Transfer principles for Galois cohomology and Serre's conjecture II

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Abstract

In this article, we prove three transfer principles for the cohomological dimension of fields. Given a fixed field K with finite cohomological dimension δ, they allow to:

- construct countable subfields of K with cohomological dimension $\leq \delta$;
- construct totally ramified extensions of K with cohomological dimension $\leq \delta - 1$ when K is a complete discrete valuation field with countable residue field;
- construct algebraic extensions of K with cohomological dimension $\leq \delta - 1$ and satisfying a norm condition when K is countable.

We then apply these results to Serre's conjecture II and to some variants for fields of any cohomological dimension that are inspired by conjectures of Kato and Kuzumaki. In particular, we prove that Serre's conjecture II for characteristic 0 fields implies Serre's conjecture II for positive characteristic fields.

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1. Introduction

Kato and Kuzumaki's conjectures

In 1986, in the article [KK86], Kato and Kuzumaki stated a set of conjectures which aimed at giving a diophantine characterization of cohomological dimension of fields. For this purpose, they introduced some properties of fields which are variants of the classical C_i-property and which involve Milnor K-theory and projective hypersurfaces of small degree. They hoped that those properties would characterize fields of small cohomological dimension.

More precisely, fix a field K and two non-negative integers q and i. Let $\mathrm{K}_q^{\mathrm{M}}(K)$ be the q-th Milnor K-group of K. For each finite extension L of K, one can define a norm morphism $N_{L/K} : \mathrm{K}_q^{\mathrm{M}}(L) \to \mathrm{K}_q^{\mathrm{M}}(K)$ (see Section 1.7 of [Kat80]). Thus, if Z is a scheme of finite type over K, one can introduce the subgroup $N_q(Z/K)$ of $\mathrm{K}_q^{\mathrm{M}}(K)$ generated by the images of the norm morphisms $N_{L/K}$ when L runs through the finite extensions of

K such that $Z(L) \neq \emptyset$. One then says that the field K is C_i^q if, for each $n \geq 1$, for each finite extension L of K and for each hypersurface Z in $\mathbb{P}_L^n$ of degree d with $d^i \leq n$, one has $N_q(Z/L) = \mathrm{K}_q^{\mathrm{M}}(L)$. For example, the field K is C_i^0 if, for each finite extension L of K, every hypersurface Z in $\mathbb{P}_L^n$ of degree d with $d^i \leq n$ has a zero-cycle of degree 1. The field K is C_0^q if, for each tower of finite extensions $M/L/K$, the norm morphism $N_{M/L} : \mathrm{K}_q^{\mathrm{M}}(M) \to \mathrm{K}_q^{\mathrm{M}}(L)$ is surjective.

Kato and Kuzumaki conjectured that, for $i \geq 0$ and $q \geq 0$, a perfect field is C_i^q if, and only if, it is of cohomological dimension at most $i+q$. This conjecture generalizes a question raised by Serre in [Ser02] asking whether the cohomological dimension of a C_i-field is at most i. As it was already pointed out at the end of Kato and Kuzumaki's original paper [KK86], Kato and Kuzumaki's conjecture for $i = 0$ follows from the Bloch-Kato conjecture (which has been established by Rost and Voevodsky, cf. [Rio14]): in other words, a perfect field is C_0^q if, and only if, it is of cohomological dimension at most q. However, it turns out that the conjectures of Kato and Kuzumaki are wrong in general. For example, Merkurjev constructed in [Mer91] a field of characteristic 0 and of cohomological dimension 2 which did not satisfy property C_2^0. Similarly, Colliot-Thélène and Madore produced in [CTM04] a field of characteristic 0 and of cohomological dimension 1 which did not satisfy property C_1^0. These counter-examples were all constructed by a method using transfinite induction due to Merkurjev and Suslin.

Higher versions of Serre's conjectures

In the article [ILA20], we introduced several variants of Kato and Kuzumaki's C_i^q properties, by replacing hypersurfaces of low degree by homogeneous spaces of linear algebraic groups. One of those variants can be stated as follows: given a non-negative integer q, we say that a field K has the C_{PHS}^q property if, for each finite extension L of K and for each principal homogeneous space Z under a smooth linear connected algebraic group over L, one has $N_q(Z/L) = K_q^M(L)$. According to the Main Theorem of [ILA20], the C_{PHS}^q property characterizes perfect fields with cohomological $\leq q+1$ and is hence a good replacement for Kato and Kuzumaki's C_1^q property.

In the particular case where $q = 0$, our result recovers the zero-cycle version of Serre's conjecture I, which is a classical theorem by Steinberg [Ste65]:

Theorem 1.1 (Serre's conjecture I). *Let K be a perfect field of cohomological dimension ≤ 1. Then every principal homogeneous space under a connected linear K-group has a rational point.*

The Main Theorem of [ILA20] is therefore in some sense a version of Serre's conjecture I for higher-dimensional fields. It is then natural to ask whether one can find good replacements for the C_2^q property that would characterize fields with cohomological dimension $\leq q+2$, while recovering a version for higher-dimensional fields of the so-called Serre's conjecture II:

Conjecture 1.2 (Serre's conjecture II). *Let K be a field of cohomological dimension ≤ 2. Then every principal homogeneous space under a semisimple simply connected K-group has a rational point.*

Contrary to Serre's conjecture I, this one remains open, although a lot has been done in particular cases, whether specifying the type of the group or the field of definition (cf. the lecture notes [Gil19] for a survey on the topic). For instance, Bayer-Fluckiger and Parimala [BP95] proved it for classical groups (types A, B, C and D with no triality) and groups of type F_4 and G_2 over perfect fields, while the same result was obtained for imperfect fields by Berhuy, Frings and Tignol [BFT07]. Quasi-split groups of exceptional types other than E_8 have been treated by Chernousov [Che03] and Gille [Gil01] (cf. also [Gil19, Thm. 6.0.1]). On the other hand, classical work by Kneser [Kne65a, Kne65b], Bruhat and Tits [BT87], Harder [Har65, Har66], and Chernousov [Che89] yields the conjecture for p-adic fields and totally imaginary number fields, while the work of de Jong, He and Starr [dJHS11] implies the conjecture for function fields of complex algebraic surfaces.

In analogy with our generalization of Serre's conjecture I in [ILA20], we are led to the following conjecture:

Conjecture 1.3 (Higher Serre's conjecture II)**.** *Let K be a field of cohomological dimension $\leq q+2$. Then for every principal homogeneous space Z under a semisimple simply connected K-group we have $N_q(Z/K) = \mathrm{K}_q^{\mathrm{M}}(K)$.*

Note that a converse to this statement will be proved to hold for perfect fields later in the article (Proposition 4.11).

As before, the case where $q = 0$ corresponds to a weakening of the original conjecture, where rational points have been replaced by zero-cycles of degree 1. However, the upshot of the higher version of Serre's conjecture II is a gain in flexibility. Indeed, it allows to work with fields of *arbitrary* cohomological dimension, and hence it allows to argue by focusing on the *fields* involved instead of *groups*. This is in full contrast with the usual approaches to this conjecture and with the methods used in [ILA20].

For instance, given a field K of positive characteristic and cohomological dimension ≤ 2, one can always construct a complete discrete valuation field $\tilde{K}$ of characteristic 0 with residue field K. One would then like to be able to deduce Serre's conjecture II for K (whether with rational points or zero-cycles of degree 1) from a statement for $\tilde{K}$. But this is a field of cohomological dimension ≤ 3, so one cannot apply the classical conjecture to $\tilde{K}$. The higher version of the conjecture provides a method to avoid this difficulty.

Transfer principles

In order to exploit the above-mentioned flexibility and to prove some instances of the Higher Serre's conjecture II, we need to construct, given a field of fixed cohomological dimension, some associated fields with prescribed cohomological dimension and suitable additional properties. The main objective of this article is to present the following "transfer principles", which go in this direction, but that are of independent interest.

Theorem A (From uncountable to countable fields, Theorem 3.1)**.**
Let K be a field of characteristic $p \geq 0$ and with cohomological dimension δ. Let K_0 be a countable subfield of K. Then there exists a countable subextension K_∞ of K/K_0 that has cohomological dimension $\leq \delta$.

Theorem B (From positive to zero characteristic, Theorem 3.4)**.**
Let $\tilde{K}$ be a complete discrete valuation field of characteristic 0 with countable residue field

K of cohomological dimension δ. Then there exists a totally ramified extension $\tilde{K}_\dagger/\tilde{K}$ with cohomological dimension δ.

Theorem C (From higher to lower cohomological dimension, Theorem 3.12)**.**
Let $\delta \geq 1$ be an integer, ℓ a prime number and K an ℓ-special countable field of characteristic 0 *and with cohomological dimension δ. For each $x \in K^\times$, there exists an algebraic extension K_x of K that has cohomological dimension $\leq (\delta - 1)$ and such that $x \in N_{L/K}(L^\times)$ for every finite subextension L of K_x/K.*

As a consequence, we also get a "transfer principle" for the norm groups $N_q(Z/K)$:

Theorem D (Transfer principle for norm groups, Theorem 4.1)**.**
Let $m, n, q \geq 0$ be three integers with $n \geq m \geq 1$ and K a field of characteristic 0 and with cohomological dimension $\leq n$. Let Z be a K-variety and assume that there exists a countable subfield K_0 of K and a form Z_0 of Z over K_0 satisfying the following condition:

($\star$) *For every countable subextension L of K/K_0, and for every algebraic extension M/L of cohomological dimension $\leq m$, we have $N_q(Z_{0,M}/M) = \mathrm{K}_q^{\mathrm{M}}(M)$.*

Then $N_{n-m+q}(Z/K) = \mathrm{K}^{\mathrm{M}}_{n-m+q}(K)$.

This result, which follows from Theorems A and C, has some natural consequences on Kato and Kuzumaki's conjectures (Corollary 4.3). A particular case of this is the following statement:

Corollary (Corollary 4.4)**.** *Let K be a field with finite transcendence degree over $\mathbb{Q}$. Assume that, for every $i \geq 1$ and for every algebraic extension L/K of cohomological dimension i, every hypersurface Z in $\mathbb{P}^n_L$ of degree d with $d^i \leq n$ has a zero-cycle of degree 1. Then Kato and Kuzumaki's conjectures hold for K.*

Theorem D also allows to immediately recover, *using completely different techniques*, the theorem of [ILA20] settling the C^q_{PHS} property for perfect fields of cohomological dimension $\leq q + 1$ (Corollary 4.7). But its most beautiful and powerful applications concern Serre's conjecture II and its higher version 1.3.

Applications to Serre's conjecture II and its higher version

The main application of Theorem D is the following conditional result concerning Conjecture 1.3:

Theorem E (Theorem 4.13)**.** *Assume that the classical Serre's conjecture II holds for countable fields of characteristic* 0*. Then for every field K of cohomological dimension $\leq q+2$ and for every principal homogeneous space Z under a semisimple simply connected K-group we have $N_q(Z/K) = \mathrm{K}_q^{\mathrm{M}}(K)$.*

This result describes the behaviour of principal homogeneous spaces under semisimple simply connected groups over fields with bounded cohomological dimension. It turns out that it also holds when one restricts it (together with the assumption on Serre's conjecture II) to semisimple simply connected isotypical groups of a given type. Hence, using the currently known cases of Conjecture 1.2, we get the following *unconditional* corollary:

Corollary (Corollary 4.14). *Let K be a field of cohomological dimension $\leq q+2$. Then for every principal homogeneous space Z under a semisimple simply connected K-group without factors of types E_6 or E_7, we have $N_q(Z/K) = \mathrm{K}_q^{\mathrm{M}}(K)$.*

The last main result of this article is an application of the transfer principles given by Theorems A and B to the classical Serre's conjecture II:

Theorem F (Theorem 4.16). *If Serre's conjecture II holds for countable fields of characteristic 0, then it holds for arbitrary fields.*

Again, this result also holds when one restricts to principal homogeneous spaces under semisimple simply connected isotypical groups of a given type. It therefore allows to *unconditionally* recover Serre's conjecture II for groups without factors of types E_6, E_7, E_8 or trialitarian D_4 over imperfect fields from the analoguous result over perfect fields, which was settled by Bayer-Fluckiger and Parimala in [BP95]. This consequence of Theorem F was already proved in [BFT07] by a completely different method based on a case by case study following the classification of semisimple simply connected groups instead of focusing on fields.

Organization of the article

In section 2, we introduce a certain number of preliminaries and notations regarding the cohomological dimension of fields, Milnor K-theory and the norm groups $N_q(Z/K)$. Section 3 is dedicated to the proofs of the transfer principles for Galois cohomology given by Theorems A, B and C. In Section 4, we deduce the transfer principle for norm groups given by Theorem D, and we explain how to get the applications to Serre's conjecture II and its higher variants given by Theorems E and F.

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2. Preliminaries and notations

2.1 The cohomological dimension of a field

The cohomological dimension $\mathrm{cd}(K)$ of a perfect field K is the cohomological dimension of its absolute Galois group. In other words, it is the smallest integer δ (or ∞ if such an integer does not exist) such that $H^n(K, M) = 0$ for every $n > \delta$ and each finite Galois module M.

It is more complicated to define a good notion for the cohomological dimension of an imperfect field. This was first done by Kato in [Kat82].

Definition 2.1. *Let K be any field.*

(i) Let ℓ be a prime number different from the characteristic of K. The ℓ-cohomological dimension $\mathrm{cd}_\ell(K)$ *and the* separable ℓ-cohomological dimension $\mathrm{sd}_\ell(K)$ *of K are both the ℓ-cohomological dimension of the absolute Galois group of K.*

(ii) (Kato, [Kat82]; Gille, [Gil00]). Assume that K has characteristic $p > 0$. Let Ω^i_K be the i-th exterior product over K of the absolute differential module $\Omega^1_{K/\mathbb{Z}}$ and let $H^{i+1}_p(K)$ be the cokernel of the morphism $\mathfrak{p}^i_K : \Omega^i_K \to \Omega^i_K/d(\Omega^{i-1}_K)$ defined by

$$x\frac{dy_1}{y_1} \wedge ... \wedge \frac{dy_i}{y_i} \mapsto (x^p - x)\frac{dy_1}{y_1} \wedge ... \wedge \frac{dy_i}{y_i} \mod d(\Omega^{i-1}_K),$$

for $x \in K$ and $y_1, ..., y_i \in K^\times$. The p-cohomological dimension $\mathrm{cd}_p(K)$ *of K is the smallest integer δ (or ∞ if such an integer does not exist) such that $[K : K^p] \leq p^\delta$ and $H^{\delta+1}_p(L) = 0$ for all finite extensions L of K. The* separable p-cohomological dimension $\mathrm{sd}_p(K)$ *of K is the smallest integer δ (or ∞ if such an integer does not exist) such that $H^{\delta+1}_p(L) = 0$ for all finite separable extensions L of K.*

(iii) The cohomological dimension $\mathrm{cd}(K)$ *of K is the supremum of all the $\mathrm{cd}_\ell(K)$'s when ℓ runs through all prime numbers. The* separable cohomological dimension $\mathrm{sd}(K)$ *of K is the supremum of all the $\mathrm{sd}_\ell(K)$'s when ℓ runs through all prime numbers.*

For further use, we recall the following classical criterion for cohomological dimension, which follows from [Ser02, I.3.3, Cor. 1, I.4.1, Prop. 21]).

Proposition 2.2. *Let K be any field and let ℓ be a prime number different from the characteristic of K. Then K has ℓ-cohomological dimension $\leq \delta$ if and only if, for every finite separable extension L/K with $[L : K]$ coprime to ℓ and containing a primitive ℓ-th root of unity, $H^{\delta+1}(L, \mu_\ell^{\otimes(\delta+1)}) = 0$.*

2.2 Milnor K-theory

Let K be any field and let q be a non-negative integer. The q-th Milnor K-group of K is by definition the group $\mathrm{K}^{\mathrm{M}}_0(K) = \mathbb{Z}$ if $q = 0$ and:

$$\mathrm{K}^{\mathrm{M}}_q(K) := \underbrace{K^\times \otimes_{\mathbb{Z}} ... \otimes_{\mathbb{Z}} K^\times}_{q \text{ times}} / \langle x_1 \otimes ... \otimes x_q | \exists i, j, i \neq j, x_i + x_j = 1\rangle$$

if $q > 0$. For $x_1, ..., x_q \in K^\times$, the symbol $\{x_1, ..., x_q\}$ denotes the class of $x_1 \otimes ... \otimes x_q$ in $\mathrm{K}^{\mathrm{M}}_q(K)$. More generally, for r and s non-negative integers such that $r + s = q$, there is a natural pairing:

$$\mathrm{K}^{\mathrm{M}}_r(K) \times \mathrm{K}^{\mathrm{M}}_s(K) \to \mathrm{K}^{\mathrm{M}}_q(K)$$

which we will denote $\{\cdot, \cdot\}$.

When L is a finite extension of K, one can construct a norm homomorphism

$$N_{L/K} : \mathrm{K}^{\mathrm{M}}_q(L) \to \mathrm{K}^{\mathrm{M}}_q(K),$$

satisfying the following properties (cf. [Kat80, §1.7] or [GS17, §7.3]):

- For $q = 0$, the map $N_{L/K} : \mathrm{K}^{\mathrm{M}}_0(L) \to \mathrm{K}^{\mathrm{M}}_0(K)$ is given by multiplication by $[L : K]$.
- For $q = 1$, the map $N_{L/K} : \mathrm{K}^{\mathrm{M}}_1(L) \to \mathrm{K}^{\mathrm{M}}_1(K)$ coincides with the usual norm $L^\times \to K^\times$.

- If r and s are non-negative integers such that $r+s=q$, we have $N_{L/K}(\{x,y\}) = \{x, N_{L/K}(y)\}$ for $x \in \mathrm{K}_r^{\mathrm{M}}(K)$ and $y \in \mathrm{K}_s^{\mathrm{M}}(L)$.

- If M is a finite extension of L, we have $N_{M/K} = N_{L/K} \circ N_{M/L}$.

Recall also that Milnor K-theory is endowed with residue maps (cf. [GS17, §7.1]). Indeed, when $\tilde{K}$ is a complete discrete valuation field with ring of integers $\mathcal{O}_{\tilde{K}}$ and residue field K, there exists a unique residue morphism:

$$\partial : \mathrm{K}_q^{\mathrm{M}}(\tilde{K}) \to \mathrm{K}_{q-1}^{\mathrm{M}}(K)$$

such that, for each uniformizer π and for all units $\tilde{u}_2, ..., \tilde{u}_q \in \mathcal{O}_{\tilde{K}}^{\times}$ whose images in K are denoted $u_2, \dots, u_q$, one has:

$$\partial(\{\pi, \tilde{u}_2, \dots, \tilde{u}_q\}) = \{u_2, \dots, u_q\}.$$

The kernel of ∂ is the subgroup $U_q(\tilde{K})$ of $\mathrm{K}_q^{\mathrm{M}}(\tilde{K})$ generated by symbols of the form $\{x_1, \dots, x_q\}$ with $x_1, \dots, x_q \in \mathcal{O}_{\tilde{K}}^{\times}$. We denote by $U_q^i(\tilde{K})$ the subgroup of $\mathrm{K}_q^{\mathrm{M}}(\tilde{K})$ generated by those symbols that lie in $U_q(K)$ and that are of the form $\{x_1, \dots, x_q\}$ with $x_1 \in 1 + \pi^i \mathcal{O}_{\tilde{K}}$ and $x_2, \dots, x_q \in \tilde{K}^{\times}$.

Let us finally recall the description given by Kato of the K-theory modulo p of $\tilde{K}$ when K is of characteristic $p > 0$ and $\tilde{K}$ contains a primitive p-th root of unity ζ_p. For that purpose, we set $\mathrm{k}_q^{\mathrm{M}}(\tilde{K}) := \mathrm{K}_q^{\mathrm{M}}(\tilde{K})/p$ and $u_q^i(\tilde{K}) := U_q^i(\tilde{K})/p$. We fix a uniformizer $\pi \in \tilde{K}$, and for each $a \in K$, we denote by $\tilde{a}$ a lift of a to $\mathcal{O}_{\tilde{K}}$. Finally, we set $\epsilon_{\tilde{K}} := \frac{v_{\tilde{K}}(p)p}{p-1}$, where $v_{\tilde{K}}$ denotes the valuation on $\tilde{K}$. Then, for every integer $q \geq 0$, we have the following isomorphisms (cf. [Kat82, p. 218]):

$$\begin{aligned} \rho_0^{q+1} : \mathrm{k}_{q+1}^{\mathrm{M}}(K) \oplus \mathrm{k}_q^{\mathrm{M}}(K) &\to \mathrm{k}_{q+1}^{\mathrm{M}}(\tilde{K})/u_{q+1}^1(\tilde{K}), \\ (\{y_1, y_2, \dots, y_{q+1}\}, 0) &\mapsto \{\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_{q+1}\}, \\ (0, \{y_1, y_2, \dots, y_q\}) &\mapsto \{\pi, \tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_q\}; \end{aligned} \tag{1}$$

for every $0 < i < \epsilon_{\tilde{K}}$ such that $(i,p) = 1$,

$$\begin{aligned} \rho_i^{q+1} : \Omega_K^q &\to u_{q+1}^i(\tilde{K})/u_{q+1}^{i+1}(\tilde{K}), \\ x \frac{dy_1}{y_1} \wedge \frac{dy_2}{y_2} \wedge \cdots \wedge \frac{dy_q}{y_q} &\mapsto \{1 + \pi^i \tilde{x}, \tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_q\}; \end{aligned} \tag{2}$$

for every $0 < i < \epsilon_{\tilde{K}}$ such that $p|i$;

$$\begin{aligned} \rho_i^{q+1} : \Omega_K^q/\mathrm{Ker}(d) \oplus \Omega_K^{q-1}/\mathrm{Ker}(d) &\to u_{q+1}^i(\tilde{K})/u_{q+1}^{i+1}(\tilde{K}), \\ \left(x \frac{dy_1}{y_1} \wedge \frac{dy_2}{y_2} \wedge \cdots \wedge \frac{dy_q}{y_q}, 0\right) &\mapsto \{1 + \pi^i \tilde{x}, \tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_q\}, \\ \left(0, x \frac{dy_1}{y_1} \wedge \frac{dy_2}{y_2} \wedge \cdots \wedge \frac{dy_{q-1}}{y_{q-1}}\right) &\mapsto \{\pi, 1 + \pi^i \tilde{x}, \tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_{q-1}\}; \end{aligned} \tag{3}$$

for $i = \epsilon_{\tilde{K}}$,

$$\begin{aligned} \rho_i^{q+1} : H_p^{q+1}(K) \oplus H_p^q(K) &\to u_{q+1}^i(\tilde{K}), \\ \left(x\frac{dy_1}{y_1} \wedge \frac{dy_2}{y_2} \wedge \cdots \wedge \frac{dy_q}{y_q}, 0\right) &\mapsto \{1+(\zeta_p-1)^p\tilde{x}, \tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_q\}, \\ \left(0, x\frac{dy_1}{y_1} \wedge \frac{dy_2}{y_2} \wedge \cdots \wedge \frac{dy_{q-1}}{y_{q-1}}\right) &\mapsto \{\pi, 1+(\zeta_p-1)^p\tilde{x}, \tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_{q-1}\}. \end{aligned} \tag{4}$$

2.3 The norm groups $N_q(Z/K)$

We conclude these preliminaries by studying the following norm groups, which were originally defined by Kato and Kuzumaki in [KK86]:

Definition 2.3. *Let K be a field and let q be a non-negative integer. For each K-scheme Z of finite type, we define the* norm group *$N_q(Z/K)$ as the subgroup of $\mathrm{K}_q^{\mathrm{M}}(K)$ generated by the images of the maps $N_{L/K} : \mathrm{K}_q^{\mathrm{M}}(L) \to \mathrm{K}_q^{\mathrm{M}}(K)$ when L runs through the finite extensions of K such that $Z(L) \neq \emptyset$.*

In this article, we will use the following elementary lemmas about the norm groups.

Lemma 2.4. *Let K be a field and let Y and Z be two K-varieties. Assume that $N_q(Y_L/L) = N_q(Z_L/L) = \mathrm{K}_q^{\mathrm{M}}(L)$ for every finite extension L of K. Then $N_q(Y \times_K Z/K) = \mathrm{K}_q^{\mathrm{M}}(K)$.*

Proof. Fix $z \in \mathrm{K}_q^{\mathrm{M}}(K)$. Since $N_q(Y/K) = \mathrm{K}_q^{\mathrm{M}}(K)$, there exist finite extensions $L_1, \dots, L_r$ of K and elements $z_1 \in \mathrm{K}_q^{\mathrm{M}}(L_1), \dots, z_r \in \mathrm{K}_q^{\mathrm{M}}(L_r)$ such that:

$$\begin{cases} \forall\, i,\ Y(L_i) \neq \emptyset, \\ z = \sum_i N_{L_i/K}(z_i). \end{cases}$$

Now, for each i, we have $N_q(Z_{L_i}/L_i) = \mathrm{K}_q^{\mathrm{M}}(L_i)$, and hence there exist finite extensions $L_{i,1}, \dots, L_{i,r_i}$ of L_i and elements $z_{i,1} \in \mathrm{K}_q^{\mathrm{M}}(L_{i,1}), \dots z_{i,r_i} \in \mathrm{K}_q^{\mathrm{M}}(L_{i,r_i})$ such that:

$$\begin{cases} \forall\, j,\ Z(L_{i,j}) \neq \emptyset, \\ z_i = \sum_j N_{L_{i,j}/L_i}(z_{i,j}). \end{cases}$$

Hence $z = \sum_{i,j} N_{L_{i,j}/K}(z_{i,j})$, and since both Y and Z have points in all the $L_{i,j}$'s, we get $z \in N_q(Y \times_K Z/K)$. ☺

Lemma 2.5. *Let K be a field, $q \geq 0$ an integer and Z a K-variety. For every prime number p, fix an extension K_p/K corresponding to a p-Sylow subgroup of $\mathrm{Gal}\,(K^{\mathrm{sep}}/K)$. Assume that $N_q(Z_{K_p}/K_p) = \mathrm{K}_q^{\mathrm{M}}(K_p)$ for every prime p. Then $N_q(Z/K) = \mathrm{K}_q^{\mathrm{M}}(K)$.*

Proof. Take $z \in \mathrm{K}_q^{\mathrm{M}}(K)$ and fix a prime number p. Since $N_q(Z_{K_p}/K_p) = \mathrm{K}_q^{\mathrm{M}}(K_p)$, we can find finite extensions $L_1, \dots, L_r$ of K_p such that

$$\begin{cases} \forall\, i,\ Z(L_i) \neq \emptyset, \\ z \in \langle N_{L_i/K_p}(\mathrm{K}_q^{\mathrm{M}}(L_i)) : 1 \leq i \leq r \rangle. \end{cases}$$

We can then also find a finite subextension K'_p of K_p/K together with finite extensions $L'_1, \dots, L'_r$ of K'_p, with $L'_i \subset L_i$, such that $z \in \langle N_{L'_i/K'_p}(\mathrm{K}_q^{\mathrm{M}}(L'_i)) : 1 \leq i \leq r\rangle$ and $Z(L'_i) \neq \emptyset$ for every $1 \leq i \leq r$. In particular:

$$[K'_p : K] \cdot z \in \langle N_{L'_i/K}(\mathrm{K}_q^{\mathrm{M}}(L'_i)) : 1 \leq i \leq r\rangle \subset N_q(Z/K).$$

Since $[K'_p : K]$ is not divisible by p and since the previous argument can be done for every prime number p, we deduce that $z \in N_q(Z/K)$. ☺

3. Transfer principles for cohomological dimension

In this section, we prove the three transfer principles for Galois cohomology given by Theorems A, B and C. These results allow to move from uncountable to countable fields, from positive characteristic to characteristic 0 fields, and from fields with higher cohomological dimension to lower cohomological dimension.

3.1 From uncountable to countable fields

Theorem 3.1. *Let K be a field of characteristic $p \geq 0$ and with cohomological dimension δ. Let K_0 be a countable subfield of K. Then there exists a countable subextension K_∞ of K/K_0 that has cohomological dimension $\leq \delta$.*

Remark 3.2. By adapting the subsequent proof of Theorem 3.1 and by using transfinite induction, it is possible to replace K_0 by any infinite subfield of K in the statement and conclude that there exists a subextension K_∞ of K/K_0 that has the same cardinality as K_0 and that has cohomological dimension $\leq \delta$.

In order to prove Theorem 3.1, we need the following result in positive characteristic.

Lemma 3.3. *Let K be a field of characteristic $p > 0$ such that $[K : K^p] = p^\delta$ and let $\alpha_1, \dots, \alpha_\delta$ be a p-basis. Let K_0 be a countable subfield of K. Then there exists a countable subextension $\Theta(K_0)$ of K/K_0 with the same p-basis as K.*

Proof. For each $x \in K$, write

$$x = \sum_{0 \leq i_1, \dots, i_\delta < p} \lambda_{x,i_1,\dots,i_\delta}^p \alpha_1^{i_1} \cdots \alpha_\delta^{i_\delta},$$

with $\lambda_{x,i_1,\dots,i_\delta} \in K$. We then define, for an arbitrary subfield L of K containing $\alpha_1, \dots, \alpha_\delta$,

$$\theta(L) := L(\lambda_{x,i_1,\dots,i_\delta} : 0 \leq i_1, \dots, i_\delta < p, x \in L).$$

Consider now the field $K_1 := K_0(\alpha_1, \dots, \alpha_\delta)$ and, for $i \geq 1$, define $K_{i+1} := \theta(K_i)$. We claim that

$$\Theta(K_0) := \bigcup_{i \in \mathbb{N}} K_i$$

satisfies the statement of the Lemma. Indeed, by construction, it is a countable subextension of K/K_0. Moreover, for $x \in \Theta(K_0)$, there exists $i_0 \in \mathbb{N}$ such that $x \in K_{i_0}$ and thus, we can always write

$$x = \sum_{0 \leq i_1, \dots, i_\delta < p} \lambda_{x,i_1,\dots,i_\delta}^p \alpha_1^{i_1} \cdots \alpha_\delta^{i_\delta},$$

with $\lambda_{x,i_1,\dots,i_\delta} \in K_{i_0+1} \subset \Theta(K_0)$. This proves that $\alpha_1, \dots, \alpha_\delta$ is a p-basis of $\Theta(K_0)$. ☺

Proof of Theorem 3.1. Fix an algebraic closure $\overline{K}$ of K and, if $p > 0$, a p-basis of K. All fields considered in this proof, in particular composite fields, will be assumed to be subfields of $\overline{K}$. We also fix the following:

- We let $\sigma = (\sigma_1, \sigma_2) : \mathbb{N} \to \mathbb{N}^2$ be a bijection such that $\sigma_1(n) \leq n$ for each $n \in \mathbb{N}$.
- For each countable subextension L of K/K_0, we define $\mathcal{E}_L$ as the set of triples (M, ℓ, a) such that M is a finite extension of L, ℓ is a prime number and a is an element in
$$H_\ell^{\delta+1}(M) := \begin{cases} H^{\delta+1}(M, \mu_\ell^{\otimes(\delta+1)}) & \text{if } \ell \neq p; \\ H_p^{\delta+1}(M) & \text{if } \ell = p. \end{cases}$$
Note that such a set is countable, so we assume that it comes with a given numbering.

Let us now inductively construct an increasing infinite sequence of subfields $(L_i)_{i\geq 1}$ of K in the following way. We set first $L_1 := K_0$. Then, for $i \geq 1$, consider the $\sigma_2(i)$-th term of the set $\mathcal{E}_{L_{\sigma_1(i)}}$, which we denote by (M_i, ℓ_i, a_i). Since $\sigma_1(i) \leq i$, we have $L_{\sigma_1(i)} \subset L_i$. Moreover, since K has cohomological dimension δ, we have $a_i|_{M_iK} = 0$. Hence we may and do define L'_{i+1} as a finitely generated extension of L_i contained in K such that $a_i|_{M_iL'_{i+1}} = 0$, and we set
$$L_{i+1} := \begin{cases} L'_{i+1} & \text{if } p = 0; \\ \Theta(L'_{i+1}) & \text{if } p > 0. \end{cases}$$
We introduce the field:
$$K_\infty := \bigcup_{i\geq 1} L_i.$$
We claim that K_∞ satisfies the conditions of the theorem. Indeed, all the L_i's are countable fields, and hence so is K_∞. Moreover, if $p > 0$, we know by Lemma 3.3 that L_i has the same p-basis as K for $i \geq 2$, and hence $[K_\infty : K_\infty^p] = [K : K^p] \leq p^\delta$. Finally, according to the definition of the cohomological dimension and to Proposition 2.2, in order to prove that $\mathrm{cd}(K_\infty) \leq \delta$, it suffices to check that $H_\ell^{\delta+1}(M_\infty) = 0$ for each finite extension M_∞/K_∞ and each prime number ℓ. We henceforth fix an element $a \in H_\ell^{\delta+1}(M_\infty)$. We can then find an integer $i \geq 1$, a finite extension M of L_i and an element $b \in H_\ell^{\delta+1}(M)$ such that $MK_\infty = M_\infty$ and $b|_{M_\infty} = a$. Now, the triple (M, ℓ, b) is the j-th element of $\mathcal{E}_{L_i}$ for some $j \in \mathbb{N}$. This implies that $b|_{ML'_{\sigma^{-1}(i,j)+1}} = 0$ by construction. By the inclusion $ML'_{\sigma^{-1}(i,j)+1} \subset ML_{\sigma^{-1}(i,j)+1} \subset MK_\infty = M_\infty$, we deduce that $a = b|_{M_\infty} = 0$, as wished. ☺

3.2 From positive to zero characteristic

Theorem 3.4. *Let $\tilde{K}$ be a complete discrete valuation field of characteristic 0 with countable residue field K of cohomological dimension δ. Then there exists a totally ramified extension $\tilde{K}_\dagger/\tilde{K}$ with cohomological dimension δ.*

Remark 3.5. It is highly likely that the countability assumption of the theorem can be removed by adapting the subsequent proof and by using transfinite induction. Doing so would however significantly increase the technicality of the proof, and for applications we will not need such a general statement.

The main ingredient in order to prove Theorem 3.4 is the following proposition.

Proposition 3.6. *Let $\tilde{K}$ be a complete discrete valuation field of characteristic 0 with infinite residue field K of characteristic $p > 0$. Let δ be the cohomological dimension of K and let $\tilde{L}/\tilde{K}$ be a finite unramified Galois extension with residue field extension L/K. Assume that $\tilde{K}$ contains a primitive p-th root of unity and that it contains $\sqrt{-1}$ if $p = 2$. Then, for any element $a \in \mathrm{k}^{\mathrm{M}}_{\delta+1}(\tilde{L}) = \mathrm{K}^{\mathrm{M}}_{\delta+1}(\tilde{L})/p$, there exists a finite and totally ramified extension $\tilde{K}'/\tilde{K}$ of p-primary degree such that a is trivial when restricted to $\tilde{K}'\tilde{L}$.*

3.2.1 Preliminaries to the proof of Proposition 3.6

In the whole Section 3.2, we fix an algebraic closure $\overline{\tilde{K}}$ of $\tilde{K}$, as well as a primitive p-th root of unity $\zeta_p \in \overline{\tilde{K}}$ and a uniformizer π in $\tilde{K}$. We also recall the notations defined in Section 2.2: for $b \in K$, we denote by $\tilde{b}$ a lift of b to $\mathcal{O}_{\tilde{K}}$ and we set $\epsilon_{\tilde{K}} := \frac{v_{\tilde{K}}(p)p}{p-1}$.

Both in Theorem 3.4 and Proposition 3.6, the field K is assumed to have cohomological dimension δ. In particular we have $\mathrm{k}^{\mathrm{M}}_{\delta+1}(K) = 0$ and $[K : K^p] = p^{\delta_0}$ with $\delta_0 \leq \delta$. We henceforth fix a family $(\alpha_1, \dots, \alpha_\delta)$ in K such that the first δ_0 terms form a p-basis of K/K^p and $\alpha_i = 0$ for $i > \delta_0$.

Note that the finite separable extension L of K also satisfies $\mathrm{k}^{\mathrm{M}}_{\delta+1}(L) = 0$ and has the same p-basis as K. In particular,

$$\Omega^\delta_L = L\left(\frac{d\alpha_1}{\alpha_1} \wedge \dots \wedge \frac{d\alpha_\delta}{\alpha_\delta}\right) \quad \text{and} \quad \Omega^{\delta+1}_L = 0,$$

and hence

$$\Omega^\delta_L/\mathrm{Ker}(d) = d(\Omega^\delta_L) = 0 \quad \text{and} \quad H^{\delta+1}_p(L) = 0.$$

The isomorphisms $\rho_i := \rho_i^{\delta+1}$ defined in Section 2.2 can therefore be described as follows:

$$\begin{aligned} \rho_0 : \mathrm{k}^{\mathrm{M}}_\delta(L) &\to \mathrm{k}^{\mathrm{M}}_{\delta+1}(\tilde{L})/u^1_{\delta+1}(\tilde{L}), \\ \{y_1, y_2, \dots, y_\delta\} &\mapsto \{\pi, \tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_\delta\}; \end{aligned} \tag{5}$$

for every $0 < i < \epsilon_{\tilde{L}}$ such that $(i, p) = 1$,

$$\begin{aligned} \rho_i : \Omega^\delta_L &\to u^i_{\delta+1}(\tilde{L})/u^{i+1}_{\delta+1}(\tilde{L}), \\ x\,\frac{d\alpha_1}{\alpha_1} \wedge \dots \wedge \frac{d\alpha_\delta}{\alpha_\delta} &\mapsto \{1 + \pi^i\tilde{x}, \tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_\delta\}; \end{aligned} \tag{6}$$

for every $0 < i < \epsilon_{\tilde{L}}$ such that $p|i$,

$$\begin{aligned} \rho_i : \Omega^{\delta-1}_L/\mathrm{Ker}(d) &\to u^i_{\delta+1}(\tilde{L})/u^{i+1}_{\delta+1}(\tilde{L}), \\ x\,\frac{dy_1}{y_1} \wedge \frac{dy_2}{y_2} \wedge \dots \wedge \frac{dy_{\delta-1}}{y_{\delta-1}} &\mapsto \{\pi, 1 + \pi^i\tilde{x}, \tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_{\delta-1}\}; \end{aligned} \tag{7}$$

for $i = \epsilon_{\tilde{L}}$,

$$\begin{aligned} \rho_i : H^\delta_p(L) &\to u^i_{\delta+1}(\tilde{L}), \\ x\,\frac{dy_1}{y_1} \wedge \frac{dy_2}{y_2} \wedge \dots \wedge \frac{dy_{\delta-1}}{y_{\delta-1}} &\mapsto \{\pi, 1 + (\zeta_p - 1)^p\tilde{x}, \tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_{\delta-1}\}. \end{aligned} \tag{8}$$

3.2.2 Proof of Proposition 3.6

We prove first the proposition in the particular case where

$$a = \{1 + \pi^{\ell_0}\tilde{d}_0, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_\delta\}, \tag{9}$$

with $v_{\tilde{L}}(p) < \ell_0 < \epsilon_{\tilde{L}}$, $(\ell_0, p) = 1$ and $\tilde{d}_0 \in \mathcal{O}_{\tilde{L}}^\times$. We start with the following two lemmas:

Lemma 3.7. *Let $\tilde{K}$ be a complete discrete valuation field of characteristic 0 whose residue field K has characteristic $p > 0$. Let $\ell > v_{\tilde{K}}(p)$ be an integer, let π be a uniformizer in $\tilde{K}$ and let $\tilde{c}, \tilde{d} \in \mathcal{O}_{\tilde{K}}$. Then we have the following equality in $\tilde{K}^\times/(\tilde{K}^\times)^p$:*

$$(1 + \pi^\ell \tilde{c})(1 + \pi^\ell \tilde{d}) = 1 + \pi^\ell(\tilde{c} + \tilde{d}).$$

Proof. There exists $\tilde{x} \in \mathcal{O}_{\tilde{K}}$ such that:

$$(1 + \pi^\ell \tilde{c})(1 + \pi^\ell \tilde{d}) = (1 + \pi^\ell(\tilde{c} + \tilde{d}))(1 + \pi^{2\ell}\tilde{x}).$$

Now, since $\ell > v_{\tilde{K}}(p)$, we have $2\ell > \epsilon_{\tilde{K}}$. Hence, isomorphism (4) from Section 2.2 applied to $q = 0$ tells us that $(1 + \pi^{2\ell}\tilde{x})$ is a p-th power in $\tilde{K}$. ☺

Lemma 3.8. *Let $\tilde{K}$ be a complete discrete valuation field of characteristic 0 whose residue field K has characteristic $p > 0$. Let $\ell < \epsilon_{\tilde{K}}$ be an integer prime to p, let π be a uniformizer in $\tilde{K}$ and let $\tilde{d}$ be a unit in $\tilde{K}$. Then the extension $\tilde{K}(\sqrt[p]{1 + \pi^\ell \tilde{d}})/\tilde{K}$ is totally ramified of degree p.*

Proof. Set $z := \sqrt[p]{1 + \pi^\ell \tilde{d}} - 1 \in \tilde{K}$. It is a zero of the polynomial:

$$\mu := (X + 1)^p - 1 - \pi^\ell \tilde{d} \in \tilde{K}[X].$$

By computing its Newton polygon (cf. for instance [Neu99, II, Prop. 6.3]) and by using the inequality $\ell < \epsilon_{\tilde{K}}$, one sees that the roots of μ have valuation ℓ/p. Since ℓ is prime to p, this shows that $\tilde{K}(\sqrt[p]{1 + \pi^\ell \tilde{d}})/\tilde{K}$ is totally ramified of degree p, as wished. ☺

Proof of Proposition 3.6 for the symbol (9). Let H be the Galois group of $\tilde{L}/\tilde{K}$ and let n be its order. Consider $\mathbb{F}_p$ as a subfield of L. For each nonzero family $\mathbf{m} := (m_\sigma)_{\sigma \in H} \in \mathbb{F}_p^H$, consider the sets:

$$X_{\mathbf{m}} := \left\{ \lambda \in L : \sum_{\sigma \in H} m_\sigma \sigma(\lambda)^p = 0 \right\} = \left\{ \lambda \in L : \sum_{\sigma \in H} m_\sigma \sigma(\lambda) = 0 \right\},$$

$$Y_{\mathbf{m}} := \left\{ \lambda \in L : \sum_{\sigma \in H} m_\sigma \sigma(\lambda)^p = -\sum_{\sigma \in H} m_\sigma \sigma(d_0) \right\}.$$

Since the elements of $\mathrm{Gal}(L/K)$ are K-linearly independent, the sets $X_{\mathbf{m}}$ are all strict sub-K-vector spaces. Moreover, each $Y_{\mathbf{m}}$ is either empty or of the form $y_{\mathbf{m}} + X_{\mathbf{m}}$ for some $y_{\mathbf{m}} \in Y_{\mathbf{m}}$. The field K being infinite, we deduce that:

$$Y := \bigcup_{\substack{\mathbf{m} \in \mathbb{F}_p^H \\ \mathbf{m} \neq 0}} Y_{\mathbf{m}} \neq L.$$

In particular, we may and do fix an element λ_0 in the complement of Y in L, as well as a lift $\tilde{\lambda}_0$ of λ_0 in $\tilde{L}$.

Set now $\tilde{d}_1 := \tilde{d}_0 + \tilde{\lambda}_0^p$ and for each $\sigma \in H$ set $y_\sigma := \sqrt[p]{1 + \pi^{\ell_0}\sigma(\tilde{d}_1)}$. Introduce also $x := \sqrt[p]{\pi^{\ell_0}\tilde{\alpha}_1}$, consider the fields:

$$\tilde{L}' := \tilde{L}\left((y_\sigma)_{\sigma\in H}\right), \qquad \tilde{L}'' := \tilde{L}'(x) = \tilde{L}\left(x, (y_\sigma)_{\sigma\in H}\right),$$

and set:

$$H' := \mathrm{Gal}(\tilde{L}'/\tilde{L}), \quad H'' := \mathrm{Gal}(\tilde{L}''/\tilde{L}).$$

We start by proving the following three facts about the fields $\tilde{L}'$ and $\tilde{L}''$:

Fact 1: The extension $\tilde{L}'/\tilde{K}$ is Galois. Observe that, for every $\varphi \in \mathrm{Gal}(\tilde{K}^{\mathrm{sep}}/\tilde{K})$, there exists $\sigma_0 \in H$ such that $\varphi|_{\tilde{L}} = \sigma_0$ and hence, for each $\sigma \in H$:

$$\varphi(y_\sigma^p) = \sigma_0(y_\sigma^p) = \sigma_0(1 + \pi^{\ell_0}\sigma(\tilde{d}_1)) = 1 + \pi^{\ell_0}\sigma_0(\sigma(\tilde{d}_1)) = y_{\sigma_0\sigma}^p.$$

Since $\tilde{K}$ contains all p-th roots of unity, we deduce that the extension $\tilde{L}'/\tilde{K}$ is Galois.

Fact 2: The extension $\tilde{L}'/\tilde{L}$ is abelian with Galois group $(\mathbb{Z}/p\mathbb{Z})^n$. Since $\tilde{L}$ contains all p-th roots of unity and $\tilde{L}' = \tilde{L}\left((y_\sigma)_{\sigma\in H}\right)$, it suffices to prove by Kummer theory that the subgroup of $\tilde{L}^\times/(\tilde{L}^\times)^p$ generated by $(y_\sigma^p)_{\sigma\in H}$ has order p^n. In other words, it suffices to prove that, for any nonzero $(m_\sigma)_{\sigma\in H} \in \{0,\dots,p-1\}^H$:

$$\prod_{\sigma\in H} y_\sigma^{pm_\sigma} \neq 1 \in \tilde{L}^\times/(\tilde{L}^\times)^p.$$

Now, using Lemma 3.7 and the assumption that $\ell_0 > v_{\tilde{L}}(p)$, we compute in $\tilde{L}^\times/(\tilde{L}^\times)^p$:

$$\begin{aligned}\prod_{\sigma\in H} y_\sigma^{pm_\sigma} &= 1 + \sum_{\sigma\in H} m_\sigma \pi^{\ell_0}\sigma(\tilde{d}_1)\\ &= 1 + \pi^{\ell_0}\sum_{\sigma\in H} m_\sigma\sigma(\tilde{d}_1)\\ &= 1 + \pi^{\ell_0}\sum_{\sigma\in H} m_\sigma(\sigma(\tilde{d}_0) + \sigma(\tilde{\lambda}_0)^p)\end{aligned}$$

By construction of $\tilde{\lambda}_0$, we have

$$\sum_{\sigma\in H} m_\sigma(\sigma(d_0) + \sigma(\lambda_0)^p) \neq 0$$

in L, and hence we get $\prod_{\sigma\in H} y_\sigma^{pm_\sigma} \neq 1 \in \tilde{L}^\times/(\tilde{L}^\times)^p$ thanks to Lemma 3.8 and the assumption that $\ell_0 < \epsilon_{\tilde{L}}$.

Fact 3: The extension $\tilde{L}''/\tilde{L}$ is totally ramified. Since x^p has valuation prime to p, Fact 2 easily implies that the subgroup of $\tilde{L}^\times/(\tilde{L}^\times)^p$ generated by x^p and by $(y_\sigma^p)_{\sigma\in H}$ has order p^{n+1}. Hence the extension $\tilde{L}''/\tilde{L}$ is abelian with Galois group $(\mathbb{Z}/p\mathbb{Z})^{n+1}$. Its degree p subextensions are given by the:

$$\tilde{L}_{\mathbf{m},r} := \tilde{L}\left(x^r \prod_{\sigma\in H} y_\sigma^{m_\sigma}\right)$$

for $\mathbf{m} := (m_\sigma) \in \{0, \dots, p-1\}^H$ and $r \in \{0, \dots, p-1\}$ (with $\mathbf{m}$ and r not both zero). Whenever $r \neq 0$, the valuation of $x^{pr} \prod_{\sigma \in H} y_\sigma^{pm_\sigma}$ in $\tilde{L}$ is prime to p and hence the extension $\tilde{L}_{\mathbf{m},r}/\tilde{L}$ is totally ramified. When $r = 0$ and $\mathbf{m}$ is nonzero, the same computations as in Fact 2 show that the product $\prod_{\sigma \in H} y_\sigma^{pm_\sigma}$ belongs to $1 + \pi^{\ell_0} \mathcal{O}_{\tilde{L}}^\times$. Hence, by Lemma 3.8, the extension $\tilde{L}_{\mathbf{m},0}/\tilde{L}$ is again totally ramified. Summing up, we have proved that the degree p subextensions of the $(\mathbb{Z}/p\mathbb{Z})^{n+1}$-extension $\tilde{L}''/\tilde{L}$ are all totally ramified, and hence $\tilde{L}''/\tilde{L}$ is also totally ramified.

Back to the proof of Proposition 3.6 for the symbol (9), Fact 1 allows to introduce the Galois group G of $\tilde{L}'/\tilde{K}$, so that we have an exact sequence:

$$1 \to H' \to G \to H \to 1. \tag{10}$$

Moreover, by Fact 2, we have:

$$H' = \bigoplus_{\sigma \in H} \mathbb{Z}/p\mathbb{Z} \cdot \tau_\sigma,$$

where

$$\tau_\sigma : y_\rho \mapsto \begin{cases} \zeta_p y_\rho & \text{if } \rho = \sigma \\ y_\rho & \text{otherwise.} \end{cases}$$

Hence, for each $\sigma, \sigma' \in H$, if we denote by $\tilde{\sigma}'$ a lifting of σ to G, we get:

$$\tilde{\sigma}' \tau_\sigma \tilde{\sigma}'^{-1} = \tau_{\sigma'\sigma}.$$

In particular, we have $H' = \mathbb{Z}/p\mathbb{Z}[H]$. But by Shapiro's Lemma, $H^2(H, \mathbb{Z}/p\mathbb{Z}[H]) = 0$. This means by [NSW08, Thm. 1.2.4] that exact sequence (10) has a splitting $s : H \to G$. We introduce the fields:

$$\tilde{K}' := \tilde{L}'^{s(H)}, \qquad \tilde{K}'' := \tilde{K}'(x).$$

By Fact 3, the extension $\tilde{L}''/\tilde{L}$ is totally ramified. Hence so is the extension $\tilde{K}''/\tilde{K}$. We conclude by using once again Lemma 3.7 to observe that:

$$\begin{aligned} a &= \{1 + \pi^{\ell_0} \tilde{d}_0, \tilde{\alpha}_1, \dots, \tilde{\alpha}_\delta\} \\ &= \{1 + \pi^{\ell_0} (\tilde{d}_1 - \tilde{\lambda}_0^p), \tilde{\alpha}_1, \dots, \tilde{\alpha}_\delta\} \\ &= \{1 + \pi^{\ell_0} \tilde{d}_1, \tilde{\alpha}_1, \dots, \tilde{\alpha}_\delta\} + \{1 - \pi^{\ell_0} \tilde{\lambda}_0^p, \tilde{\alpha}_1, \dots, \tilde{\alpha}_\delta\} \\ &= \{1 + \pi^{\ell_0} \tilde{d}_1, \tilde{\alpha}_1, \dots, \tilde{\alpha}_\delta\} + \{1 - \pi^{\ell_0} \tilde{\lambda}_0^p, \pi^{\ell_0} \tilde{\lambda}_0^p \tilde{\alpha}_1, \alpha_2, \dots, \tilde{\alpha}_\delta\} \\ &= \{1 + \pi^{\ell_0} \tilde{d}_1, \tilde{\alpha}_1, \dots, \tilde{\alpha}_\delta\} + \{1 - \pi^{\ell_0} \tilde{\lambda}_0^p, \pi^{\ell_0} \tilde{\alpha}_1, \alpha_2, \dots, \tilde{\alpha}_\delta\} \end{aligned}$$

and hence that $a|_{\tilde{L}\tilde{K}''} = a|_{\tilde{L}''} = 0$. ☺

In order to prove Proposition 3.6 in the general case, we need some explicit computations, summarized in the following lemma.

Lemma 3.9. *Keep the hypotheses and notation of Proposition 3.6 and Preliminaries 3.2.1.*

(i) Consider a symbol

$$b_1 = \{1 + \pi^j \tilde{c}, \tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_\delta\} \in \mathrm{k}^{\mathrm{M}}_{\delta+1}(\tilde{L}),$$

where $\tilde{c}$ is a unit in $\tilde{L}$ and $j > 0$. Then b_1 can be written as a sum of symbols of the form

$$\{1 + \pi^i \tilde{\lambda}^p \tilde{\alpha}_1^{j_1} \ldots \tilde{\alpha}_\delta^{j_\delta}, \tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_\delta\},$$

where $0 \leq j_1, \ldots, j_\delta < p$, $\tilde{\lambda} \in \tilde{L}$ is a unit and $j \leq i < \epsilon_{\tilde{L}}$.

(ii) Consider a symbol

$$b_2 = \{1 + \pi^i \tilde{\lambda}^p \tilde{\alpha}_1^{j_1} \ldots \tilde{\alpha}_\delta^{j_\delta}, \tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_\delta\} \in \mathrm{k}^{\mathrm{M}}_{\delta+1}(\tilde{L}),$$

where $0 \leq j_1, \ldots, j_\delta < p$, $\tilde{\lambda} \in \tilde{L}$ is a unit and $0 < i < \epsilon_{\tilde{L}}$. Assume that $p|i$. Then:

- *if $(j_1, \ldots, j_\delta) \neq (0, \ldots, 0)$, then b_2 is trivial;*
- *if $(j_1, \ldots, j_\delta) = (0, \ldots, 0)$, then b_2 is equal to a symbol of the form*

$$\{1 + \pi^\ell \tilde{d}, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_\delta\},$$

for some $\ell \geq v_{\tilde{L}}(p) + \frac{i}{p}$ and some unit $\tilde{d}$ in $\tilde{L}$.

Proof. We prove (i). Since $\{\alpha_1, \alpha_2, \ldots, \alpha_\delta\}$ contains a p-basis of the residue field L, we can write

$$\tilde{c} \equiv \sum_{0 \leq j_1, \ldots, j_\delta < p} \tilde{\lambda}^p_{j_1, \ldots, j_\delta} \tilde{\alpha}_1^{j_1} \ldots \tilde{\alpha}_\delta^{j_\delta} \mod \pi,$$

for some $\tilde{\lambda}_{j_1, \ldots, j_\delta} \in \mathcal{O}^\times_{\tilde{L}} \cup \{0\}$. This implies that we may always write

$$1 + \pi^j \tilde{c} = \left(\prod_{0 \leq j_1, \ldots, j_\delta < p} (1 + \pi^j \tilde{\lambda}^p_{j_1, \ldots, j_\delta} \tilde{\alpha}_1^{j_1} \ldots \tilde{\alpha}_\delta^{j_\delta}) \right) \cdot (1 + \pi^{j'} \tilde{d}),$$

for some $j' > j$ and $\tilde{d}$ a unit in $\tilde{L}$. In particular, we get

$$b_1 = \sum_{0 \leq j_1, \ldots, j_\delta < p} \{1 + \pi^j \tilde{\lambda}^p_{j_1, \ldots, j_\delta} \tilde{\alpha}_1^{j_1} \ldots \tilde{\alpha}_\delta^{j_\delta}, \tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_\delta\} + \{1 + \pi^{j'} \tilde{d}, \tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_\delta\}.$$

Iterating this process, we see that we can always write b_1 as a sum of a symbol b_1' of the form:

$$\{1 + \pi^\epsilon \tilde{d}_\epsilon, \tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_\delta\}$$

with $\epsilon = \epsilon_{\tilde{L}}$ and $\tilde{d}_\epsilon$ an integer in $\tilde{L}$, and of symbols of the form

$$\{1 + \pi^i \tilde{\lambda}^p \tilde{\alpha}_1^{j_1} \ldots \tilde{\alpha}_\delta^{j_\delta}, \tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_\delta\},$$

where $0 \leq j_1, \ldots, j_\delta < p$, $\tilde{\lambda} \in \tilde{L}$ is a unit and $j \leq i < \epsilon_{\tilde{L}}$. We conclude by observing that b_1' lies in $u^\epsilon_{\delta+1}(\tilde{L})$ and that it corresponds by isomorphism (4) to an element of $H^{\delta+1}_p(L)$, which is trivial.

We now prove (ii). Assume first that $(j_1, \dots, j_\delta) \neq (0, \dots, 0)$. Without loss of generality, we may and do assume that $j_1 \neq 0$. For arbitrary $\ell \in \mathbb{N}$, we have

$$\begin{aligned} b_2 &= \{1 + \pi^i \tilde{\lambda}^p \tilde{\alpha}_1^{j_1} \cdots \tilde{\alpha}_\delta^{j_\delta}, (-\pi^i \tilde{\lambda}^p \tilde{\alpha}_1^{j_1} \cdots \tilde{\alpha}_\delta^{j_\delta})^\ell \tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_\delta\}, \\ &= \{1 + \pi^i \tilde{\lambda}^p \tilde{\alpha}_1^{j_1} \cdots \tilde{\alpha}_\delta^{j_\delta}, (-1)^\ell \pi^{i\ell} \tilde{\lambda}^{p\ell} \tilde{\alpha}_1^{\ell j_1 + 1}, \tilde{\alpha}_2, \dots, \tilde{\alpha}_\delta\} \\ &\quad + \sum_{k=2}^{\delta} \ell j_k \{1 + \pi^i \tilde{\lambda}^p \tilde{\alpha}_1^{j_1} \cdots \tilde{\alpha}_\delta^{j_\delta}, \tilde{\alpha}_k, \tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_\delta\} \\ &= \{1 + \pi^i \tilde{\lambda}^p \tilde{\alpha}_1^{j_1} \cdots \tilde{\alpha}_\delta^{j_\delta}, (-1)^\ell \pi^{i\ell} \tilde{\lambda}^{p\ell} \tilde{\alpha}_1^{\ell j_1 + 1}, \tilde{\alpha}_2, \dots, \tilde{\alpha}_\delta\} \end{aligned}$$

and since -1 is a p-th power in $\tilde{L}$ and $p|i$, the second entry in the last symbol is a p-th power provided that $\ell j_1 \equiv -1 \pmod{p}$. Hence b_2 is trivial in $\mathrm{k}_{\delta+1}^{\mathrm{M}}(\tilde{L})$.

Assume now that $(j_1, \dots, j_\delta) = (0, \dots, 0)$. In this case,

$$1 + \pi^i \tilde{\lambda}^p = (1 + \pi^{i/p} \tilde{\lambda})^p (1 + p\pi^{i/p} \tilde{x}),$$

for some $\tilde{x} \in \tilde{L}$ with $v_{\tilde{L}}(\tilde{x}) \geq 0$. Hence, in $\mathrm{k}_{\delta+1}^{\mathrm{M}}(\tilde{L})$,

$$b_2 = \{1 + p\pi^{i/p} \tilde{x}, \tilde{\alpha}_1, \dots, \tilde{\alpha}_\delta\}.$$

☺

Proof of Proposition 3.6. From the isomorphisms (5) to (8), we deduce that the element $a \in \mathrm{k}_{\delta+1}^{\mathrm{M}}(\tilde{L})$ can always be written as a sum of symbols of the forms

(a) $\{\pi, \tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_\delta\}$, where $\tilde{y}_1, \tilde{y}_2 \dots, \tilde{y}_\delta$ are units in $\tilde{L}$; and

(b) $\{1 + \pi^i \tilde{c}, \tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_\delta\}$, where $\tilde{c}$ is a unit in $\tilde{L}$ and $0 < i < \epsilon_{\tilde{L}}$ with $(i, p) = 1$.

By applying Lemma 3.9.(i) to the terms of type (b) that appear in a, we see that a can always be written as a sum of symbols of the form (a) and of the form

$$\{1 + \pi^i \tilde{\lambda}^p \tilde{\alpha}_1^{j_1} \dots \tilde{\alpha}_\delta^{j_\delta}, \tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_\delta\},$$

where $0 \leq j_1, \dots, j_\delta < p$, $\lambda \in \tilde{L}$ is a unit and $0 < i < \epsilon_{\tilde{L}}$. Observe now that the finite extension $\tilde{L}' := \tilde{L}(\sqrt[p]{\pi})$ of $\tilde{L}$ trivializes any symbol of the form (a), so the restriction of a to $\tilde{L}'$ is a sum of symbols of the form

$$\{1 + \pi'^{pi} \tilde{\lambda}^p \tilde{\alpha}_1^{j_1} \dots \tilde{\alpha}_\delta^{j_\delta}, \tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_\delta\},$$

where $0 \leq j_1, \dots, j_\delta < p$, $\tilde{\lambda} \in \tilde{L}$ is a unit, $0 < i < \epsilon_{\tilde{L}}$ and $\pi' := \sqrt[p]{\pi}$. We may then apply Lemma 3.9.(ii) to these and deduce that the restriction of a to $\tilde{L}'$ is a sum of symbols of the form

$$\{1 + \pi'^\ell \tilde{d}, \tilde{\alpha}_1, \dots, \tilde{\alpha}_\delta\},$$

for some $\ell > v_{\tilde{L}'}(p)$ and some unit $\tilde{d}$ in $\tilde{L}'$. And since all these symbols share the components $\tilde{\alpha}_1, \dots, \tilde{\alpha}_\delta$, we may put them together and finally conclude that

$$a|_{\tilde{L}'} = \{1 + \pi'^{\ell_0} \tilde{d}_0, \tilde{\alpha}_1, \dots, \tilde{\alpha}_\delta\},$$

for some $\ell_0 > v_{\tilde{L}'}(p)$ and some $\tilde{d}_0 \in \tilde{L}'$.

Observe now that, if $p|\ell_0$, we may apply successively parts (i) and (ii) of Lemma 3.9 to this symbol and obtain that $a|_{\tilde{L}'}$ is of the form:

$$\{1+\pi'^{\ell_0'}\tilde{d}_0', \tilde{\alpha}_1, \dots, \tilde{\alpha}_\delta\},$$

for some $\ell_0' \geq v_{\tilde{L}'}(p) + \frac{\ell_0}{p}$. But, if $\ell_0 < \epsilon_{\tilde{L}'} = v_{\tilde{L}'}(p)p/(p-1)$, then $\ell_0' > \ell_0$. Iterating this argument, we may therefore assume that either $(\ell_0, p) = 1$ or $\ell_0 \geq \epsilon_{\tilde{L}'}$. But in the latter case, the isomorphism (4) and the fact that $H_p^{q+1}(L) = 0$ immediately imply that $a|_{\tilde{L}'} = 0$.

We are hence reduced to the case where $v_{\tilde{L}'}(p) < \ell_0 < \epsilon_{\tilde{L}'}$ and $(\ell_0, p) = 1$, so that, up to replacing $\tilde{K}$ by $\tilde{K}(\sqrt[p]{\pi})$ and $\tilde{L}$ by $\tilde{L}(\sqrt[p]{\pi}) = \tilde{L}(\sqrt[p]{\pi})$, we only need to prove the proposition for

$$a = \{1+\pi^{\ell_0}\tilde{d}_0, \tilde{\alpha}_1, \dots, \tilde{\alpha}_\delta\},$$

which is the case we have already dealt with. $\square$

3.2.3 Proof of Theorem 3.4

Let p be the characteristic exponent of K and let $\pi \in \tilde{K}$ be a uniformizer. The case of a finite residue field is easy to settle. Thus, if $p \neq 1$, up to replacing $\tilde{K}$ by $\tilde{K}(\zeta_p)$ (and by $\tilde{K}(\sqrt{-1})$ if $p = 2$), we may assume that $\tilde{K}$ satisfies the hypotheses of Proposition 3.6.

All extensions of $\tilde{K}$ considered in this proof, in particular composite fields, will be assumed to be subfields of a fixed algebraic closure of $\tilde{K}$.

Consider a compatible system $(\sqrt[n]{\pi})_{n\geq 1}$ of n-th roots of π in the fixed algebraic closure of $\tilde{K}$, set $\tilde{K}_n := \tilde{K}(\sqrt[n]{\pi})$ for each $n \geq 1$, and introduce the field

$$\tilde{K}_{(p')} := \bigcup_{(n,p)=1} \tilde{K}_n,$$

which is a totally ramified extension of $\tilde{K}$.

We claim that $\mathrm{cd}_\ell(\tilde{K}_{(p')}) \leq \delta$ for every prime $\ell \neq p$. Fix such a prime number ℓ and a finite extension $\tilde{L}_{(p')}$ of $\tilde{K}_{(p')}$. There exists an integer n_0 prime to p and a finite extension $\tilde{L}_{n_0}$ of $\tilde{K}_{n_0}$ such that $\tilde{L}_{(p')} = \tilde{L}_{n_0}\tilde{K}_{(p')}$. For each $n \geq n_0$, we set $\tilde{L}_n := \tilde{L}_{n_0}(\sqrt[n]{\pi})$ and we denote by L_n the residue field of $\tilde{L}_n$. Note that, up to increasing n_0, we may assume that $\tilde{L}_n \cap \tilde{K}_{(p')} = \tilde{K}_n$ for every $n \geq n_0$. Then, for each $n \geq n_0$ and $m \geq 1$, the residue maps induce the following commutative diagram:

$$\begin{array}{ccc}
H^{\delta+1}(\tilde{L}_{mn}, \mu_\ell^{\otimes(\delta+1)}) & \xrightarrow[\cong]{\partial} & H^{\delta}(L_{mn}, \mu_\ell^{\otimes\delta}) \\
\uparrow{\scriptstyle \mathrm{Res}_{\tilde{L}_{mn}/\tilde{L}_n}} & & \uparrow{\scriptstyle e_{\tilde{L}_{mn}/\tilde{L}_n}\cdot \mathrm{Res}_{L_{mn}/L_n}} \\
H^{\delta+1}(\tilde{L}_{n}, \mu_\ell^{\otimes(\delta+1)}) & \xrightarrow[\cong]{\partial} & H^{\delta}(L_{n}, \mu_\ell^{\otimes\delta})
\end{array}$$

in which the lines are isomorphisms since L_r has cohomological dimension δ for every r (cf. [Ser02, II. App. §2] and [CT95, Prop. 3.3.1]). And since $\tilde{L}_n \cap \tilde{K}_{(p')} = \tilde{K}(\sqrt[n]{\pi})$, the integer $e_{\tilde{L}_{mn}/\tilde{L}_n}$ is always divisible by m. Hence, the group

$$H^{\delta+1}(\tilde{L}_{(p')}, \mu_\ell^{\otimes(\delta+1)}) = \varinjlim_n H^{\delta+1}(\tilde{L}_n, \mu_\ell^{\otimes(\delta+1)}),$$

is trivial. This being true for every finite extension $\tilde{L}_{(p')}$ of $\tilde{K}_{(p')}$, we get $\mathrm{cd}_\ell(\tilde{K}_{(p')}) \leq \delta$.

This deals with the case where K has characteristic 0, so we assume hereafter that K has characteristic $p > 0$. We construct now a second extension $\tilde{K}_{(p)}$ of $\tilde{K}$ as follows. We proceed as in Theorem 3.1. In particular, we fix the following notations:

- We let $\sigma = (\sigma_1, \sigma_2) : \mathbb{N} \to \mathbb{N}^2$ be a bijection such that $\sigma_1(n) \leq n$ for each $n \in \mathbb{N}$.
- For each finite extension $\tilde{L}$ of $\tilde{K}$, we define $\mathcal{E}_{\tilde{L}}$ as the set of pairs $(\tilde{M}, a)$ such that $\tilde{M}$ is a finite, tamely ramified, Galois extension of $\tilde{L}$ with separable residue field extension M/L, and a is an element in $\mathrm{k}^{\mathrm{M}}_{\delta+1}(\tilde{M}) = H^{\delta+1}(\tilde{M}, \mu_p^{\otimes(\delta+1)})$.

We claim that, for each finite extension $\tilde{L}$ of $\tilde{K}$, the set $\mathcal{E}_{\tilde{L}}$ is countable. Indeed, since K is countable, so is the set of unramified extensions of $\tilde{L}$. Moreover, since every tamely ramified extension of the maximal unramified extension of $\tilde{L}$ consists in taking a root of a fixed uniformizer, we see that tamely ramified extensions of $\tilde{L}$ are countable as well. Furthermore, for each such extension $\tilde{M}/\tilde{L}$, the group $\mathrm{k}^{\mathrm{M}}_{\delta+1}(\tilde{M})$ is countable, according to the isomorphisms (1) to (4). Thus, we may assume that $\mathcal{E}_{\tilde{L}}$ comes with a given numbering.

Let us now inductively construct an increasing infinite sequence of finite and totally ramified extensions $(\tilde{L}_i)_{i\geq 1}$ of $\tilde{K}$ satisfying the following property: the maximal tamely ramified extension $\tilde{L}_i^{\mathrm{tam}}$ of $\tilde{K}$ in $\tilde{L}_i$ is of the form $\tilde{K}(\sqrt[e_i]{\pi})$ for some e_i prime to p. To do so, we set first $\tilde{L}_1 := \tilde{K}$. Then, for $i \geq 1$, consider the $\sigma_2(i)$-th term of the set $\mathcal{E}_{L_{\sigma_1(i)}}$, which we denote by $(\tilde{M}_i, a_i)$. Of course, since $\sigma_1(i) \leq i$, we have $\tilde{L}_{\sigma_1(i)} \subset \tilde{L}_i$.

Since $\tilde{M}_i/\tilde{L}_i$ is tamely ramified, its ramification degree e'_i is prime to p. Set now $e_{i+1} := e_i e'_i$. By the inductive assumption, $\tilde{L}_i^{\mathrm{tam}} = \tilde{K}(\sqrt[e_i]{\pi})$, and hence the extension $\tilde{L}_i(\sqrt[e_{i+1}]{\pi})/\tilde{K}$ is totally ramified while the extension $\tilde{M}_i(\sqrt[e_{i+1}]{\pi})/\tilde{L}_i(\sqrt[e_{i+1}]{\pi})$ is unramified. Applying Proposition 3.6 to the unramified extension $\tilde{M}_i(\sqrt[e_{i+1}]{\pi})/\tilde{L}_i(\sqrt[e_{i+1}]{\pi})$ and to the symbol $a_i|_{\tilde{M}_i(\sqrt[e_{i+1}]{\pi})}$, we know that there exists a finite and totally ramified extension $\tilde{L}_{i+1}/\tilde{L}_i(\sqrt[e_{i+1}]{\pi})$ of p-primary degree such that $a_i|_{\tilde{L}_{i+1}\tilde{M}_i} = 0$. By construction, we have $\tilde{L}_{i+1}^{\mathrm{tam}} = \tilde{K}(\sqrt[e_{i+1}]{\pi})$. We summarize this in Figure 1, where we input the *ramification degrees* for each extension.

Let us now introduce the field:

$$\tilde{K}_{(p)} := \bigcup_{i\geq 1} \tilde{L}_i.$$

Observe that the extension $\tilde{K}_{(p)}/\tilde{K}$ is totally ramified and that the maximal tamely ramified extension of $\tilde{K}$ contained in $\tilde{K}_{(p)}$ is contained in $\tilde{K}_{(p')}$. Hence the composite $\tilde{K}_\dagger := \tilde{K}_{(p)}\tilde{K}_{(p')}$ is totally ramified over $\tilde{K}$.

Recall that, for each prime number ℓ other than p, we have $\mathrm{cd}_\ell(\tilde{K}_{(p')}) \leq \delta$, and hence $\mathrm{cd}_\ell(\tilde{K}_\dagger) \leq \delta$. To conclude that $\mathrm{cd}(\tilde{K}_\dagger) \leq \delta$, it therefore suffices to check that $\mathrm{cd}_p(\tilde{K}_{(p)}) \leq \delta$. But according to Lemma 3.10 below, that amounts to check that the group $H^{\delta+1}(\tilde{M}_{(p)}, \mu_p^{\otimes(\delta+1)})$ is trivial for each finite tamely ramified Galois extension $\tilde{M}_{(p)}/\tilde{K}_{(p)}$ with separable residue field extension. We henceforth fix an element $a \in H^{\delta+1}(\tilde{M}_{(p)}, \mu_p^{\otimes(\delta+1)})$. We can then find an integer $i \geq 1$, a finite tamely ramified Galois extension $\tilde{M}$ of $\tilde{L}_i$ with separable residue field extension and an element

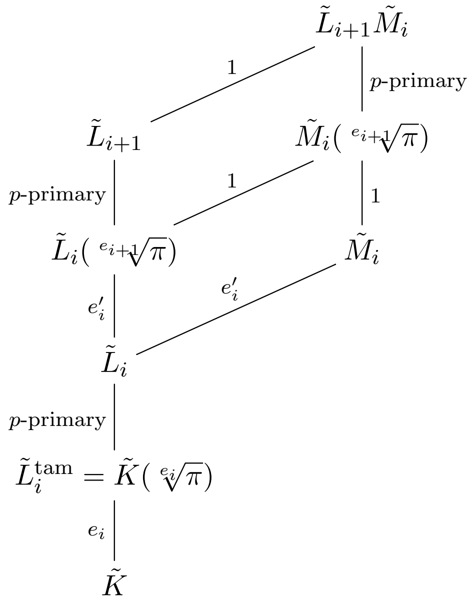

Figure 1: Extensions involved in the construction of $\tilde{K}_{(p)}$ and their corresponding ramification degrees.

$b \in H^{\delta+1}(\tilde{M}, \mu_p^{\otimes(\delta+1)})$ such that $\tilde{M}\tilde{K}_{(p)} = \tilde{M}_{(p)}$ and $b|_{\tilde{M}_{(p)}} = a$. Now, the pair $(\tilde{M}, b)$ is the j-th element of $\mathcal{E}_{\tilde{L}_i}$ for some $j \in \mathbb{N}$. This implies that $b|_{\tilde{M}\tilde{L}_{\sigma^{-1}(i,j)+1}} = 0$ by construction. By the inclusion $\tilde{M}\tilde{L}_{\sigma^{-1}(i,j)+1} \subset \tilde{M}\tilde{K}_{(p)} = \tilde{M}_{(p)}$, we deduce that $a = b|_{\tilde{M}_{(p)}} = 0$, as wished. ☺

Lemma 3.10. *Let $\tilde{K}$ be a complete discrete valuation field of characteristic 0 whose residue field K has characteristic $p > 0$. Let δ be the cohomological dimension of K and let $\tilde{L}$ be an algebraic extension of $\tilde{K}$ such that, for each finite tamely ramified Galois extension $\tilde{M}$ of $\tilde{L}$ with separable residue field extension M/L, the cohomology group $H^{\delta+1}(\tilde{M}, \mu_p^{\otimes(\delta+1)})$ vanishes. Then $\operatorname{cd}_p(\tilde{L}) \leq \delta$.*

Proof. According to Proposition 2.2, it suffices to prove that $H^{\delta+1}(\tilde{N}, \mu_p^{\otimes(\delta+1)}) = 0$ for each finite extension of $\tilde{N}$ of $\tilde{L}$ of degree prime to p and containing ζ_p. For that purpose, denote by $\tilde{M}$ the Galois closure of $\tilde{N}/\tilde{L}$ and write the exact sequence of finite Galois modules over $\tilde{L}$:

$$0 \to F \to \mu_p^{\otimes(\delta+1)}[\tilde{M}/\tilde{L}] \to \mu_p^{\otimes\delta+1}[\tilde{N}/\tilde{L}] \to 0.$$

It induces an exact sequence of cohomology groups:

$$H^{\delta+1}(\tilde{M}, \mu_p^{\otimes(\delta+1)}) \to H^{\delta+1}(\tilde{N}, \mu_p^{\otimes(\delta+1)}) \to H^{\delta+2}(\tilde{L}, F).$$

Note now that the maximal tamely ramified extension (with separable residue field extension) $\tilde{K}^{\mathrm{tr}}/\tilde{K}$ is Galois. Moreover, since $\tilde{N}/\tilde{L}$ has degree prime to p, it is tamely ramified and has separable residue field extension. Thus, the extension $\tilde{M}/\tilde{L}$ has the same two properties, and is Galois, and hence $H^{\delta+1}(\tilde{M}, \mu_p^{\otimes(\delta+1)}) = 0$. Moreover, by [Kat82, Cor. to Thm. 3], we have $\operatorname{cd}(\tilde{K}) = \delta + 1$, so that $H^{\delta+2}(\tilde{L}, F) = 0$. We deduce that $H^{\delta+1}(\tilde{N}, \mu_p^{\otimes(\delta+1)}) = 0$. ☺

3.3 From higher to lower cohomological dimension

Our final transfer principle for the cohomological dimension of fields involves the notion of a universal norm:

Definition 3.11. *Let L/K be an algebraic field extension and let $x \in K^\times$. We say that x is a* universal norm *for L/K if $x \in N_{K'/K}(K'^\times)$ for every finite extension K' of K contained in L.*

Theorem 3.12. *Let $\delta \geq 1$ be an integer, ℓ a prime number and K an ℓ-special countable field of characteristic 0 and with cohomological dimension δ. For each $x \in K^\times$, there exists an algebraic extension K_x of K that has cohomological dimension $\leq (\delta - 1)$ and for which x is a universal norm.*

Proof. Let $\mathcal{E}_K$ be the set of pairs (L, a) such that L is a finite extension of K and a a nonzero symbol in $H^\delta(L, \mu_\ell^{\otimes \delta})$. Since K is countable, so is the set $\mathcal{E}_K$ and hence we can number its elements: $(L_1, a_1), (L_2, a_2), \dots$

Let us now inductively construct a sequence of pairs $(K_i, x_i)_{i \geq 0}$ in the following way. For $i = 0$, we set $K_0 = K$ and $x_0 = x$. For $i \geq 0$, we let n_i be the smallest integer such that $L_{n_i} \subset K_i$ and $a_{n_i}|_{K_i} \neq 0$. According to Theorem 1.21(1-2) of [SJ06], there exists a geometrically irreducible projective generic splitting $\nu_{\delta-1}$-variety X_i over K_i of dimension $\ell^{\delta-1} - 1$ for the symbol $a_{n_i}|_{K_i}$ (see Definitions 1.10 and 1.20 of [SJ06]). Moreover, by Theorem A.1 of [SJ06], we have an exact sequence:

$$\bigoplus_{p \in X_i \text{ closed}} K_i(p)^\times \xrightarrow{\bigoplus N_{K_i(p)/K_i}} K_i^\times \xrightarrow{a_{n_i}|_{K_i} \cup \cdot} \mathrm{K}^{\mathrm{M}}_{\delta+1}(K_i)/\ell.$$

Since K has cohomological dimension δ, we deduce that x_i is the image of some element $\alpha_i \in \bigoplus_{p \in X_i \text{ closed}} K_i(p)^\times$ by the morphism $\bigoplus N_{K_i(p)/K_i}$. But, if we set:

$$A_0(X_i, \mathcal{K}_1) := \operatorname{coker} \left(\bigoplus_{\substack{q \in X_i \\ \dim \overline{\{q\}} = 1}} \mathrm{K}^{\mathrm{M}}_2(K_i(q)) \xrightarrow{\bigoplus \partial} \bigoplus_{p \in X_i \text{ closed}} K_i(p)^\times \right),$$

the morphism $\bigoplus N_{K_i(p)/K_i}$ factors through:

$$\overline{A}_0(X_i, \mathcal{K}_1) := \operatorname{coker} \left(A_0(X_i \times X_i, \mathcal{K}_1) \xrightarrow{(p_1)_* - (p_2)_*} A_0(X_i, \mathcal{K}_1) \right).$$

Moreover, Theorem 1.21(3) of [SJ06] shows that there exist a closed point p_i of X_i and an element $\lambda_i \in K_i(p_i)^\times$ such that its image via the inclusion:

$$K_i(p_i)^\times \hookrightarrow \bigoplus_{p \in X_i \text{ closed}} K_i(p)^\times,$$

has the same image as α_i in $\overline{A}_0(X_i, \mathcal{K}_1)$. We deduce that:

$$x_i = N_{K_i(p_i)/K_i}(\lambda_i).$$

We then set $K_{i+1} := K_i(p_i)$ and $x_{i+1} := \lambda_i$. For the sequel, it will be important to note that for each $i \geq 0$, we have the following properties:

$$\begin{cases} K_i \subset K_{i+1}, & (11) \\ x_i = N_{K_{i+1}/K_i}(x_{i+1}), & (12) \\ a_{n_i}|_{K_i} \neq 0, & (13) \\ a_{n_i}|_{K_{i+1}} = 0. & (14) \end{cases}$$

In particular, (13) and (14) imply that the n_i's are pairwise distinct.

Let us now introduce the field:

$$K_x := \bigcup_{i \geq 0} K_i.$$

We claim that K_x satisfies the conditions of the theorem.

Indeed, equation (12), together with the equality $x_0 = x$, shows that x is a universal norm for K_x/K. It remains to show that $\mathrm{cd}(K_x) \leq \delta - 1$. By Proposition 2.2 and the Bloch-Kato conjecture, it is enough to prove that every nonzero symbol in $H^\delta(K_x, \mu_\ell^{\otimes \delta})$ is trivial. Let a be such a symbol. We can then find an integer $i \geq 0$ and a symbol $b \in H^\delta(K_i, \mu_\ell^{\otimes \delta})$ such that $b|_{K_x} = a$. Since the pair (K_i, b) belongs to $\mathcal{E}_K$, we can find an integer N such that $(K_i, b) = (L_N, a_N)$. Then we have two cases:

(a) If $N = n_{j_0}$ for some j_0, then we have $a_{n_{j_0}}|_{K_{j_0+1}} = 0$ by equation (14), and hence $a = b|_{K_x} = a_{n_{j_0}}|_{K_x} = 0$.

(b) If $N \neq n_j$ for all j, since the n_j's are pairwise distinct, there exists an integer $j_0 > i$ such that $n_{j_0} > N$. Then $L_N = K_i \subset K_{j_0}$ and hence, by definition of n_{j_0}, we deduce that $a_N|_{K_{j_0}} = 0$. Hence $a = b|_{K_x} = a_N|_{K_x} = 0$.

☺

Remark 3.13. The assumption that the field K is of characteristic 0 in Theorem 3.12 is needed in the proof to apply the results of [SJ06].

4. Applications

In this section, we give several applications of the previous transfer principles for Galois cohomology. We start by proving the transfer principle for norm groups given by Theorem D and then we settle Theorems E and F about Serre's conjecture II and its higher versions.

4.1 A transfer principle for norm groups

Theorem 4.1. *Let $m, n, q \geq 0$ be three integers with $n \geq m \geq 1$ and K a field of characteristic 0 and with cohomological dimension $\leq n$. Let Z be a K-variety and assume that there exists a countable subfield K_0 of K and a form Z_0 of Z over K_0 satisfying the following condition:*

($\star$) *For every countable subextension L of K/K_0, and for every algebraic extension M/L of cohomological dimension $\leq m$, we have $N_q(Z_{0,M}/M) = \mathrm{K}_q^{\mathrm{M}}(M)$.*

Then $N_{n-m+q}(Z/K) = \mathrm{K}_{n-m+q}^{\mathrm{M}}(K)$.

Proof. We prove first the particular case where $K = K_0$. For that purpose, we may assume, by Lemma 2.5, that K is ℓ-special for some prime number ℓ. We then proceed by induction on $r := n - m$. The result is obvious when $r = 0$ since then $m = n$ and K has cohomological dimension $\leq n$. We henceforth fix some $r_0 \geq 1$, assume to have proven the theorem whenever $r < r_0$ and study the case when $r = r_0$.

To do so, consider a symbol $a := \{a_1, \ldots, a_{r+q}\} \in \mathrm{K}^{\mathrm{M}}_{r+q}(K)$. Introduce the field $K_{a_{r+q}}$ given by Theorem 3.12. It has cohomological dimension $\leq n-1$. By the inductive assumption, we have $N_{r+q-1}(Z_{K_{a_{r+q}}}/K_{a_{r+q}}) = \mathrm{K}^{\mathrm{M}}_{r+q-1}(K_{a_{r+q}})$. In particular, $\{a_1, \ldots, a_{r+q-1}\} \in N_{r+q-1}(Z_{K'}/K')$ for some finite subextension K' of $K_{a_{r+q}}/K$. This means that there exist finite extensions $K'_1, \ldots, K'_r$ of K' and elements $z_1 \in \mathrm{K}^{\mathrm{M}}_q(K'_1), \ldots, z_r \in \mathrm{K}^{\mathrm{M}}_q(K'_r)$ such that:

$$\begin{cases} \forall\, i,\ Z(K'_i) \neq \emptyset, \\ \{a_1, \ldots, a_{r+q-1}\} = \sum_i N_{K'_i/K'}(z_i). \end{cases}$$

Now, since a_{r+q} is a universal norm in $K_{a_{r+q}}$, we get that $a_{r+q} = N_{K'/K}(z_0)$ for some $z_0 \in K'^{\times}$. Thus,

$$\begin{aligned} a = \{a_1, \ldots, a_{r+q-1}, a_{r+q}\} &= \{a_1, \ldots, a_{r+q-1}, N_{K'/K}(z_0)\} = N_{K'/K}(\{a_1, \ldots, a_{r+q-1}, z_0\}) \\ = N_{K'/K}(\{\sum_i N_{K'_i/K'}(z_i), z_0\}) &= \sum_i N_{K'/K}(N_{K'_i/K'}(\{z_i, z_0\})) = \sum_i N_{K'_i/K}(\{z_i, z_0\}), \end{aligned}$$

and hence $a \in N_{r+q}(Z/K)$, as wished.

Now we move on to the general case. Consider an element $a \in \mathrm{K}^{\mathrm{M}}_{n-m+q}(K)$. There exists a countable subextension K' of K/K_0 and an element $a' \in \mathrm{K}^{\mathrm{M}}_{n-m+q}(K')$ such that $a'|_K = a$. By Theorem 3.1, we can find a countable subextension L of K/K' that has cohomological dimension $\leq n$. Moreover, by assumption $(\star)$, we have $N_q(Z_{0,M}/M) = \mathrm{K}^{\mathrm{M}}_q(M)$ for every algebraic extension M of L with cohomological dimension $\leq m$. By the case $K = K_0$ applied to L, we deduce that $N_{n-m+q}(Z_{0,L}/L) = \mathrm{K}^{\mathrm{M}}_{n-m+q}(L)$. In particular, $a'|_L \in N_{n-m+q}(Z_{0,L}/L)$, and hence $a = a'|_K \in N_{n-m+q}(Z/K)$. ☺

A natural application of this transfer principle for norm groups concerns Kato and Kuzumaki's conjectures. To see that, let us recall the definition of the C_i^q properties.

Definition 4.2 (Kato-Kuzumaki, [KK86]). *Let K be a field and let $i, q \geq 0$ be two integers. The field K is said to have the C_i^q property if, for each $n \geq 1$, for each finite extension L of K and for each hypersurface Z in $\mathbb{P}^n_L$ of degree d with $d^i \leq n$, one has $N_q(Z/L) = \mathrm{K}_q(L)$.*

Kato and Kuzumaki conjectured in [KK86] that a field has the C_i^q property if, and only if, its cohomological dimension is $\leq q + i$. Even though it is known nowadays that these conjectures are false in general, they are still open questions for fields that appear naturally in arithmetic geometry. Indeed, the only known counterexamples are built by means of transfinite induction [Mer91, CTM04]. Moreover, some instances of these conjectures are known to hold for number fields and p-adic fields [Wit15], function fields of complex varieties and fields of Laurent series over these function fields [Izq18], and function fields of p-adic curves [ILA22].

In this context, Theorem 4.1 has the following immediate consequence.

Corollary 4.3. *Let $i, m, n, q \geq 0$ be four integers with $n \geq m \geq 1$ and $i + q \geq m$. Let K a field of characteristic 0 and with cohomological dimension $\leq n$. Assume that:*

($\diamond$) *For every large enough countable subfield L of K, every algebraic extension M/L of cohomological dimension $\leq m$ satisfies the C_i^q property.*

Then K satisfies the C_i^{n-m+q} property.

In particular, we get the following two small direct applications:

Corollary 4.4. *Let K be a field with finite transcendence degree over $\mathbb{Q}$. If, for every $i \geq 1$, algebraic extensions of K with cohomological dimension i satisfy the C_i^0 property, then Kato and Kuzumaki's conjectures hold for K.*

Corollary 4.5. *Let K be a countable C_1 field of characteristic 0 and let L be a finitely generated extension of K of cohomological dimension i. If Kato and Kuzumaki's conjectures hold for algebraic extensions of L with cohomological dimension $< i$, then Kato and Kuzumaki's conjectures also hold for L.*

Now, in [ILA20], we introduced several variants of Kato and Kuzumaki's C_i^q properties, by replacing hypersurfaces of low degree by homogeneous spaces of linear algebraic groups. Let us recall one of them.

Definition 4.6. *Let q be a non-negative integer. We say that a field K has the C_{HS}^q property if, for each finite extension L of K and for each homogeneous space Z under a smooth linear connected algebraic group over L, one has $N_q(Z/L) = K_q^M(L)$.*

Using Theorem 4.1, we can recover one of the main results in [ILA20] with a whole new approach, which focuses on fields rather than groups:

Corollary 4.7. *Let q be a non-negative integer. Every characteristic 0 field with cohomological dimension at most $q + 1$ has the C_{HS}^q-property.*

Together with the converse statement, which was easily settled in [ILA20, Prop. 3.2], this result shows that property C_{HS}^q is a good replacement for Kato and Kuzumaki's C_1^q property, in the sense that it characterizes perfect fields of cohomological dimension $\leq q + 1$. In the particular case where $q = 0$, it recovers the zero-cycle version of Serre's conjecture I (Theorem 1.1), which was proved by Steinberg [Ste65], as well as an extension to homogeneous spaces with nontrivial stabilizers due to Springer [Ser02, III.2.4, Thm. 3]. Corollary 4.7 is therefore in some sense a generalization of Serre's conjecture I and of Springer's theorem to higher-dimensional fields.

Thanks to the transfer principles, we now view this corollary under a new light, as a direct consequence of the case $q = 0$. Indeed, Steinberg's and Springer's Theorems imply that fields of cohomological dimension ≤ 1 have the C_{HS}^0 property. Then Theorem 4.1 tells us immediately that fields of cohomological dimension $\leq q + 1$ have the C_{HS}^q property.

4.2 Higher Serre's conjecture II

The C_{sc}^q property

Given the last application from the previous section, it is natural to ask whether one can find good replacements for the C_2^q property that would characterize fields with cohomological dimension $\leq q + 2$, while recovering a version of Serre's conjecture II (Conjecture 1.2) for higher-dimensional fields. This is the purpose of the following definition.

Definition 4.8. *Let q be a non-negative integer. We say that a field K has the C^q_{sc} property if, for any finite extension L/K and any principal homogeneous space Z under a semisimple simply connected L-group G, we have $N_q(Z/L) = \mathrm{K}^{\mathrm{M}}_q(L)$.*

Similarly, given a type Λ in the classification of semisimple absolutely almost simple simply connected groups (e.g. $\Lambda = A_n$, or $\Lambda = E_6$, or $\Lambda = {}^1A_n$), we say that a field K has the C^q_Λ property if, for any finite extension L/K and any principal homogeneous space Z under a simply connected isotypical L-group G of type Λ, we have $N_q(Z/L) = \mathrm{K}^{\mathrm{M}}_q(L)$.

Recall that a simply connected L-group G is said to be isotypical of type Λ if it is isomorphic to a finite product of Weil restrictions of the form $R_{M/L}(H)$ with H an absolutely almost simple simply connected M-group H of type Λ and M/L a finite separable extension. Properties C^q_{sc} and C^q_Λ are then easily related to each other in the following way:

Proposition 4.9. *Let q be a non-negative integer. Assume that a field K has the C^q_Λ property for every type Λ. Then K has the C^q_{sc} property.*

Proof. This follows from Lemma 2.4 and the fact that any simply connected K-group G is a product of Weil restrictions of semisimple almost simple simply connected groups. ☺

We are now ready to suggest the following higher variant of Serre's conjecture II:

Conjecture 4.10 (Higher Serre's conjecture II)**.** *Let q be a non-negative integer and K a field. If K has cohomological dimension at most $q+2$, then the field K has the C^q_{sc} property.*

Before studying this conjecture in detail, we prove the following proposition, that shows that the converse holds for perfect fields (and that a partial converse holds for imperfect fields). In particular, the C^q_{sc} property should characterize perfect fields of cohomological dimension $\leq q+2$ and hence should be a good replacement for the C^q_2 property.

Proposition 4.11. *If a field K has the $C^q_{{}^1A_n}$ property, then all of its finite extensions have separable cohomological dimension at most $q+2$. In particular, if K is perfect, then its cohomological dimension is $\leq q+2$.*

Proof. Let K be a field with the $C^q_{{}^1A_n}$ property. Fix a prime number ℓ, and assume first that ℓ is different from the characteristic of K. Consider a finite extension L of K containing a primitive ℓ-th root of unity and a symbol $\{a_1, \dots, a_{q+3}\} \in H^{q+3}(L, \mu_\ell^{\otimes(q+3)})$. Let A be the cyclic L-algebra $(a_1, a_2)_\ell$ and consider the L-variety Z given by the equation $\mathrm{Nrd}_A(\mathbf{x}) = a_3$. It is a principal homogeneous space under the group $\mathrm{SL}_1(A)$. Since K has the $C^q_{{}^1A_n}$ property, one can find finite extensions $L_1, \dots, L_r$ of L and elements $b_i \in \mathrm{K}^{\mathrm{M}}_q(L_i)$ for $1 \leq i \leq r$ such that:

$$\begin{cases} \forall\, i,\ Z(L_i) \neq \emptyset \\ \{a_4, \dots, a_{q+3}\} = \sum_{i=1}^r N_{L_i/L}(b_i). \end{cases}$$

By Theorem 24.4 of [Sus85], the condition $Z(L_i) \neq \emptyset$ implies that the restriction of the symbol $\{a_1, a_2, a_3\} \in H^3(L, \mu_\ell^{\otimes 3})$ to L_i is trivial. Hence:

$$\{a_1, \dots, a_{q+3}\} = \sum_{i=1}^r N_{L_i/L}(\{a_1, a_2, a_3, b_i\}) = 0.$$

By the Bloch-Kato conjecture ([Rio14]), we deduce that the group $H^{q+3}(L, \mu_\ell^{\otimes(q+3)})$ is trivial. This being true for each finite extension L of K containing a primitive ℓ-th root of unity, the field K has ℓ-cohomological dimension at most $q+2$ by Proposition 2.2.

Assume now that ℓ is equal to the characteristic of K. Consider a finite extension L of K and an element $x\frac{dy_1}{y_1} \wedge ... \wedge \frac{dy_{q+2}}{y_{q+2}} \in H_\ell^{q+3}(L)$. Let $A \in \mathrm{Br}(K)$ be the cyclic central simple algebra $[x, y_1)$ and introduce the $\mathrm{SL}_1(A)$-torsor Z given by the equation $\mathrm{Nrd}_A(\mathbf{x}) = y_2$. Since K has the $C^q_{1_{A_n}}$ property, one can find finite extensions $L_1, \dots, L_r$ of L and elements $b_1 \in \mathrm{K}_q^{\mathrm{M}}(L_1), \dots, b_r \in \mathrm{K}_q^{\mathrm{M}}(L_r)$ such that:

$$\begin{cases} \forall i,\ Z(L_i) \neq \emptyset \\ \{y_3, \dots, y_{q+2}\} = \sum_{i=1}^r N_{L_i/L}(b_i). \end{cases}$$

According to Theorem 6 of [Gil00], the condition $Z(L_i) \neq \emptyset$ implies that:

$$\mathrm{Res}_{L_i/L}\left(x\frac{dy_1}{y_1} \wedge \frac{dy_2}{y_2}\right) = 0$$

for each i. By denoting by $\nu(q)_{L_i}$ the kernel of $\mathfrak{p}^q_{L_i}$ and by $\psi^q_{L_i} : \mathrm{K}^{\mathrm{M}}_q(L_i) \to \nu(q)_{L_i}$ the differential symbol, and by using Lemma 9.5.7 of [GS17], we get:

$$\begin{aligned} x\frac{dy_1}{y_1} \wedge ... \wedge \frac{dy_{q+2}}{y_{q+2}} &= x\frac{dy_1}{y_1} \wedge \frac{dy_2}{y_2} \wedge \left(\sum_i \mathrm{Tr}_{L_i/L}(\psi^q_{L_i}(b_i))\right) \\ &= \sum_i \mathrm{Tr}_{L_i/L}\left(\mathrm{Res}_{L_i/L}\left(x\frac{dy_1}{y_1} \wedge \frac{dy_2}{y_2}\right) \wedge \psi^q_{L_i}(b_i)\right) = 0. \end{aligned}$$

We deduce that $H_\ell^{q+3}(L) = 0$. ☺

Remark 4.12. Proposition 4.11 shows that, if a field K has the C_2^q property, then all of its finite extensions have separable cohomological dimension at most $q+2$. Indeed, if A is a cyclic algebra of index d, the equation $\mathrm{Nrd}_A(\mathbf{x}) = a$ has degree d and d^2 variables, hence it naturally defines a degree d hypersurface in $\mathbb{P}_K^{d^2}$.

Application of transfer principles to Higher Serre's conjecture II

In order to study Conjecture 4.10, we may apply Theorem 4.1 to torsors under semisimple simply connected groups and get the following conditional result:

Theorem 4.13. *If every countable field of characteristic* 0 *and cohomological dimension* ≤ 2 *has the* C^0_{sc} *property (resp. the* C^0_Λ *property for* Λ *a type in the classification of semisimple absolutely almost simple simply connected groups), then every field of cohomological dimension* $\leq q+2$ *has the* C^q_{sc} *property (resp.* C^q_Λ *property), for any* $q \geq 0$.

Proof. Assume first that the fields we are considering are of characteristic 0. Then we can apply directly Theorem 4.1 with $n = q+2$, $m = 2$, and Z an arbitrary torsor under a semisimple simply connected group. The result follows in this case.

We are left then with the case of positive characteristic, which will follow by an *ad hoc* transfer principle that will reduce this case to the one of characteristic 0. By Proposition 4.9, we may reduce our study to a fixed type Λ. And since finite extensions of a field have the same cohomological dimension, it will suffice to prove that, for any field K of cohomological dimension $q+2$ and characteristic $p>0$, any type Λ in the classification of semisimple absolutely almost simple simply connected groups and any principal homogeneous space Z under a simply connected isotypical K-group G of type Λ, we have $N_q(Z/K) = \mathrm{K}_q^{\mathrm{M}}(K)$.

We henceforth fix such a field K and type Λ. Let G be a semisimple simply connected isotypical K-group of type Λ. Using Lemma 2.4, we may assume that $G = R_{L/K}(H)$ for some finite separable extension L/K and an absolutely almost simple simply connected L-group H of type Λ. Let H_0 be the Chevalley group over $\mathbb{Z}$ such that H is a twisted form of H_0 over L. According to Proposition 5 of [Bou83, IX.2, Prop. 5], there exists a complete discrete valuation ring A that has p as a uniformizer, whose fraction field $\tilde{K}$ has characteristic 0 and whose residue field is K. Let $\tilde{L}$ be the unramified extension of $\tilde{K}$ with residue field L and let B be its valuation ring. By [SGA3, XXIV, Thm. 1.3], the group scheme $\mathrm{Aut}(H_0)$ is smooth over $\mathbb{Z}$, and hence, by [SGA3, XXIV, Prop. 8.1], the map $H^1(B, \mathrm{Aut}(H_0)) \to H^1(L, \mathrm{Aut}(H_0))$ is surjective. We deduce that there exists a semisimple simply connected absolutely almost simple group $\mathcal{H}$ of type Λ over B whose special fiber is H. We denote by $\tilde{H}$ its generic fiber and we set $\mathcal{G} := R_{B/A}(\mathcal{H})$ and $\tilde{G} := R_{\tilde{L}/\tilde{K}}(\tilde{H})$.

Now let Z be a torsor under G. By [Gro68, Thm. 11.7, Rem. 11.8], the map $H^1(A, \mathcal{G}) \to H^1(K, G)$ is bijective, and hence there exists a $\mathcal{G}$-torsor $\mathcal{Z}$ lifting Z. We denote by $\tilde{Z}$ the generic fiber of $\mathcal{Z}$. Then [Kat82, Cor. to Thm. 3] shows that $\tilde{K}$ has cohomological dimension $q+3$. Since $\tilde{K}$ has characteristic 0, it has the C_Λ^{q+1} property, hence we have:

$$N_{q+1}(\tilde{Z}/\tilde{K}) = \mathrm{K}_{q+1}^{\mathrm{M}}(\tilde{K}).$$

Thus, given a symbol $\{u_1, ..., u_q\}$ in $\mathrm{K}_q^{\mathrm{M}}(K)$ and some liftings $\tilde{u}_1, ..., \tilde{u}_q \in A$ of $u_1, ..., u_q$, we can find finite extensions $\tilde{K}_1, \dots, \tilde{K}_r$ of $\tilde{K}$ and elements $x_i \in \mathrm{K}_{q+1}^{\mathrm{M}}(\tilde{K}_i)$ for each i such that:

$$\begin{cases} \forall i, \ \tilde{Z}(\tilde{K}_i) \neq \emptyset, \\ \{p, \tilde{u}_1, ..., \tilde{u}_q\} = \sum_{i=1}^r N_{\tilde{K}_i/\tilde{K}}(x_i). \end{cases}$$

Denote by $\mathcal{O}_i, K_i$ the corresponding valuation rings and residue fields. By using the compatibility of the norm morphism in Milnor K-theory with the residue maps (cf. [GS17, Prop. 7.4.1]), we deduce that $\{u_1, ..., u_q\}$ is a sum of norms coming from the K_i's. Moreover, for each $i \in \{1, ..., r\}$, the Grothendieck-Serre conjecture for discrete valuation rings ([Nis84]) implies that the restriction morphism $H^1(\mathcal{O}_i, \mathcal{G}) \to H^1(\tilde{K}_i, \tilde{G})$ is injective, so that $\mathcal{Z}(\mathcal{O}_i) \neq \emptyset$. We deduce that $Z(K_i) \neq \emptyset$, so that $\{u_1, ..., u_q\} \in N_q(Z/K)$. This being true for any symbol in $\mathrm{K}_q^{\mathrm{M}}(K)$, we get:

$$N_q(Z/K) = \mathrm{K}_q^{\mathrm{M}}(K).$$

This proves that K has the C_Λ^q property, as wished. ☺

As a consequence, we get the following unconditional result:

Corollary 4.14. *Let K be a field of cohomological dimension $\leq q+2$ with $q \geq 0$. Then K satisfies properties $C_{A_n}^q$, $C_{B_n}^q$, $C_{C_n}^q$, $C_{D_n}^q$, $C_{G_4}^q$, $C_{F_2}^q$ and $C_{E_8}^q$.*

In particular, the only remaining cases left concerning Conjecture 4.10 are those about groups of types E_6 and E_7.

Proof. The result follows from Theorem 4.13 and [BP95] for every case with the exception of D_4 in the trialitarian case and E_8. For E_8, we use [Gil19, Thm. 8.4.1] and Lemma 2.5 for $q = 0$ instead of [BP95]. In the case of trialitarian D_4, we use the following result, which is well-known to experts, but which seems not to be published anywhere. ☺

Proposition 4.15. *Let K be a field of characteristic 0 and cohomological dimension ≤ 2. Let G be an absolutely almost simple simply connected group of trialitarian type D_4 and let Z be a G-torsor. Then Z admits a zero-cycle of degree 1.*

Proof. Applying Lemma 2.5 with $q = 0$, we immediately reduce to the case where either K is 3-special or G is not trialitarian anymore. In the latter case, we are done by [BP95]. Thus, we may assume that K is 3-special and hence we are in the case of cyclic triality.

Let L/K be the cyclic cubic extension such that G_L is not trialitarian anymore. Then G_L is an inner twist of the split group G_0 of type D_4, corresponding to a class $\alpha \in H^1(L, G_0^{\mathrm{ad}})$. Now, since the center of G_0 is 2-torsion and L is 3-special, we know that its cohomology is trivial and hence the map $H^1(L, G_0) \to H^1(L, G_0^{\mathrm{ad}})$ is surjective. However, by [BP95] we know that $H^1(L, G_0)$ is trivial, and hence so is α. This tells us that G_L is split and then [Gil19, Prop. 3.2.11.(3)] shows that G admits a maximal K-torus T split by the cyclic cubic extension L.

Finally, let G_1 be the quasi-split form of G. By the same argument from above, we deduce that G can be seen as a twist of G_1 defined by a class $\beta \in H^1(K, G_1)$. Applying [BGL15, Thm. 2.4.1] to the embedding $T \to G$, we deduce that there exists an embedding $T \to G_1$ such that β comes from $H^1(K, T)$. However, by [Gil19, Cor. 5.5.2.(1)], we know that the map $H^1(K, T) \to H^1(K, G_1)$ is trivial, implying the triviality of β. We conclude then that G is quasi-split and thus the result follows once again from [BP95]. ☺

4.3 Classical Serre's conjecture II in positive characteristic

To finish the article, we apply our transfer principles to prove that Serre's conjecture II in characteristic 0 (over countable fields) implies Serre's conjecture II in positive characteristic:

Theorem 4.16. *Let Λ be a type in the classification of semisimple absolutely almost simple simply connected groups. If Serre's conjecture II (Conjecture 1.2) holds for torsors under semisimple simply connected groups of type Λ over characteristic 0 countable fields with cohomological dimension 2, then it holds for torsors under semisimple simply connected groups of type Λ over arbitrary fields.*

Proof. Let K be a field with cohomological dimension ≤ 2, and let Z be a torsor under a semisimple simply connected group G of type Λ over K. Let K_0 be a countable subfield of K over which G and Z have forms G_0 and Z_0. According to Theorem 3.1, we can find a countable extension L of K_0 contained in K and of cohomological dimension ≤ 2. If K has characteristic 0, then, by assumption, $Z_{0,L}$ is trivial, and hence so is Z.

We henceforth assume that K has characteristic $p > 0$. As in the proof of Theorem 4.13, one then can find:

- a complete discrete valuation ring B that has p as a uniformizer, whose fraction field $\tilde{L}$ has characteristic 0 and whose residue field is L;
- a semisimple simply connected group $\mathcal{G}_0$ of type Λ over B with generic fiber $\tilde{G}_0$ and special fiber $G_{0,L}$;

- a torsor $\mathcal{Z}_0$ under $\mathcal{G}_0$ with generic fiber $\tilde{Z}_0$ and special fiber $Z_{0,L}$.

Theorem 3.4 allows us to consider a totally ramified extension $\tilde{L}_\dagger$ of $\tilde{L}$ with cohomological dimension ≤ 2 and integer ring $B_\dagger$. By the case where K has characteristic 0, that we have already solved, the class $[\tilde{Z}_{0,\tilde{L}_\dagger}] \in H^1(\tilde{L}_\dagger, \tilde{G}_0)$ is trivial. Hence there exists a finite subextension $\tilde{L}_1/\tilde{L}$ of $\tilde{L}_\dagger$, with integer ring B_1 such that $[\tilde{Z}_{0,\tilde{L}_1}]$ is trivial in $H^1(\tilde{L}_1, \tilde{G}_0)$. Now, by the Grothendieck-Serre conjecture (cf. [Nis84]), we know that the map $H^1(B_1, \mathcal{G}_{0,B_1}) \to H^1(\tilde{L}_1, \tilde{G}_{0,\tilde{L}_1})$ is injective, and hence $[\mathcal{Z}_{0,B_1}]$ is trivial as well. Finally, since $\tilde{L}_1$ has residue field L, the specialization of $\mathcal{Z}_{0,B_1}$ at the closed point is $Z_{0,L}$, which is then trivial. The torsor Z is therefore also trivial. ☺

Again, as a consequence of Theorem 4.16, we deduce the following unconditional result from [BP95].

Corollary 4.17. *Let K be a field of cohomological dimension $q+2$ with $q \geq 0$. Then $H^1(K,G) = 1$ for every semisimple simply connected K-group with no factors of types E_6, E_7, E_8 or trialitarian D_4.*

This result was already proved in [BFT07] in a completely different way based on a case by case study following the classification of simply connected semisimple groups instead of focusing on fields.

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