RTG 2553:
Symmetries and classifying spaces:
Analytic, arithmetic and derived

UNIVERSITÄT DUISBURG ESSEN

Open-Minded

Hodge–Newton filtration for p-divisible groups with ramified endomorphism structure

Andrea Marrama

Doctoral Researcher (04/2020-09/2020),
Funding: other
Master ALGANT: Leiden, Duisburg-Essen

1. Advisor: Prof. Dr. Ulrich Görtz,

Essen Seminar for Algebraic Geometry and Arithmetic

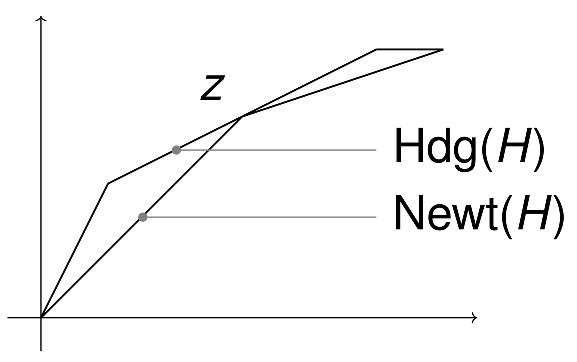
Where does the problem come from?

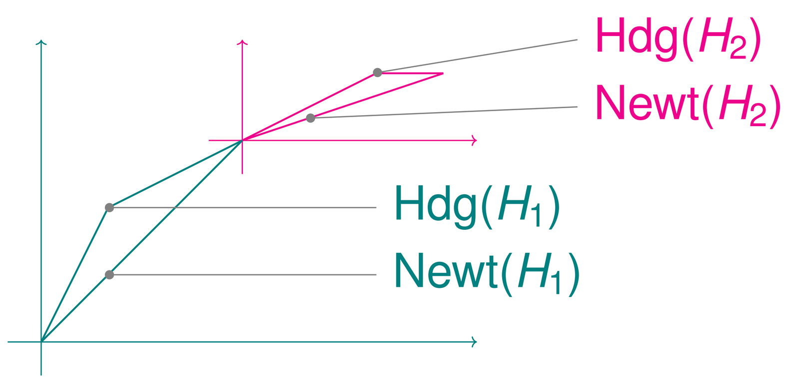
Classical invariants have been attached to a p-divisible group H over k:

- the *Newton polygon* $\mathrm{Newt}(H)$;
- the *Hodge polygon* $\mathrm{Hdg}(H)$.

These invariants tell something about the structure of H, e.g.:

- if $H = \mathcal{A}[p^\infty]$ for an abelian variety $\mathcal{A}$ over k, then $\mathrm{Newt}(H)$ detects whether $\mathcal{A}$ is supersingular, ordinary, etc.;
- if a break point z of $\mathrm{Newt}(H)$ lies on $\mathrm{Hdg}(H)$, then there exists a corresponding *Hodge–Newton decomposition* $H = H_1 \oplus H_2$ (Katz 1979).





Suppose that H is endowed with additional endomorphism structure $\iota\colon \mathcal{O}_F \to \mathrm{End}(H)$. Then:

- the Newton polygon and the Hodge polygon can be refined to new invariants $\mathrm{Newt}(H,\iota)$ and $\mathrm{Hdg}(H,\iota)$ taking the $\mathcal{O}_F$-action into account;
- the Hodge–Newton decomposition with respect to the refined invariants upgrades to an "ι-equivariant" decomposition $(H,\iota) = (H_1,\iota_1) \oplus (H_2,\iota_2)$ (Mantovan–Viehmann '10, Bijakowski–Hernandez '17).

Question. What can be said for p-divisible groups over $\mathcal{O}_K$?

Background and notation

- p a prime number;
- k a perfect field of characteristic p, e.g. $k = \mathbb{F}_p$;
- $\mathcal{O}_K$ a complete discrete valuation ring with residue field k and fraction field K of characteristic 0, e.g. $\mathcal{O}_K = \mathbb{Z}_p$;
- a *p-divisible group* over k (respectively $\mathcal{O}_K$) is an inductive system of p-power-torsion finite (flat) group schemes over k (respectively $\mathcal{O}_K$) satisfying a certain regularity, e.g.:
  - the p-power roots of unity $\mu_{p^\infty} = (\mu_{p^i})_{i\geq 1}$,
  - the p-power-torsion $\mathcal{A}[p^\infty] = (\mathcal{A}[p^i])_{i\geq 1}$ of an abelian scheme $\mathcal{A}$;
- the endomorphisms of a p-divisible group H always form a $\mathbb{Z}_p$-algebra $\mathrm{End}(H)$;
- let $F|\mathbb{Q}_p$ be a finite field extension with ring of integers $\mathcal{O}_F$;
- a p-divisible group with *endomorphism structure* for $\mathcal{O}_F$ is a pair (H,ι) consisting of a p-divisible group H and a map of $\mathbb{Z}_p$-algebras $\iota\colon \mathcal{O}_F \to \mathrm{End}(H)$.

Main result

Let H be a p-divisible group over $\mathcal{O}_K$ with endomorphism structure $\iota\colon \mathcal{O}_F \to \mathrm{End}(H)$.
One may consider suitable invariants $\mathrm{Newt}(H,\iota)$ and $\mathrm{Hdg}(H,\iota)$ as before.

If a break point of $\mathrm{Newt}(H,\iota)$ lies on $\mathrm{Hdg}(H,\iota)$, then in general one does not have a decomposition of (H,ι), however one can still obtain a *Hodge–Newton filtration*.

Theorem ([Ma22]). If a break point z of $\mathrm{Newt}(H,\iota)$ lies on $\mathrm{Hdg}(H,\iota)$, then there exists a unique subobject $(H_1,\iota_1) \subseteq (H,\iota)$ such that:

- $\mathrm{Newt}(H_1,\iota_1)$ and $\mathrm{Hdg}(H_1,\iota_1)$ equal respectively the part of $\mathrm{Newt}(H,\iota)$ and $\mathrm{Hdg}(H,\iota)$ between the origin and z;
- if (H_2,ι_2) denotes the quotient of (H,ι) by (H_1,ι_1), then $\mathrm{Newt}(H_2,\iota_2)$ and $\mathrm{Hdg}(H_2,\iota_2)$ equal respectively the rest of $\mathrm{Newt}(H,\iota)$ and $\mathrm{Hdg}(H,\iota)$ after z (up to a shift of coordinates setting the origin in z).

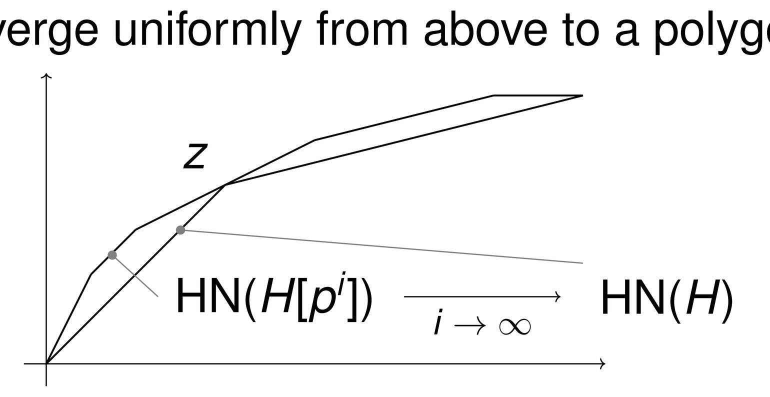
The result was previously know only for $F|\mathbb{Q}_p$ unramified (Mantovan–Viehmann '10, Shen '13, Hong '19).

Remark. If $F|\mathbb{Q}_p$ is unramified, one may define $\mathrm{Newt}(H,\iota)$ and $\mathrm{Hdg}(H,\iota)$ as those of the reduction of (H,ι) to the residue field k of $\mathcal{O}_K$. In general, this only works for $\mathrm{Newt}(H,\iota)$ but extra care is needed for $\mathrm{Hdg}(H,\iota)$.

Sketch of proof

The proof makes use of the *Harder–Narasimhan theory* for finite flat group schemes over $\mathcal{O}_K$ developed by Fargues:

to the p-power-torsion parts $H[p^i]$ of H are associated a sequence of *Harder–Narasimhan polygons* $\mathrm{HN}(H[p^i])$, $i \geq 1$, which converge uniformly from above to a polygon $\mathrm{HN}(H)$.



In the situation of the theorem, one can show that z is also a break point of $\mathrm{HN}(H)$.

Key Lemma. $\mathrm{HN}(H[p])$ lies between $\mathrm{HN}(H)$ and $\mathrm{Hdg}(H,\iota)$.

Thus, in the situation of the theorem z also lies on $\mathrm{HN}(H[p])$. To conclude, we prove the following.

Proposition. If z is a break point of $\mathrm{HN}(H)$ lying on $\mathrm{HN}(H[p])$, then there exists a corresponding filtration $H_1 \subseteq H$.

One finally checks that $H_1 \subseteq H$ is ι-stable and satisfies the conditions of the theorem.

Where does the problem go?

Taken in families, p-divisible groups with endomorphism structure (or more general additional structures) give rise to interesting geometric objects called *Rapoport–Zink spaces*.

1. The Hodge–Newton filtration allows to relate certain "non basic" Rapoport–Zink spaces $\mathcal{M}_G$ to simpler ones $\mathcal{M}_M$, passing through a space $\mathcal{M}_P$ parametrising the filtered objects. This ultimately gives new insight into the *Harris–Viehmann conjecture*, which is part of the local Langlands program.

$$\begin{tikzcd} & \mathcal{M}_P \arrow[dl] \arrow[dr, "\sim"] & \\ \mathcal{M}_M \arrow[ur, bend left] & & \mathcal{M}_G \end{tikzcd}$$

2. An alternative way to generalise the definition of the Hodge polygon from the unramified case to the general situation is considered in [BM23], leading to the notion of *integral Hodge polygon*. This:
   - provides a finer bound than $\mathrm{Hdg}(H,\iota)$ for $\mathrm{HN}(H[p])$ as in the key lemma;
   - defines a continuous function on $\mathcal{M}_G$ describing its geometry in the ramified case.

References

[Ma22] A. Marrama, *Hodge–Newton filtration for p-divisible groups with ramified endomorphism structure*; Documenta Mathematica 27 : 1805-1863, 2022.

[BM23] S. Bijakowski, A. Marrama, *The integral Hodge polygon for p-divisible groups with endomorphism structure*; preprint, arXiv:2303.06166.

Background photo: Timo Bobert, Max Greve

Funded by DFG Deutsche Forschungsgemeinschaft German Research Foundation