Filtrations of Barsotti–Tate groups via Harder–Narasimhan theory

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Harder–Narasimhan theory for finite flat group schemes

Let $p\text{–gr}_{\mathcal{O}_K}$ be the category of finite flat group schemes $X = \operatorname{Spec} A$ over $\operatorname{Spec} \mathcal{O}_K$ with $\operatorname{ord} X = \operatorname{rank}_{\mathcal{O}_K} A \in p^{\mathbb{N}}$.
Recall: the $\mathcal{O}_K$-module of invariant differentials ω_X is a finitely presented torsion $\mathcal{O}_K$-module. We have:

- $p\text{–gr}_{\mathcal{O}_K}$ exact category (as a full subcategory, closed under extensions, of the fppf abelian sheaves on $\operatorname{Spec} \mathcal{O}_K$);
- $p\text{–gr}_{\mathcal{O}_K} \longrightarrow p\text{–gr}_K$, $X \mapsto X_K = X \otimes_{\mathcal{O}_K} K$ exact, faithful functor inducing $\{ Y \subseteq X \} \xrightarrow{1:1} \{ \mathcal{Y} \subseteq X_K \}$ for any $X \in p\text{–gr}_{\mathcal{O}_K}$;
- $\operatorname{ht}\colon p\text{–gr}_{\mathcal{O}_K} \longrightarrow \mathbb{N}$, $X \mapsto \log_p \operatorname{ord} X$ additive function which factors through $X \mapsto X_K$ and such that $X = 0 \Leftrightarrow \operatorname{ht} X = 0$;
- $\deg\colon p\text{–gr}_{\mathcal{O}_K} \longrightarrow \mathbb{R}$, $X \mapsto \sum_i v(a_i)$ if $\omega_X \cong \bigoplus_i \mathcal{O}_K/a_i\mathcal{O}_K$, $a_i \in \mathcal{O}_K \setminus \{0\}$ additive function satisfying $\deg \leq \operatorname{ht}$;
- for all $X \to Y$ such that $X_K \xrightarrow{\sim} Y_K$, we have $\deg X \leq \deg Y$, with $\deg X = \deg Y \Leftrightarrow X \cong Y$ ([Fa10]).

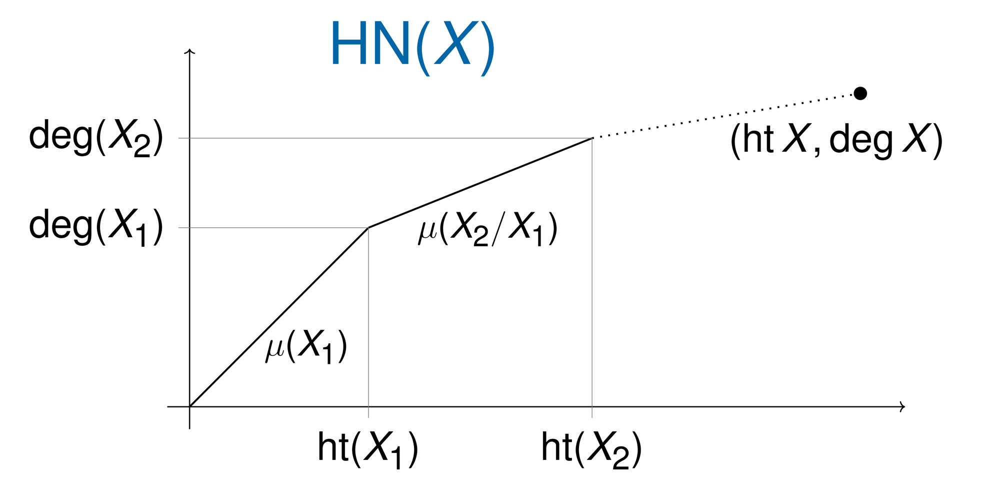
These properties make $p\text{–gr}_{\mathcal{O}_K}$ a *Harder–Narasimhan category* with respect to the slope function $\mu = \frac{\deg}{\operatorname{ht}}$:

- $X \in p\text{–gr}_{\mathcal{O}_K}$ is called *semi-stable* if for all $Y \subseteq X$ we have $\mu(Y) \leq \mu(X)$;
- in general, if $X \neq 0$ there exists a unique *Harder–Narasimhan filtration* $0 = X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_r = X$ such that $\forall i$, X_i/X_{i-1} is semi-stable and $\mu(X_1) > \cdots > \mu(X/X_{r-1})$, e.g. $\mu = 1 \Leftrightarrow$ multiplicative part, $\mu = 0 \Leftrightarrow$ étale part.

We obtain a *Harder–Narasimhan polygon* $\operatorname{HN}(X)\colon [0, \operatorname{ht} X] \longrightarrow [0, \deg X]$.

Setup

- p a prime number;
- $(\mathcal{O}_K, v)$ a complete valuation ring of rank 1, with perfect residue field of characteristic p, $v(p) = 1$, fraction field K of characteristic 0.



Filtrations of Barsotti–Tate groups

Let $p\text{–div}_{\mathcal{O}_K}$ be the category of Barsotti–Tate groups $H = (H[p^i])_{i\geq 1}$ over $\operatorname{Spec} \mathcal{O}_K$. Recall:

- $\forall i \geq 1$ we have $H[p^i] \in p\text{–gr}_{\mathcal{O}_K}$, $\operatorname{ht} H[p^i] = ih$ for a fixed $h = \operatorname{ht} H \in \mathbb{N}$;
- $\forall j > i$ we have $0 \longrightarrow H[p^i] \longrightarrow H[p^j] \xrightarrow{p^i} h[p^{j-i}] \longrightarrow 0$ exact;
- $\omega_H := \varprojlim_i \omega_{H[p^i]}$ is a finite free $\mathcal{O}_K$-module, $\omega_H/p^i\omega_H = \omega_{H[p^i]}$;
- $\dim_H = \operatorname{rank}_{\mathcal{O}_K} \omega_H$, $\deg H[p^i] = i \dim H$.

HN(H) arises also from other related Harder–Narasimhan theories, such as for filtered φ-modules.

Theorem (Fargues). The sequence of functions

$$\widetilde{\operatorname{HN}}(H[p^i]) := \frac{1}{i}\operatorname{HN}(H[p^i])(i\,\cdot\,)\colon [0, \operatorname{ht} H] \longrightarrow [0, \dim H]$$

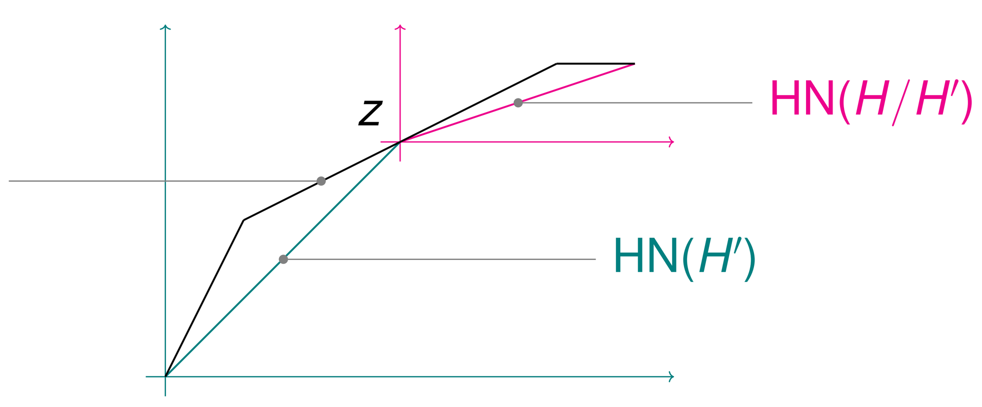
converges uniformly from above, for $i \to \infty$, to a continuous, concave, piecewise affine linear function $\operatorname{HN}(H)\colon [0, \operatorname{ht} H] \longrightarrow [0, \dim H]$ with integral break points.

Question. When does a break point of HN(H) "correspond" to a subobject $H' \subseteq H$?

If H is *of HN type*, i.e. if the $\widetilde{\operatorname{HN}}(H[p^i])$ form a constant sequence of functions, then the Harder–Narasimhan filtrations of $H[p^i]$ build up to a filtration of H. More generally:

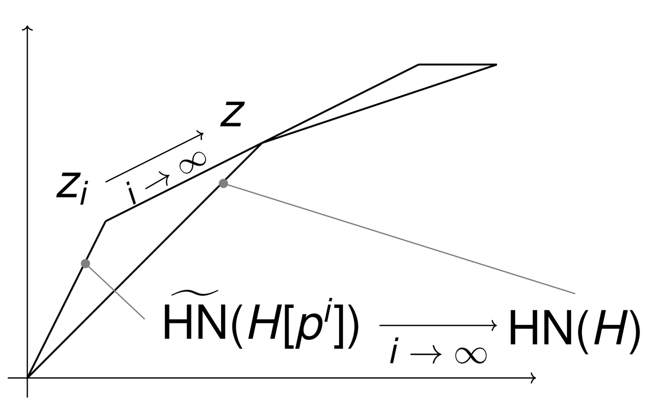
Theorem ([Ma22]). Let $z = (x, y)$ be a break point of HN(H) which also lies on HN($H[p]$). Then, there exists a unique subobject $H' \subseteq H$ in $p\text{–div}_{\mathcal{O}_K}$ such that:

- HN(H') equals the restriction of HN(H) to $[0, x]$,
- HN(H/H') equals the rest of HN(H) after z (up to shifting the origin to z).



Idea of the proof

Find a sequence of break points z_i of $\widetilde{\operatorname{HN}}(H[p^i])$ converging to z as $i \to \infty$, let $H'(i)$ be the respective subobjects of $H[p^i]$ in $p\text{–gr}_{\mathcal{O}_K}$.



Key point: $\forall j > i$, $H'(i) = H'(j)[p^i]$;

$\Rightarrow \operatorname{ht} H'(i)/H'(i-1)$ eventually stabilises;

$H' := (H'(i)/H'(i_0))_{i \geq i_0} \subseteq H/H[p^{i_0}] \cong H$, $i_0 \gg 0$.

Comparison with Hodge polygons

In order to verify the sufficient condition of the theorem, it is helpful to know other polygons which control HN($H[p]$) from above.

- Let $\operatorname{Hdg}(H)\colon [0, \operatorname{ht} H] \to [0, \dim H]$ be the polygon with slope 1 on $[0, \dim H]$ and slope 0 on $[\dim H, \operatorname{ht} H]$. Because $\deg \leq \operatorname{ht}$, we always have:

$$\operatorname{HN}(H[p]) \leq \operatorname{Hdg}(H),$$

i.e. Hdg(H) lies above HN($H[p]$) and the two polygons share the end point.

Suppose that H is endowed with *additional endomorphisms* $\iota\colon \mathcal{O}_F \to \operatorname{End}(H)$, where $\mathcal{O}_F$ is the ring of integers of a finite extension F of $\mathbb{Q}_p$ of degree d. Then, the situation can be improved. Set:

$$\operatorname{HN}(H[p], \iota) := \frac{1}{d}\operatorname{HN}(H[p])(d\,\cdot\,)\colon [0, \operatorname{ht} H/d] \longrightarrow [0, \dim H/d].$$

- If $F|\mathbb{Q}_p$ is unramified, we have an eigenspace decomposition

$$\omega_H = \bigoplus_{\tau\colon F \to K} \omega_{H,\tau} \qquad \text{with } d_\tau := \operatorname{rank}_{\mathcal{O}_K} \omega_{H,\tau} \leq \operatorname{ht} H/d.$$

Let $\operatorname{Hdg}(H, \iota)_\tau\colon [0, \operatorname{ht} H/d] \to [0, \dim H/d]$ be the polygon with slope 1 on $[0, d_\tau]$ and slope 0 on $[d_\tau, \operatorname{ht} H/d]$. We have (Shen '13):

$$\operatorname{HN}(H[p], \iota) \leq \operatorname{Hdg}(H, \iota) := \frac{1}{d}\sum_\tau \operatorname{Hdg}(H, \iota)_\tau.$$

- If $F|\mathbb{Q}_p$ is totally ramified with uniformiser $\pi \in \mathcal{O}_F$, we may write

$$\pi \circlearrowright \omega_H \twoheadrightarrow \omega_H/\pi\omega_H = \omega_{H[\pi]} \cong \bigoplus_{i=1}^{\operatorname{ht} H/d} \mathcal{O}_K/a_i\mathcal{O}_K, \quad v(a_1) \geq \cdots \geq v(a_{\operatorname{ht} H/d}).$$

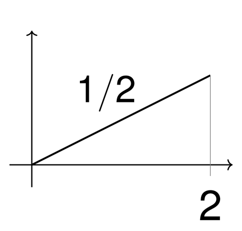
Let $\operatorname{Hdg}^{\text{int}}(H, \iota)\colon [0, \operatorname{ht} H/d] \to [0, \dim H/d]$ be the polygon with slope $v(a_i)$ on $[i-1, i]$. Then ([BM23]):

$$\operatorname{HN}(H[p], \iota) \leq \operatorname{Hdg}^{\text{int}}(H, \iota).$$

Example. $F|\mathbb{Q}_p$ quadratic ramified, $\operatorname{ht} H = 4$, $\dim H = 2$, $\pi \mapsto \pi_1, \pi_2 \in \mathcal{O}_K$.

$$\pi \circlearrowright \omega_H \quad \approx \quad \begin{pmatrix} \pi_1 & 0 \\ 0 & \pi_1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \pi_1 & m \\ 0 & \pi_2 \end{pmatrix} \text{ with } v(m) \geq \frac{1}{2} \quad \text{or} \quad 0 \leq v(m) \leq \frac{1}{2}$$

$\operatorname{Hdg}^{\text{int}}(H, \iota) =$



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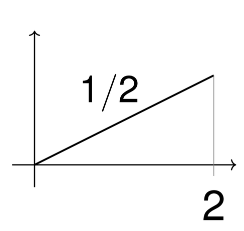

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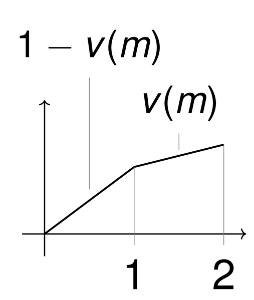

References

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[Ma22] A. Marrama, *Hodge–Newton filtration for p-divisible groups with ramified endomorphism structure*; Documenta Mathematica 27 : 1805-1863, 2022.
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