) = \mathrm{Hom}_{\mathrm{Tors}(U,G_U)}(X_U, Y_U)$$

is bijective and this concludes the proof. □

References

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