Barsotti-Tate groups with ramified endomorphism structure

Andrea Marrama

École Polytechnique

Journées Arithmétiques
Nancy, 3 July 2023

Introduction

In joint work with Stéphane Bijakowski (arXiv:2303.06166), we associate a new invariant, called the *integral Hodge polygon*, to objects as in the title.

Introduction

In joint work with Stéphane Bijakowski (arXiv:2303.06166), we associate a new invariant, called the *integral Hodge polygon*, to objects as in the title.

The aim of this talk is to introduce this invariant focusing on a particular example of these objects.

Dieudonné theory

Setup:

- k perfect field of characteristic $p > 0$;
- $W(k)$ Witt vectors with coefficients in k;
- $\sigma\colon W(k) \to W(k)$ Frobenius lift.

Dieudonné theory

Setup:

k perfect field of characteristic $p > 0$;

$W(k)$ Witt vectors with coefficients in k;

$\sigma \colon W(k) \to W(k)$ Frobenius lift.

$$(\text{Barsotti-Tate groups over } k)^{op} \xrightarrow{\sim} (\text{Dieudonné modules over } k)$$
$$H \mapsto (\mathbb{D}, \varphi)$$

Here:

$\mathbb{D}$ finite free $W(k)$-module;

$\varphi \colon \mathbb{D} \to \mathbb{D}$ injective, σ-linear, satisfying $p\mathbb{D} \subseteq \varphi\mathbb{D}$;

height of H: $\operatorname{ht} H = \operatorname{rk}_{W(k)} \mathbb{D} \in \mathbb{N}$;

invariant differentials of H: $\omega \cong \mathbb{D}/\varphi\mathbb{D}$, finite k-vector-space.

Example in special fibre

Let H be the Barsotti-Tate group over k corresponding to:

$$\mathbb{D} = W(k)^6, \qquad \varphi = \begin{pmatrix} 0 & p & 0 & & & \\ 0 & 0 & p & & \mathbf{0} & \\ 1 & 0 & 0 & & & \\ & & & 0 & 0 & p \\ & \mathbf{0} & & 1 & 0 & 0 \\ & & & 0 & 1 & 0 \end{pmatrix} \sigma.$$

Example in special fibre

Let H be the Barsotti-Tate group over k corresponding to:

$$\mathbb{D} = W(k)^6, \qquad \varphi = \begin{pmatrix} 0 & p & 0 & & & \\ 0 & 0 & p & & \mathbf{0} & \\ 1 & 0 & 0 & & & \\ & & & 0 & 0 & p \\ & \mathbf{0} & & 1 & 0 & 0 \\ & & & 0 & 1 & 0 \end{pmatrix} \sigma.$$

We have:

$\operatorname{ht} H = 6$;

$\dim H := \dim_k \omega = \dim_k \mathbb{D}/\varphi\mathbb{D} = 3$.

Example in special fibre

Let H be the Barsotti-Tate group over k corresponding to:

$$\mathbb{D} = W(k)^6, \qquad \varphi = \begin{pmatrix} 0 & p & 0 & & & \\ 0 & 0 & p & & \mathbf{0} & \\ 1 & 0 & 0 & & & \\ & & & 0 & 0 & p \\ & \mathbf{0} & & 1 & 0 & 0 \\ & & & 0 & 1 & 0 \end{pmatrix} \sigma.$$

We have:

$$\operatorname{ht} H = 6;$$

$$\dim H := \dim_k \omega = \dim_k \mathbb{D}/\varphi\mathbb{D} = 3.$$

Let us put some *ramified endomorphism structure* on H.

Ramified endomorphism structure

More setup:

$F := \mathbb{Q}_p(\pi)$ with $\pi^3 = p$;

$\mathcal{O}_F \subseteq F$ ring of integers;

$d := [F : \mathbb{Q}_p] = 3$.

Ramified endomorphism structure

More setup:

$F := \mathbb{Q}_p(\pi)$ with $\pi^3 = p$;

$\mathcal{O}_F \subseteq F$ ring of integers;

$d := [F : \mathbb{Q}_p] = 3$.

We obtain an action of $\mathcal{O}_F$ on H by letting π act on $(\mathbb{D}, \varphi)$ via:

$$[\pi] = \begin{pmatrix} 0 & 0 & p & & & \\ 1 & 0 & 0 & & \mathbf{0} & \\ 0 & 1 & 0 & & & \\ & & & 0 & 0 & p \\ & \mathbf{0} & & 1 & 0 & 0 \\ & & & 0 & 1 & 0 \end{pmatrix}.$$

Note: $[\pi] \circ \varphi = \varphi \circ [\pi]$, $\quad [\pi]^3 = p \cdot \mathrm{id}_{\mathbb{D}}$, $\quad \mathbb{Z}_p$ acts linearly.

Ramified endomorphism structure

More setup:

$F := \mathbb{Q}_p(\pi)$ with $\pi^3 = p$;

$\mathcal{O}_F \subseteq F$ ring of integers;

$d := [F : \mathbb{Q}_p] = 3$.

We obtain an action of $\mathcal{O}_F$ on H by letting π act on $(\mathbb{D}, \varphi)$ via:

$$[\pi] = \begin{pmatrix} \begin{matrix} 0 & 0 & p \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{matrix} & \Large 0 \\ \Large 0 & \begin{matrix} 0 & 0 & p \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{matrix} \end{pmatrix}.$$

Note: $[\pi] \circ \varphi = \varphi \circ [\pi]$, $\quad [\pi]^3 = p \cdot \mathrm{id}_{\mathbb{D}}$, $\quad \mathbb{Z}_p$ acts linearly.

We would like to lift H to a mixed characteristic base ring.

Grothendieck-Messing theory

Setup:

- $K|\mathbb{Q}_p$ completely valued field with residue field k;
- $v\colon K^\times \to \mathbb{R}$ normalised at $v(p) = 1$;
- $\mathcal{O}_K \subseteq K$ valuation ring;
- assume that K contains a Galois closure of F;
- embeddings $\tau_i\colon F \to K$, $\pi \mapsto \tau_i(\pi)$, $i = 1, 2, 3$.

Grothendieck-Messing theory

Setup:

$K|\mathbb{Q}_p$ completely valued field with residue field k;

$v\colon K^\times \to \mathbb{R}$ normalised at $v(p) = 1$;

$\mathcal{O}_K \subseteq K$ valuation ring;

assume that K contains a Galois closure of F;

embeddings $\tau_i\colon F \to K$, $\pi \mapsto \tau_i(\pi)$, $i = 1, 2, 3$.

$$(\text{Lifts of } H \text{ to } \mathcal{O}_K)^{op} \xrightarrow{\sim} (\mathcal{O}_K\text{-direct-summands of } \mathbb{D} \otimes \mathcal{O}_K \ (\dots))$$
$$\tilde{H} \mapsto \tilde{\omega}$$

Here:

$\tilde{\omega}$: invariant differentials of $\tilde{H}$, free $\mathcal{O}_K$-module, $\operatorname{rk} \tilde{\omega} = \dim H$;

$(\dots)$ stands for: $\tilde{\omega} \otimes k \cong p\varphi^{-1}\mathbb{D}/p\mathbb{D} \subseteq \mathbb{D}/p\mathbb{D}$.

Example in mixed characteristic

Let $c \in K$ with $0 < v(c) < \frac{1}{2}$, write $\mathbb{D} \otimes \mathcal{O}_K = \bigoplus_{i=1}^{6} \mathcal{O}_K \cdot f_i$.

Let H_c be the lift of H to $\mathcal{O}_K$ corresponding to:

$$\omega_c := < f_2 + cf_4,\ f_3 + cf_5,\ \frac{p}{c} f_1 + f_6 > \subseteq \mathbb{D} \otimes \mathcal{O}_K.$$

Example in mixed characteristic

Let $c \in K$ with $0 < v(c) < \frac{1}{2}$, write $\mathbb{D} \otimes \mathcal{O}_K = \bigoplus_{i=1}^{6} \mathcal{O}_K \cdot f_i$.

Let H_c be the lift of H to $\mathcal{O}_K$ corresponding to:

$$\omega_c := < f_2 + cf_4,\ f_3 + cf_5,\ \frac{p}{c} f_1 + f_6 > \subseteq \mathbb{D} \otimes \mathcal{O}_K.$$

Since $\omega_c \subseteq \mathbb{D} \otimes \mathcal{O}_K$ is stable under $[\pi]$, the $\mathcal{O}_F$-action on H lifts to an $\mathcal{O}_F$-action on H_c.

We obtain a family of Barsotti-Tate groups H_c over $\mathcal{O}_K$ with an action $\iota_c \colon \mathcal{O}_F \to \mathrm{End}(H_c)$, lifting H and its $\mathcal{O}_F$-action.

Example in mixed characteristic

Let $c \in K$ with $0 < v(c) < \frac{1}{2}$, write $\mathbb{D} \otimes \mathcal{O}_K = \bigoplus_{i=1}^{6} \mathcal{O}_K \cdot f_i$.

Let H_c be the lift of H to $\mathcal{O}_K$ corresponding to:

$$\omega_c := < f_2 + cf_4,\ f_3 + cf_5,\ \frac{p}{c} f_1 + f_6 > \subseteq \mathbb{D} \otimes \mathcal{O}_K.$$

Since $\omega_c \subseteq \mathbb{D} \otimes \mathcal{O}_K$ is stable under $[\pi]$, the $\mathcal{O}_F$-action on H lifts to an $\mathcal{O}_F$-action on H_c.

We obtain a family of Barsotti-Tate groups H_c over $\mathcal{O}_K$ with an action $\iota_c \colon \mathcal{O}_F \to \mathrm{End}(H_c)$, lifting H and its $\mathcal{O}_F$-action.

Let us attach some invariants to (H_c, ι_c) and see if they record some variation along the family.

The Pappas-Rapoport polygon

We have an (eigenspace) decomposition:

$$\omega_c \otimes_{\mathcal{O}_K} K = \bigoplus_{i=1}^{3} \omega_{c,K,i}$$

such that $[\pi]$ acts as $\tau_i(\pi)$ on $\omega_{c,K,i}$.

In fact: $\dim_K \omega_{c,K,i} = 1$ for all i's.

The Pappas-Rapoport polygon

We have an (eigenspace) decomposition:

$$\omega_c \otimes_{\mathcal{O}_K} K = \bigoplus_{i=1}^{3} \omega_{c,K,i}$$

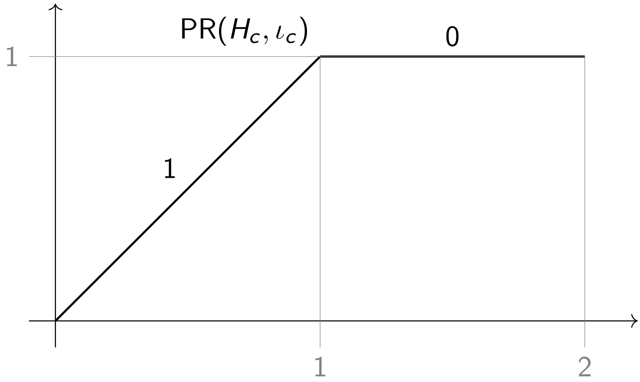
such that $[\pi]$ acts as $\tau_i(\pi)$ on $\omega_{c,K,i}$.

In fact: $\dim_K \omega_{c,K,i} = 1$ for all i's.

$$\mathrm{PR}(H_c, \iota_c) := \frac{1}{d} \sum_i (\underbrace{\overbrace{1, \ldots, 1}^{\dim_K \omega_{c,K,i}}, 0, \ldots, 0}_{\mathrm{ht}\, H/d}) = (1, 0)$$

The facts that $\mathrm{ht}\, H/d \in \mathbb{N}$ and that $\dim_K \omega_{c,K,i} \leq \mathrm{ht}\, H/d$ are general phenomena.

Picture



The integral Hodge polygon

$[\pi]\colon \omega_c \longrightarrow \omega_c$ is an injective map of free $\mathcal{O}_K$-modules. Thus:

$$[\pi] \approx \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_{dim_H} \end{pmatrix} \text{ for suitable bases of } \omega_c,$$

the a_j's uniquely determined up to units and permutations.

In fact:

$$\begin{aligned} f_2 + cf_4 &=: e_1 \mapsto e_2 \\ f_3 + cf_5 &=: e_2 \mapsto ce_3 \\ \frac{p}{c} f_1 + f_6 &=: e_3 \mapsto \frac{p}{c} e_1 \end{aligned}$$

The integral Hodge polygon

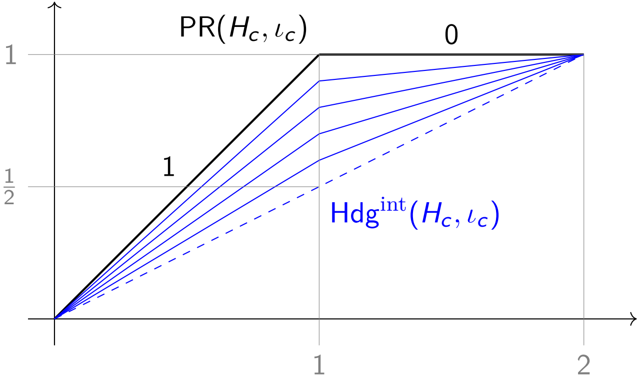
$$[\pi] \colon \omega_c \longrightarrow \omega_c, \qquad [\pi] \approx \begin{pmatrix} \frac{p}{c} & & \\ & c & \\ & & 1 \end{pmatrix}$$

$$\mathrm{Hdg}^{\mathrm{int}}(H_c, \iota_c) := (\underbrace{\text{valuations of non units } a_j\text{'s in} \geq \text{order}, 0, \ldots, 0}_{\text{ht}\, H/d})$$

$$= \left(v\left(\frac{p}{c}\right), v(c)\right) = (1 - v(c), v(c))$$

The fact that $\#\{\text{ non units } a_j\text{'s }\} \leq \text{ht}\, H/d$ is a general phenomenon.

Picture



$\mathrm{PR}(H_c, \iota_c)$
0
1
1
$\frac{1}{2}$
$\mathrm{Hdg}^{\mathrm{int}}(H_c, \iota_c)$
1
2

In general

Let F be a finite, totally ramified extension of $\mathbb{Q}_p$.

Let (H, ι) be a p-divisible group over $\mathcal{O}_K$ with an action of $\mathcal{O}_F$. Then $\mathrm{PR}(H, \iota)$ and $\mathrm{Hdg}^{\mathrm{int}}(H, \iota)$ have the same end point

$$\left(\frac{\mathrm{ht}\, H}{d}, \frac{\dim H}{d}\right)$$

and $\mathrm{PR}(H, \iota)$ lies above $\mathrm{Hdg}^{\mathrm{int}}(H, \iota)$.

In general

Let F be a finite, totally ramified extension of $\mathbb{Q}_p$.

Let (H, ι) be a p-divisible group over $\mathcal{O}_K$ with an action of $\mathcal{O}_F$.
Then $\mathrm{PR}(H, \iota)$ and $\mathrm{Hdg}^{\mathrm{int}}(H, \iota)$ have the same end point

$$\left(\frac{\mathrm{ht}\, H}{d}, \frac{\dim H}{d}\right)$$

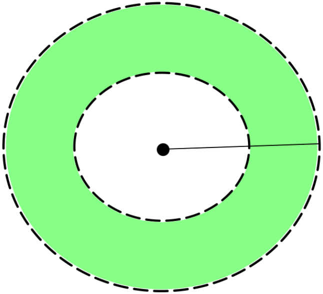
and $\mathrm{PR}(H, \iota)$ lies above $\mathrm{Hdg}^{\mathrm{int}}(H, \iota)$.

If (H, ι) is a family of objects over a p-adic analytic space, then:

- $\mathrm{PR}(H, \iota)$ is locally constant;
- $\mathrm{Hdg}^{\mathrm{int}}(H, \iota)$ varies continuously below $\mathrm{PR}(H, \iota)$.

Geometric picture

In our example, the family (H_c, ι_c) is defined over a p-adic open annulus: the distance from the center is measured by $p^{-v(c)}$ and it is detected by $\mathrm{Hdg}^{\mathrm{int}}(H_c, \iota_c)$.



Where to find this in nature: integral models of Shimura varieties, Rapoport-Zink spaces, . . .

Thank you for your attention!