On Kato and Kuzumaki's properties for the Milnor K_2 of function fields of p-adic curves

Diego Izquierdo and Giancarlo Lucchini Arteche

Abstract

Let K be the function field of a curve C over a p-adic field k. We prove that, for each $n, d \geq 1$ and for each hypersurface Z in $\mathbb{P}^n_K$ of degree d with $d^2 \leq n$, the second Milnor K-theory group of K is spanned by the images of the norms coming from finite extensions L of K over which Z has a rational point. When the curve C has a point in the maximal unramified extension of k, we generalize this result to hypersurfaces Z in $\mathbb{P}^n_K$ of degree d with $d \leq n$.

MSC Classes: 11E76, 11G20, 12G05, 12G10, 14G05, 14G27, 14J45, 14J70, 19C99, 19F05.

Keywords: Milnor K-theory, zero-cycles, Fano hypersurfaces, p-adic function fields, C_i property, Galois cohomology, cohomological dimension.

1. Introduction

In 1986, in the article [KK86], Kato and Kuzumaki stated a set of conjectures which aimed at giving a diophantine characterization of cohomological dimension of fields. For this purpose, they introduced some properties of fields which are variants of the classical C_i-property and which involve Milnor K-theory and projective hypersurfaces of small degree. They hoped that those properties would characterize fields of small cohomological dimension.

More precisely, fix a field K and two non-negative integers q and i. Let $K_q(K)$ be the q-th Milnor K-group of K. For each finite extension L of K, one can define a norm morphism $N_{L/K} : K_q(L) \to K_q(K)$ (see Section 1.7 of [Kat80]). Thus, if Z is a scheme of finite type over K, one can introduce the subgroup $N_q(Z/K)$ of $K_q(K)$ generated by the images of the norm morphisms $N_{L/K}$ when L runs through the finite extensions of K such that $Z(L) \neq \emptyset$. One then says that the field K is C_i^q if, for each $n \geq 1$, for each finite extension L of K and for each hypersurface Z in $\mathbb{P}^n_L$ of degree d with $d^i \leq n$, one has $N_q(Z/L) = K_q(L)$. For example, the field K is C_i^0 if, for each finite extension L of K, every hypersurface Z in $\mathbb{P}^n_L$ of degree d with $d^i \leq n$ has a 0-cycle of degree 1. The field K is C_0^q if, for each tower of finite extensions $M/L/K$, the norm morphism $N_{M/L} : K_q(M) \to K_q(L)$ is surjective.

Kato and Kuzumaki conjectured that, for $i \geq 0$ and $q \geq 0$, a perfect field is C_i^q if, and only if, it is of cohomological dimension at most $i+q$. This conjecture generalizes a question raised by Serre in [Ser65] asking whether the cohomological dimension of a C_i-field is at most i. As it was already pointed out at the end of Kato and Kuzumaki's original paper [KK86], Kato and Kuzumaki's conjecture for $i = 0$ follows from the Bloch-Kato conjecture (which has been established by Rost and Voevodsky, cf. [Rio14]): in other words, a perfect field is C_0^q if, and only if, it is of cohomological dimension at most q. However, it turns out that the conjectures of Kato and Kuzumaki are wrong in general. For example, Merkurjev constructed in [Mer91] a field of characteristic 0 and of cohomological dimension 2 which did not satisfy property C_2^0. Similarly, Colliot-Thélène and Madore produced in [CTM04] a field of characteristic 0 and of cohomological dimension 1 which did not satisfy property C_1^0. These counter-examples were all constructed by a method using transfinite induction due to Merkurjev and Suslin. The conjecture of Kato and Kuzumaki is therefore still completely open for fields that usually appear in number theory or in algebraic geometry.

In 2015, in [Wit15], Wittenberg proved that totally imaginary number fields and p-adic fields have the C_1^1 property. In 2018, in [Izq18], the first author also proved that, given a positive integer n, finite extensions of $\mathbb{C}(x_1, ..., x_n)$ and of $\mathbb{C}(x_1, ..., x_{n-1})((t))$ are C_i^q for any $i, q \geq 0$ such that $i + q = n$. These are essentially the only known cases of Kato and Kuzumaki's conjectures. Note however that a variant of the C_1^q-property involving homogeneous spaces under connected linear groups is proved to characterize fields with cohomological dimension at most $q + 1$ in [ILA21].

In the present article, we are interested in Kato and Kuzumaki's conjectures for the function field K of a smooth projective curve C defined over a p-adic field k. The field K has cohomological dimension 3, and hence it is expected to satisfy the C_i^q-property for $i + q \geq 3$. As already mentioned, the Bloch-Kato conjecture implies this result when $q \geq 3$. In this article, we make progress in the case $q = 2$.

Our first main result is the following:

Main Theorem A. *Function fields of p-adic curves satisfy the C_2^2-property.*

Of course, this implies that function fields of p-adic curves also satisfy the C_i^2-property for each $i \geq 2$. It therefore only remains to prove the C_1^2-property. In that direction, we prove the following main result:

Main Theorem B. *Let K be the function field of a smooth projective curve C defined over a p-adic field k. Assume that C has a point in the maximal unramified extension of k. Then, for each $n, d \geq 1$ and for each hypersurface Z in $\mathbb{P}_K^n$ of degree d with $d \leq n$, we have $K_2(K) = N_2(Z/K)$.*

This theorem applies for instance when K is the rational function field $k(t)$ or the function field of a curve that has a rational point.

The article is structured as follows. In Section 2, we introduce all the notations and basic definitions we will need in the sequel. In Section 3, we prove Theorem 3.1, which widely generalizes Main Theorem A. Finally, in Section 4, we prove Theorem 4.8, and its Corollaries 4.9 and 4.10, which widely generalize Main Theorem B.

Acknowledgements. We thank Jean-Louis Colliot-Thélène and Olivier Wittenberg for their comments and suggestions.

The second author's research was partially supported by ANID via FONDECYT Grant 1210010.

2. Notations and preliminaries

In this section we fix the notations that will be used throughout this article.

Milnor K-theory Let K be any field and let q be a non-negative integer. The q-th Milnor K-group of K is by definition the group $K_0(K) = \mathbb{Z}$ if $q = 0$ and:

$$K_q(K) := \underbrace{K^\times \otimes_{\mathbb{Z}} ... \otimes_{\mathbb{Z}} K^\times}_{q \text{ times}} / \langle x_1 \otimes ... \otimes x_q | \exists i, j, i \neq j, x_i + x_j = 1 \rangle$$

if $q > 0$. For $x_1, ..., x_q \in K^\times$, the symbol $\{x_1, ..., x_q\}$ denotes the class of $x_1 \otimes ... \otimes x_q$ in $K_q(K)$. More generally, for r and s non-negative integers such that $r + s = q$, there is a natural pairing:

$$K_r(K) \times K_s(K) \to K_q(K)$$

which we will denote $\{\cdot, \cdot\}$.

When L is a finite extension of K, one can construct a norm homomorphism

$$N_{L/K} : K_q(L) \to K_q(K),$$

satisfying the following properties (see Section 1.7 of [Kat80] or Section 7.3 of [GS17]):

- For $q = 0$, the map $N_{L/K} : K_0(L) \to K_0(K)$ is given by multiplication by $[L : K]$.
- For $q = 1$, the map $N_{L/K} : K_1(L) \to K_1(K)$ coincides with the usual norm $L^\times \to K^\times$.
- If r and s are non-negative integers such that $r + s = q$, we have $N_{L/K}(\{x, y\}) = \{x, N_{L/K}(y)\}$ for $x \in K_r(K)$ and $y \in K_s(L)$.
- If M is a finite extension of L, we have $N_{M/K} = N_{L/K} \circ N_{M/L}$.

Recall also that Milnor K-theory is endowed with residue maps (see Section 7.1 of [GS17]). Indeed, when K is a henselian discrete valuation field with ring of integers R, maximal ideal $\mathfrak{m}$ and residue field κ, there exists a unique residue morphism:

$$\partial : K_q(K) \to K_{q-1}(\kappa)$$

such that, for each uniformizer π and for all units $u_2, ..., u_q \in R^\times$ whose images in κ are denoted $\overline{u_2}, ..., \overline{u_q}$, one has:

$$\partial(\{\pi, u_2, ..., u_q\}) = \{\overline{u_2}, ..., \overline{u_q}\}.$$

The kernel of ∂ is the subgroup $U_q(K)$ of $K_2(K)$ generated by symbols of the form $\{x_1, ..., x_q\}$ with $x_1, ..., x_q \in R^\times$. If $U_q^1(K)$ stands for the subgroup of $K_q(K)$ generated by those symbols that lie in $U_q(K)$ and that are of the form $\{x_1, ..., x_q\}$ with $x_1 \in \mathfrak{m}$ and $x_2, ..., x_q \in K^\times$, then $U_q^1(K)$ is ℓ-divisible for each prime ℓ different from the characteristic of κ and $U_q(K)/U_q^1(K)$ is canonically isomorphic to $K_q(\kappa)$. Moreover, if

L/K is a finite extension with ramification degree e and residue field λ, then the norm map $N_{L/K} : K_q(L) \to K_q(K)$ sends $U_q(L)$ to $U_q(K)$ and $U_q^1(L)$ to $U_q^1(K)$, and the following diagrams commute:

$$\begin{tikzcd} K_q(L)/U_q(L) \arrow[r, "\cong"] \arrow[d, "N_{L/K}"] & K_{q-1}(\lambda) \arrow[d, "N_{\lambda/\kappa}"] \\ K_q(K)/U_q(K) \arrow[r, "\cong"] & K_{q-1}(\kappa), \end{tikzcd} \qquad \begin{tikzcd} U_q(L)/U_q^1(L) \arrow[r, "\cong"] \arrow[d, "N_{L/K}"] & K_q(\lambda) \arrow[d, "eN_{\lambda/\kappa}"] \\ U_q(K)/U_q^1(K) \arrow[r, "\cong"] & K_q(\kappa). \end{tikzcd}$$

The C_i^q properties Let K be a field and let i and q be two non-negative integers. For each K-scheme Z of finite type, we denote by $N_q(Z/K)$ the subgroup of $K_q(K)$ generated by the images of the maps $N_{L/K} : K_q(L) \to K_q(K)$ when L runs through the finite extensions of K such that $Z(L) \neq \emptyset$. The field K is said to have the C_i^q property if, for each $n \geq 1$, for each finite extension L of K and for each hypersurface Z in $\mathbb{P}^n_L$ of degree d with $d^i \leq n$, one has $N_q(Z/L) = K_q(L)$.

Motivic complexes Let K be a field. For $i \geq 0$, we denote by $z^i(K, \cdot)$ Bloch's cycle complex defined in [Blo86]. The étale motivic complex $\mathbb{Z}(i)$ over K is then defined as the complex of Galois modules $z^i(-, \cdot)[-2i]$. By the Nesterenko-Suslin-Totaro Theorem and the Beilinson-Lichtenbaum Conjecture, it is known that :

$$H^i(K, \mathbb{Z}(i)) \cong K_i(K), \tag{1}$$

and

$$H^{i+1}(K, \mathbb{Z}(i)) = 0, \tag{2}$$

for all $i \geq 0$. Statement (1) was originally proved in [NS89] and [Tot92], and statement (2) was deduced from the Bloch-Kato conjecture in [SV00], [GL00] and [GL01]. The Bloch-Kato conjecture itself was proved in [SJ06] and [Voe11]. For the convenience of the reader, we also provide more tractable references: statement (1) follows from Theorem 5.1 of [HW19] and Theorem 1.2(2) of [Gei04], and statement (2) can be deduced from the Bloch-Kato conjecture as explained in Lemma 1.6 and Theorem 1.7 of [HW19].

Fields of interest From now on and until the end of the article, p stands for a prime number and k for a p-adic field. We let C be a smooth projective geometrically integral curve over k, and we let K be its function field. We denote by $C^{(1)}$ the set of closed points in C. The residual index $i_{\text{res}}(C)$ of C is defined to be the g.c.d. of the residual degrees of the $k(v)/k$ with $v \in C^{(1)}$. The ramification index $i_{\text{ram}}(C)$ of C is defined to be the g.c.d. of the ramification degrees of the $k(v)/k$ with $v \in C^{(1)}$.

Tate-Shafarevich groups When M is a complex of Galois modules over K and $i \geq 0$ is an integer, we define the i-th Tate-Shafarevich group of M as:

$$Ш^i(K, M) := \ker\left(H^i(K, M) \to \prod_{v \in C^{(1)}} H^i(K_v, M)\right).$$

When a suitable regular model $\mathcal{C}$ of C is given, we also introduce the following smaller Tate-Shafarevich groups:

$$Ш^i_{\mathcal{C}}(K, M) := \ker \left(H^i(K, M) \to \prod_{v \in \mathcal{C}^{(1)}} H^i(K_v, M) \right),$$

where $\mathcal{C}^{(1)}$ is the set of codimension 1 points of $\mathcal{C}$.

Poitou-Tate duality for motivic cohomology We recall the Poitou-Tate duality for motivic complexes over the field K (Theorem 0.1 of [Izq16] in the case $d = 1$). Let $\hat{T}$ be a finitely generated free Galois module over K. Set $\check{T} := \mathrm{Hom}(\hat{T}, \mathbb{Z})$ and $T = \check{T} \otimes \mathbb{Z}(2)$. Then there is a perfect pairing of finite groups:

$$\overline{Ш^2(K, \hat{T})} \times Ш^3(K, T) \to \mathbb{Q}/\mathbb{Z}, \tag{3}$$

where $\overline{A}$ denotes the quotient of A by its maximal divisible subgroup.

Note that, in the case $\hat{T} = \mathbb{Z}$, the Beilinson-Lichtenbaum conjecture (2) implies the vanishing of $Ш^3(K, \mathbb{Z}(2))$ and hence the group $Ш^2(K, \mathbb{Z})$ is divisible. By Shapiro's Lemma, the same holds for the group $Ш^2(K, \mathbb{Z}[E/K])$ for every étale K-algebra E.

3. On the C_2^2-property for p-adic function fields

The goal of this section is to prove the following theorem:

Theorem 3.1. *Let l/k be a finite unramified extension and set $L := lK$. Let Z be a proper K-variety. Then the quotient:*

$$K_2(K)/\langle N_{L/K}(K_2(L)), N_2(Z/K)\rangle$$

is $\chi_K(Z, E)^2$-torsion for each coherent sheaf E on Z.

At the end of the section, we explain how to deduce Main Theorem A.

3.1 Proof of Theorem 3.1

3.1.1 Step 0: Interpreting norms in Milnor K-theory in terms of motivic cohomology

The following lemma, which will be extensively used in the sequel, allows to interpret quotients of $K_2(K)$ by norm subgroups as twisted motivic cohomology groups.

Lemma 3.2. *Let L be a field and let $L_1, ..., L_r$ be finite separable extensions of L. Consider the étale L-algebra $E := \prod_{i=1}^r L_i$ and let $\check{T}$ be the Galois module defined by the following exact sequence:*

$$0 \to \check{T} \to \mathbb{Z}[E/L] \to \mathbb{Z} \to 0. \tag{4}$$

Then:

$$H^3(L, \check{T} \otimes \mathbb{Z}(2)) \cong K_2(L)/\langle N_{L_i/L}(K_2(L_i)) \mid 1 \le i \le r\rangle.$$

Proof. Exact sequence (4) induces a distinguished triangle:

$$\check{T} \otimes \mathbb{Z}(2) \to \mathbb{Z}[E/L] \otimes \mathbb{Z}(2) \to \mathbb{Z}(2) \to \check{T} \otimes \mathbb{Z}(2)[1].$$

By taking cohomology, we get an exact sequence:

$$H^2(L, \mathbb{Z}[E/L] \otimes \mathbb{Z}(2))) \to H^2(L, \mathbb{Z}(2)) \to H^3(L, \check{T} \otimes \mathbb{Z}(2)) \to H^3(L, \mathbb{Z}[E/L] \otimes \mathbb{Z}(2)).$$

By Shapiro's Lemma, we have:

$$H^2(L, \mathbb{Z}[E/L] \otimes \mathbb{Z}(2))) \cong H^2(E, \mathbb{Z}(2)),$$
$$H^3(L, \mathbb{Z}[E/L] \otimes \mathbb{Z}(2))) \cong H^3(E, \mathbb{Z}(2)).$$

Moreover, as recalled in section 2, the Nesterenko-Suslin-Totaro Theorem and the Beilinson-Lichtenbaum conjecture give the following isomorphisms:

$$H^2(L, \mathbb{Z}(2)) \cong K_2(L),$$
$$H^2(E, \mathbb{Z}(2)) \cong \prod_{i=1}^{r} K_2(L_i),$$
$$H^3(E, \mathbb{Z}(2)) = 0.$$

We therefore get an exact sequence:

$$\prod_{i=1}^{r} K_2(L_i) \to K_2(L) \to H^3(L, \check{T} \otimes \mathbb{Z}(2)) \to 0,$$

in which the first map is the product of the norms. □

3.1.2 Step 1: Reducing to curves with residual index 1

In this step, we prove the following proposition, that allows to reduce to the case when the curve C has residual index 1:

Proposition 3.3. *Let k'/k be the unramified extension of k of degree $i_{\text{res}}(C)$ and set $K' := k'K$. Then the norm morphism $N_{K'/K} : K_2(K') \to K_2(K)$ is surjective.*

Proof. Consider the Galois module $\check{T}$ defined by the following exact sequence:

$$0 \to \check{T} \to \mathbb{Z}[K'/K] \to \mathbb{Z} \to 0,$$

Since K'/K is cyclic, we also have an exact sequence:

$$0 \to \mathbb{Z} \to \mathbb{Z}[K'/K] \to \check{T} \to 0,$$

and hence a distinguished triangle:

$$\mathbb{Z}(2) \to \mathbb{Z}[K'/K] \otimes \mathbb{Z}(2) \to \check{T} \otimes \mathbb{Z}(2) \to \mathbb{Z}(2)[1].$$

By the Beilinson-Lichtenbaum conjecture, the group $H^3(K', \mathbb{Z}(2))$ is trivial. Hence we get an inclusion:

$$Ш^3_{\mathcal{C}}(K, \check{T} \otimes \mathbb{Z}(2)) \subseteq Ш^4_{\mathcal{C}}(K, \mathbb{Z}(2)),$$

where $\mathcal{C}$ is a fixed regular, proper and flat model of C whose reduced special fiber C_0 is a strict normal crossing divisor. Now, by Proposition 5.2 of [Kat86], the group $Ш^4_{\mathcal{C}}(K,\mathbb{Z}(2))$ is trivial, and hence so is the former group.

Now observe that, by Lemma 3.2, we have:

$$Ш^3_{\mathcal{C}}(K,\check{T}\otimes\mathbb{Z}(2)) \cong \ker\left(K_2(K)/\mathrm{im}(N_{K'/K}) \to \prod_{v\in\mathcal{C}^{(1)}} K_2(K_v)/\mathrm{im}(N_{K'_v/K_v})\right).$$

We claim that the extension K'/K totally splits at each place $v\in\mathcal{C}^{(1)}$. From this, we deduce that:

$$0 = Ш^3_{\mathcal{C}}(K,\check{T}\otimes\mathbb{Z}(2)) \cong K_2(K)/\mathrm{im}(N_{K'/K}),$$

and hence the norm morphism $N_{K'/K} : K_2(K') \to K_2(K)$ is surjective.

It remains to check the claim. It is obviously satisfied for $v\in C^{(1)}$, so we may and do assume $v\in\mathcal{C}^{(1)}\setminus C^{(1)}$. If κ and κ' denote the residue fields of k and k', we then have to prove that all the irreducible components of C_0 are κ'-curves. To do so, consider an infinite sequence of finite unramified field extensions $k=k_0\subset k_1\subset k_2\subset ...$ all with degrees prime to $[k':k]$ and denote by $\kappa=\kappa_0\subset\kappa_1\subset\kappa_2\subset ...$ the corresponding residue fields. Let k_∞ (resp. κ_∞) be the union of all the k_i's (resp. κ_i's). Since κ_∞ is infinite, Lemma 4.6 of [Wit15] and the definition of $i_{\mathrm{res}}(C)$ imply that each irreducible component of $C_0\times_{\kappa_0}\kappa_\infty$ has index divisible by $[k':k]$. Hence the same is true for all the irreducible components of C_0. The Weil conjectures then imply that these irreducible components are κ'-curves. □

3.1.3 Step 2: Solving the problem locally

In this step, we prove that the analogous statement to Theorem 3.1 over the completions of K holds. For that purpose, we first need to settle a simple lemma:

Lemma 3.4. *Let l/k be a finite extension and set $K_0 := k((t))$ and $L_0 := l((t))$. The residue map $\partial : K_2(K_0)\to k^\times$ induces an isomorphism:*

$$K_2(K_0)/N_{L_0/K_0}(K_2(L_0)) \cong k^\times/N_{l/k}(l^\times).$$

Proof. We have the following commutative diagram:

$$\begin{tikzcd} K_2(L_0) \arrow[r, "\partial"] \arrow[d, "N_{L_0/K_0}"'] & l^\times \arrow[d, "N_{l/k}"] \\ K_2(K_0) \arrow[r, "\partial"] & k^\times. \end{tikzcd}$$

Hence the residue map induces an exact sequence:

$$0\to \frac{U_2(K_0)}{U_2(K_0)\cap N_{L_0/K_0}(K_2(L_0))} \to \frac{K_2(K_0)}{N_{L_0/K_0}(K_2(L_0))} \to \frac{k^\times}{N_{l/k}(l^\times)} \to 0.$$

It therefore suffices to prove that $U_2(K_0) = U_2(K_0)\cap N_{L_0/K_0}(K_2(L_0))$. For that purpose, recall that we have a commutative diagram with exact lines:

$$\begin{tikzcd} 0 \arrow[r] & U_2^1(L_0) \arrow[r] \arrow[d, "N_{L_0/K_0}"] & U_2(L_0) \arrow[r] \arrow[d, "N_{L_0/K_0}"] & K_2(l) \arrow[r] \arrow[d, "N_{l/k}"] & 0 \\ 0 \arrow[r] & U_2^1(K_0) \arrow[r] & U_2(K_0) \arrow[r] & K_2(k) \arrow[r] & 0. \end{tikzcd}$$

But the map $N_{l/k} : K_2(l) \to K_2(k)$ is surjective since p-adic fields have the C_0^2-property, and the map $N_{L_0/K_0} : U_2^1(L_0) \to U_2^1(K_0)$ is surjective since the group $U_2^1(K_0)$ is divisible. We deduce that $N_{L_0/K_0} : U_2(L_0) \to U_2(K_0)$ is also surjective, as wished. $\square$

Proposition 3.5. *Let l/k be a finite unramified extension and set $K_0 := k((t))$ and $L_0 := l((t))$. Let Z be a proper K_0-variety. Then the quotient:*

$$K_2(K_0)/\langle N_{L_0/K_0}(K_2(L_0)), N_2(Z/K_0)\rangle$$

is $\chi_{K_0}(Z, E)$-torsion for each coherent sheaf E on Z.

Proof. For each proper K_0-scheme Z, we denote by n_Z the exponent of the quotient group $K_2(K_0)/\langle N_{L_0/K_0}(K_2(L_0)), N_2(Z/K_0)\rangle$. We say that Z satisfies property (P) if it has a model over $\mathcal{O}_{K_0}$ that is irreducible, regular, proper and flat. To prove the proposition, it suffices to check assumptions (1), (2) and (3) of Proposition 2.1 of [Wit15].

Assumption (1) is obvious. Assumption (3) is a direct consequence of Gabber and de Jong's Theorem (Theorem 3 of Exposé 0 of [ILO14]). It remains to check assumption (2). For that purpose, we proceed in the same way as in the proof of Theorem 4.2 of [Wit15]. Indeed, consider a proper K_0-scheme X together with a model $\mathcal{X}$ that is irreducible, regular, proper and flat and denote by Y its special fiber. Let m be the multiplicity of Y and let D be the effective divisor on $\mathcal{X}$ such that $Y = mD$.

The residue map induces an exact sequence:

$$0 \to \frac{U_2(K_0)}{U_2(K_0) \cap N_2(X/K_0)} \to \frac{K_2(K_0)}{N_2(X/K_0)} \to \frac{K_1(k)}{\partial(N_2(X/K_0))} \to 0. \tag{5}$$

Moreover:

(a) since k satisfies the C_0^2 property, the proof of Lemma 4.4 of [Wit15] still holds in our context, and hence the group $\frac{U_2(K_0)}{U_2(K_0)\cap N_2(X/K_0)}$ is killed by the multiplicity m of the special fiber Y of $\mathcal{X}$;

(b) the proof of Lemma 4.5 of [Wit15] also holds in our context, and hence $\partial(N_2(X/K_0)) = N_1(Y/k) = N_1(D/k)$;

(c) by Corollary 5.4 of [Wit15] applied to the proper k-scheme $D \sqcup \mathrm{Spec}(l)$, the group $k^\times/\langle N_{l/k}(l^\times), N_1(D/k)\rangle$ is killed by $\chi_k(D, \mathcal{O}_D)$.

By using exact sequence (5), facts (b) and (c) and Lemma 3.4, we deduce that:

$$\chi_k(D, \mathcal{O}_D) \cdot K_2(K_0) \subset \langle N_{L_0/K_0}(K_2(L_0)), N_2(X/K_0), U_2(K_0)\rangle.$$

Hence, by fact (a), we get:

$$m\chi_k(D, \mathcal{O}_D) \cdot K_2(K_0) \subset \langle N_{L_0/K_0}(K_2(L_0)), N_2(X/K_0)\rangle.$$

But $m\chi_k(D, \mathcal{O}_D) = \chi_{K_0}(X, \mathcal{O}_X)$ by Proposition 2.4 of [ELW15], and hence the quotient $K_2(K_0)/\langle N_{L_0/K_0}(K_2(L_0)), N_2(X/K_0)\rangle$ is killed by $\chi_{K_0}(X, \mathcal{O}_X)$. $\square$

3.1.4 Step 3: Globalizing local field extensions

In rest of the proof, we will show how one can deduce the global Theorem 3.1 from the local Proposition 3.5. For that purpose, we first need to find a suitable way to globalize local extension: more precisely, given a place $w \in C^{(1)}$ and a finite extension $M^{(w)}$ of K_w such that $Z(M^{(w)}) \neq \emptyset$, we want to find a suitable finite extension M of K that can be seen as a subfield of $M^{(w)}$ and such that $Z(M) \neq \emptyset$. For technical reasons related to the failure of Cébotarev's Theorem over the field K, we also need M to be linearly disjoint from a given finite extension of K. The following proposition is the key statement allowing to do that:

Proposition 3.6. *Let Z be a smooth geometrically integral K-variety. Let T be a finite subset of $C^{(1)}$. Fix a finite extension L of K and, for each $w \in T$, a finite extension $M^{(w)}$ of K_w such that $Z(M^{(w)}) \neq \emptyset$. Then there exists a finite extension M of K satisfying the following properties:*

(i) $Z(M) \neq \emptyset$;

(ii) for each $w \in T$, the field M is a subfield of $M^{(w)}$;

(iii) the extensions L/K and M/K are linearly disjoint.

Proof. Before starting the proof, we introduce the following notations for each $w \in T$:

$$n^{(w)} := [M^{(w)} : K_w],$$
$$m^{(w)} := \prod_{w' \in T \setminus \{w\}} n^{(w)},$$

so that the integer $n := n^{(w)} m^{(w)}$ is independent of w. We now proceed in three substeps.

Substep 1. Let Z' be a projective hypersurface in $\mathbb{P}^m_K$ given by a non-zero equation

$$f(x_0, ..., x_m) = 0$$

that is birationally equivalent to Z. Let U and U' be non-empty open sub-schemes of Z and Z' that are isomorphic. Up to reordering the variables and shrinking U', we may and do assume that the polynomial $\partial f / \partial x_0$ is non-zero and that:

$$U' \cap \{\partial f / \partial x_0 (x_0, ..., x_m) = 0\} = \emptyset.$$

Given an element $w \in T$, the variety Z is smooth, $Z(M^{(w)}) \neq \emptyset$ and $M^{(w)}$ is large (for the definition of this notion, please refer to [Pop14]). Hence the sets $U(M^{(w)})$ and $U'(M^{(w)})$ are non-empty. We can therefore find a non-trivial solution $(y_0^{(w)}, ..., y_m^{(w)})$ of the equation $f(x_0, ..., x_m) = 0$ in $M^{(w)}$ such that:

$$\begin{cases} (y_0^{(w)}, ..., y_m^{(w)}) \in U' \\ \partial f / \partial x_0 (y_0^{(w)}, ..., y_m^{(w)}) \neq 0. \end{cases}$$

Substep 2. Given $w \in T$, there exist $m^{(w)}$ elements $\alpha_1, ..., \alpha_{m^{(w)}} \in M^{(w)}$ whose respective minimal polynomials $\mu_{\alpha_1}, ..., \mu_{\alpha_{m^{(w)}}}$ are pairwise distinct and such that $M^{(w)} = K_w(\alpha_i)$ for each $1 \leq i \leq m^{(w)}$. Recalling that $n = n^{(w)} m^{(w)}$, introduce the degree n

monic polynomial $\mu^{(w)} := \prod_{i=1}^{m^{(w)}} \mu_{\alpha_i}$ and consider the set H of n-tuples $(a_0, ..., a_{n-1}) \in K^n$ such that the polynomial $T^n + \sum_{i=0}^{n-1} a_i T^i$ is irreducible over L. By Corollary 12.2.3 of [FJ08], the set H contains a Hilbertian subset of K^n, and hence, according to Proposition 19.7 of [Jar91], if we fix some $\epsilon > 0$, we can find an n-tuple $(b_0, ..., b_{n-1})$ in H such that the polynomial $\mu := T^n + \sum_{i=0}^{n-1} b_i T^i$ is coefficient-wise ϵ-close to $\mu^{(w)}$ for each $w \in T$. Consider the field $K' := K[T]/(\mu)$. If ϵ is chosen small enough, then K' is contained in $M^{(w)}$ for each $w \in T$. Moreover, since $(b_0, \dots, b_{n-1}) \in H$, the polynomial μ is irreducible over L, and hence the extensions K'/K and L/K are linearly disjoint.

Substep 3. According to Substep 1, for each $w \in T$, $y_0^{(w)}$ is a simple root of the polynomial

$$g^{(w)}(T) := f(T, y_1^{(w)}, ..., y_m^{(w)}).$$

Let H' be the set of m-tuples $(z_1, ..., z_m)$ in K' such that $f(T, z_1, ..., z_m)$ is irreducible over LK'. By Corollary 12.2.3 of [FJ08], the set H' contains a Hilbertian subset of K'^m. Hence, by Proposition 19.7 of [Jar91], we can find $(y_1, ..., y_m)$ in H' such that the polynomial

$$g(T) := f(T, y_1, ..., y_m)$$

is coefficient-wise ϵ-close to $g^{(w)}$ for each $w \in T$. Introduce the field $M := K'[T]/(g(T))$. We check that M satisfies the conditions of the proposition, provided that ϵ is chosen small enough:

(i) Fix $w \in T$. By Substep 1, the m-tuple $(y_0^{(w)}, ..., y_m^{(w)})$ lies in U'. Hence, for ϵ small enough, if $y_{0,w}$ stands for the root of g that is closest to $y_0^{(w)}$, then the m-tuple $(y_0, ..., y_m)$ lies in U'. We deduce that $U'(M) \neq \emptyset$, and hence $Z(M) \neq \emptyset$.

(ii) For each $w \in T$, the polynomial $g^{(w)}$ has a simple root in $M^{(w)}$, and hence so does $g(T)$ if ϵ is chosen small enough. The field M can therefore be seen as a subfield of $M^{(w)}$.

(iii) Since $(y_1, ..., y_m) \in H'$, the polynomial $g(T)$ is irreducible over LK'. Hence the extensions M/K' and LK'/K' are linearly disjoint. Moreover, by Substep 2, the extensions K'/K and L/K are linearly disjoint. We deduce that L/K and M/K are linearly disjoint.

□

3.1.5 Step 4: Computation of a Tate-Shafarevich group

This step, which is quite technical, consists in computing the Tate-Shafarevich groups of some finitely generated free Galois modules over K. Recall that for each abelian group A, we denote by $\overline{A}$ the quotient of A by its maximal divisible subgroup.

Proposition 3.7. *Let $r \geq 2$ be an integer and let $L, K_1, ..., K_r$ be finite extensions of K contained in $\overline{K}$. Consider the composite fields $K_{\mathcal{I}} := K_1...K_r$ and $K_{\hat{i}} := K_1...K_{i-1}K_{i+1}...K_r$ for each i, and denote by n the degree of L/K. Consider the Galois module $\hat{T}$ defined by the following exact sequence:*

$$0 \to \mathbb{Z} \to \mathbb{Z}[E/K] \to \hat{T} \to 0, \tag{6}$$

where $E := L \times K_1 \times ... \times K_r$. Given two positive integers m and m', make the following assumptions:

(LD1) *the Galois closure of L/K and the extension $K_{\mathcal{I}}/K$ are linearly disjoint;*

(LD2) *for each $i \in \{1, ..., r\}$, the fields K_i and $K_{\hat{i}}$ are linearly disjoint over K;*

(H1) *the restriction map:*

$$Ш^2(K, \hat{T}) \to Ш^2(L, \hat{T}) \oplus Ш^2(K_{\mathcal{I}}, \hat{T})$$

is injective;

(H2) *the restriction map*

$$\mathrm{Res}_{LK_{\mathcal{I}}/K_{\mathcal{I}}} : Ш^2(K_{\mathcal{I}}, \mathbb{Z}) \to Ш^2(LK_{\mathcal{I}}, \mathbb{Z})$$

is surjective and its kernel is m-torsion;

(H3) *for each i, the restriction maps*

$$\mathrm{Res}_{LK_i/K_i} : Ш^2(K_i, \mathbb{Z}) \to Ш^2(LK_i, \mathbb{Z})$$

and

$$\mathrm{Res}_{LK_{\hat{i}}/K_{\hat{i}}} : Ш^2(K_{\hat{i}}, \mathbb{Z}) \to Ш^2(LK_{\hat{i}}, \mathbb{Z})$$

are surjective;

(H4) *for each finite extension L' of L contained in the Galois closure of L/K, the kernel of the restriction map*

$$\mathrm{Res}_{L'K_{\mathcal{I}}/L} : Ш^2(L', \mathbb{Z}) \to Ш^2(L'K_{\mathcal{I}}, \mathbb{Z})$$

is m'-torsion.

Then $\overline{Ш^2(K, \hat{T})}$ is $((m \vee m') \wedge n)$-torsion.

Recall that $\overline{A}$ denotes the quotient of A by its maximal divisible subgroup.

Remark 3.8. In the sequel of the article, we will only use the proposition in the case when L/K is Galois. However, this assumption does not simplify the proof.

Proof. Consider the following sequence:

$$\begin{array}{ccccc} Ш^2(K, \hat{T}) & \xrightarrow{f_0} & Ш^2(L, \hat{T}) \oplus Ш^2(K_{\mathcal{I}}, \hat{T}) & \xrightarrow{g_0} & Ш^2(LK_{\mathcal{I}}, \hat{T}) \\ x & \longmapsto & (\mathrm{Res}_{L/K}(x), \mathrm{Res}_{K_{\mathcal{I}}/K}(x)) & & \\ & & (x, y) & \longmapsto & \mathrm{Res}_{LK_{\mathcal{I}}/L}(x) - \mathrm{Res}_{LK_{\mathcal{I}}/K_{\mathcal{I}}}(y). \end{array} \tag{7}$$

It is obviously a complex, and the first arrow is injective by (H1). In order to give further information about the complex (7), let us consider the following commutative diagram, in which the first and second rows are obtained in the same way as the third:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] \\
Ш^2(K,\mathbb{Z}) \arrow[r, "f_1"] \arrow[d, "\phi_0"] & Ш^2(L,\mathbb{Z}) \oplus Ш^2(K_{\mathcal{I}},\mathbb{Z}) \arrow[r, two heads, "g_1"] \arrow[d, "\phi_1"] & Ш^2(LK_{\mathcal{I}},\mathbb{Z}) \arrow[d, "\phi_2"] \\
Ш^2(K,\mathbb{Z}[E/K]) \arrow[r, "f"] \arrow[d, "\psi_0"] & Ш^2(L,\mathbb{Z}[E/K]) \oplus Ш^2(K_{\mathcal{I}},\mathbb{Z}[E/K]) \arrow[r, "g"] \arrow[d, "\psi_1"] & Ш^2(LK_{\mathcal{I}},\mathbb{Z}[E/K]) \arrow[d, "\psi_2"] \\
Ш^2(K,\hat{T}) \arrow[r, hook, "f_0"] & Ш^2(L,\hat{T}) \oplus Ш^2(K_{\mathcal{I}},\hat{T}) \arrow[r, "g_0"] \arrow[d] & Ш^2(LK_{\mathcal{I}},\hat{T}) \arrow[d] \\
 & 0 & 0
\end{tikzcd} \tag{8}$$

The second and third columns are exact since the exact sequence (6) splits over L, $K_{\mathcal{I}}$ and $LK_{\mathcal{I}}$. Moreover, all the lines are complexes, and in the first one, the arrow g_1 is surjective since the restriction map:

$$Ш^2(K_{\mathcal{I}},\mathbb{Z}) \to Ш^2(LK_{\mathcal{I}},\mathbb{Z})$$

is surjective by assumption (H2).

The next two lemmas constitute the core of the proof of Proposition 3.7.

Lemma 3.9. *Let* $a \in Ш^2(K,\hat{T})$ *and* $b = (b_L, b_{K_{\mathcal{I}}}) \in Ш^2(L,\mathbb{Z}[E/K]) \oplus Ш^2(K_{\mathcal{I}},\mathbb{Z}[E/K]))$ *such that* $f_0(a) = \psi_1(b)$ *and* $g(b) = 0$. *Then* $mb_{K_{\mathcal{I}}}$ *comes by restriction from* $Ш^2(K_{\hat{i}},\mathbb{Z}[E/K])$ *for each* i.

Proof. Consider the following commutative diagram, constructed exactly in the same way as diagram (8):

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] \\
Ш^2(K,\mathbb{Z}) \arrow[r, "f_1^i"] \arrow[d, "\phi_0"] & Ш^2(L,\mathbb{Z}) \oplus Ш^2(K_{\hat{i}},\mathbb{Z}) \arrow[r, two heads, "g_1^i"] \arrow[d, "\phi_1^i"] & Ш^2(LK_{\hat{i}},\mathbb{Z}) \arrow[d, "\phi_2^i"] \\
Ш^2(K,\mathbb{Z}[E/K]) \arrow[r, "f^i"] \arrow[d, "\psi_0"] & Ш^2(L,\mathbb{Z}[E/K]) \oplus Ш^2(K_{\hat{i}},\mathbb{Z}[E/K]) \arrow[r, "g^i"] \arrow[d, "\psi_1^i"] & Ш^2(LK_{\hat{i}},\mathbb{Z}[E/K]) \arrow[d, "\psi_2^i"] \\
Ш^2(K,\hat{T}) \arrow[r, hook, "f_0^i"] & Ш^2(L,\hat{T}) \oplus Ш^2(K_{\hat{i}},\hat{T}) \arrow[r, "g_0^i"] \arrow[d] & Ш^2(LK_{\hat{i}},\hat{T}) \arrow[d] \\
 & 0 & 0
\end{tikzcd}$$

The last two columns are exact since the exact sequence (6) splits over L, $K_{\hat{i}}$ and $LK_{\hat{i}}$, and the restriction morphism $Ш^2(K_{\hat{i}},\mathbb{Z}) \to Ш^2(LK_{\hat{i}},\mathbb{Z})$ is surjective by (H3). Hence there exists $b_{K_{\hat{i}}} \in Ш^2(K_{\hat{i}}, \mathbb{Z}[E/K])$ such that $\psi_1^i(b_L, b_{K_{\hat{i}}}) = f_0^i(a)$ and $g^i(b_L, b_{K_{\hat{i}}}) = 0$. The pair:

$$(0, b_{K_{\mathcal{I}}} - \mathrm{Res}_{K_{\mathcal{I}}/K_{\hat{i}}}(b_{K_{\hat{i}}})) \in Ш^2(L, \mathbb{Z}[E/K]) \oplus Ш^2(K_{\mathcal{I}}, \mathbb{Z}[E/K])$$

then lies in $\ker(g) \cap \ker(\psi_1)$ and a diagram chase in (8) shows that there exists $c \in Ш^2(K_{\mathcal{I}}, \mathbb{Z})$ such that:

$$\begin{cases} \phi_1(0, c) = (0, b_{K_{\mathcal{I}}} - \mathrm{Res}_{K_{\mathcal{I}}/K_{\hat{i}}}(b_{K_{\hat{i}}})) \\ \mathrm{Res}_{LK_{\mathcal{I}}/K_{\mathcal{I}}}(c) = 0. \end{cases}$$

By (H2), we have $mc = 0$, and hence: $m \cdot (b_{K_{\mathcal{I}}} - \mathrm{Res}_{K_{\mathcal{I}}/K_{\hat{i}}}(b_{K_{\hat{i}}})) = 0$. □

Lemma 3.10. *Set $\mu := m \vee m'$ and take $a \in Ш^2(K, \hat{T})$. Then $\mu a \in \mathrm{Im}(\psi_0)$.*

Before proving the lemma, let us introduce some notation.

Notation 3.11. (i) For each i, we can find a family $(K_{ij})_j$ of finite extensions of $K_{\mathcal{I}}$ together with embeddings $\sigma_{ij} : K_i \hookrightarrow K_{ij}$ so that $K_{i,1} = K_{\mathcal{I}}$, the embedding $\sigma_{i,1}$ is the identity of $K_{\mathcal{I}}$, and the K-algebra homomorphism:

$$\begin{aligned} K_i \otimes_K K_{\mathcal{I}} &\to \prod_j K_{ij} \\ x \otimes y &\mapsto (\sigma_{ij}(x)y)_j \end{aligned}$$

is an isomorphism. We denote by $\tilde{\sigma}_{ij} : K_{\mathcal{I}} \to K_{ij}$ the embedding obtained by tensoring σ_{ij} with the identity of $K_{\hat{i}}$. This is well-defined by (LD2).

(ii) For each i, j, we can find a family $(L_{ijj'})_{j'}$ of finite extensions of K_{ij} together with embeddings $\sigma_{ijj'} : L \hookrightarrow L_{ijj'}$ so that the K-algebra homormorphism:

$$\begin{aligned} L \otimes_K K_{ij} &\to \prod_{j'} L_{ijj'} \\ x \otimes y &\mapsto (\sigma_{ijj'}(x)y)_{j'}, \end{aligned} \tag{9}$$

is an isomorphism. We denote by $\tilde{\sigma}_{ijj'} : LK_i \to L_{ijj'}$ the embedding obtained by tensoring $\sigma_{ijj'}$ with σ_{ij}. Observe that, when $j = 1$, the K-algebra homomorphism (9) is simply the isomorphism $L \otimes_K K_{\mathcal{I}} \cong LK_{\mathcal{I}}$, so that $\sigma_{i,1,1}$ is none other than the inclusion of L in $LK_{\mathcal{I}}$.

(iii) We can find a family of finite extensions $(L_\alpha)_\alpha$ of L together with embeddings $\tau_\alpha : L \hookrightarrow L_\alpha$ so that $L_1 = L$, the embedding τ_1 is the identity of L, and the K-algebra homormorphism:

$$\begin{aligned} L \otimes_K L &\to \prod_\alpha L_\alpha \\ x \otimes y &\mapsto (\tau_\alpha(x)y)_\alpha \end{aligned}$$

is an isomorphism. For each α, we denote by $\tilde{\tau}_\alpha : LK_{\mathcal{I}} \to L_\alpha K_{\mathcal{I}}$ the embedding obtained by tensoring τ_α with the identity of $K_{\mathcal{I}}$. This is well-defined by (LD1).

Proof. By Shapiro's Lemma, one can identify the second line of diagram (8) with the following complex:

$$\begin{tikzcd}
\text{Ш}^2(L,\mathbb{Z}) \oplus \bigoplus_i \text{Ш}^2(K_i,\mathbb{Z}) \arrow[d, "f"] \\
\bigoplus_\alpha \text{Ш}^2(L_\alpha,\mathbb{Z}) \oplus \bigoplus_i \text{Ш}^2(LK_i,\mathbb{Z}) \oplus \text{Ш}^2(LK_{\mathcal{I}},\mathbb{Z}) \oplus \bigoplus_{i,j} \text{Ш}^2(K_{ij},\mathbb{Z}) \arrow[d, "g"] \\
\bigoplus_\alpha \text{Ш}^2(L_\alpha K_{\mathcal{I}},\mathbb{Z}) \oplus \bigoplus_{i,j,j'} \text{Ш}^2(L_{ijj'},\mathbb{Z})
\end{tikzcd}$$

where f is given by:

$$(x,(y_i)_i) \mapsto \left(((\mathrm{Res}_{\tau_\alpha : L \hookrightarrow L_\alpha}(x))_\alpha, \left(\mathrm{Res}_{LK_i/K_i}(y_i)\right)_i, \mathrm{Res}_{LK_{\mathcal{I}}/L}(x), \left(\mathrm{Res}_{\sigma_{ij}:K_i \hookrightarrow K_{ij}}(y_i)\right)_{i,j} \right),$$

and g:

$$((x_\alpha)_\alpha,(y_i)_i,z,(t_{ij})_{i,j}) \mapsto$$
$$\left(\left(\mathrm{Res}_{L_\alpha K_{\mathcal{I}}/L_\alpha}(x_\alpha) - \mathrm{Res}_{\tilde{\tau}_\alpha : LK_{\mathcal{I}} \hookrightarrow L_\alpha K_{\mathcal{I}}}(z)\right)_\alpha, \left(\mathrm{Res}_{\tilde{\sigma}_{ijj'}:LK_i \hookrightarrow L_{ijj'}}(y_i) - \mathrm{Res}_{L_{ijj'}/K_{ij}}(t_{ij})\right)_{i,j} \right).$$

Now take:

$$((x_\alpha)_\alpha,(y_i)_i,z,(t_{ij})_{i,j}) \in \bigoplus_\alpha \text{Ш}^2(L_\alpha,\mathbb{Z}) \oplus \bigoplus_i \text{Ш}^2(LK_i,\mathbb{Z}) \oplus \text{Ш}^2(LK_{\mathcal{I}},\mathbb{Z}) \oplus \bigoplus_{i,j} \text{Ш}^2(K_{ij},\mathbb{Z})$$

such that:

$$\psi_1((x_\alpha)_\alpha,(y_i)_i,z,(t_{ij})_{i,j}) = f_0(a).$$

Since $g_0(f_0(a)) = 0$ and g_1 is surjective, a diagram chase in (8) allows to assume that:

$$((x_\alpha)_\alpha,(y_i)_i,z,(t_{ij})_{i,j}) \in \ker(g).$$

This implies that:

$$\begin{cases} \mathrm{Res}_{L_\alpha K_{\mathcal{I}}/L_\alpha}(x_\alpha) = \mathrm{Res}_{\tilde{\tau}_\alpha : LK_{\mathcal{I}} \hookrightarrow L_\alpha K_{\mathcal{I}}}(z), \quad \forall\, \alpha, & (10) \\ \mathrm{Res}_{\tilde{\sigma}_{ijj'}:LK_i \hookrightarrow L_{ijj'}}(y_i) = \mathrm{Res}_{L_{ijj'}/K_{ij}}(t_{ij}), \quad \forall\, i,j,j'. & (11) \end{cases}$$

In particular, $\mathrm{Res}_{L_1 K_{\mathcal{I}}/L_1}(x_1) = \mathrm{Res}_{LK_{\mathcal{I}}/L}(x_1) = z$, and hence the commutativity of the following diagram of field extensions:

$$\begin{tikzcd}[column sep=small]
 & L_\alpha K_{\mathcal{I}} & \\
L_\alpha \arrow[ur, dash] & & LK_{\mathcal{I}} \arrow[ul, dash, "\tilde{\tau}_\alpha"'] \\
 & L \arrow[ul, dash, "\tau_\alpha"] \arrow[ur, dash] &
\end{tikzcd}$$

shows that:

$$\begin{aligned}\mathrm{Res}_{L_\alpha K_{\mathcal{I}}/L_\alpha}(\mathrm{Res}_{\tau_\alpha:L\hookrightarrow L_\alpha}(x_1)) &= \mathrm{Res}_{\tilde{\tau}_\alpha:LK_{\mathcal{I}}\hookrightarrow L_\alpha K_{\mathcal{I}}}(\mathrm{Res}_{LK_{\mathcal{I}}/L}(x_1))\\ &= \mathrm{Res}_{\tilde{\tau}_\alpha:LK_{\mathcal{I}}\hookrightarrow L_\alpha K_{\mathcal{I}}}(z)\\ &= \mathrm{Res}_{L_\alpha K_{\mathcal{I}}/L_\alpha}(x_\alpha).\end{aligned}$$

Since the kernel of $\mathrm{Res}_{L_\alpha K_{\mathcal{I}}/L_\alpha}$ is m'-torsion by (H4), we have:

$$m'\mathrm{Res}_{\tau_\alpha:L\hookrightarrow L_\alpha}(x_1) = m'x_\alpha$$

for all α. Moreover, by (H3), one can find for each i an element $\tilde{y}_i \in Ш^2(K_i,\mathbb{Z})$ such that:

$$y_i = \mathrm{Res}_{LK_i/K_i}(\tilde{y}_i).$$

Let us check that:

$$\mu((x_\alpha)_\alpha,(y_i)_i,z,(t_{ij})_{i,j}) = \mu f\left(x_1,(\tilde{y}_i)_i\right). \tag{12}$$

By construction, we have:

$$\begin{gathered}\mu(\mathrm{Res}_{\tau_\alpha:L\hookrightarrow L_\alpha}(x_1))_\alpha = \mu(x_\alpha)_\alpha,\\ (y_i)_i = \left(\mathrm{Res}_{LK_i/K_i}(\tilde{y}_i)\right)_i,\\ \mu\mathrm{Res}_{LK_{\mathcal{I}}/L}(x_1) = \mu z.\end{gathered}$$

To finish the proof of (12), it is therefore enough to check that:

$$mt_{ij} = m\mathrm{Res}_{\sigma_{ij}:K_i\hookrightarrow K_{ij}}(\tilde{y}_i) \tag{13}$$

for each i and j. For that purpose, fix $i = i_0$, and consider first the case $j = 1$. We then have $K_{i_0,1} = K_{\mathcal{I}}$, and hence, by using (11):

$$\mathrm{Res}_{LK_{\mathcal{I}}/K_{\mathcal{I}}}(t_{i_0,1}) = \mathrm{Res}_{L_{i_0,1,1}/LK_{i_0}}(y_{i_0}) = \mathrm{Res}_{L_{i_0,1,1}/K_{i_0}}(\tilde{y}_{i_0}) = \mathrm{Res}_{LK_{\mathcal{I}}/K_{\mathcal{I}}}(\mathrm{Res}_{K_{i_0,1}/K_{i_0}}(\tilde{y}_{i_0})).$$

By (H2), we deduce that:

$$mt_{i_0,1} = m\mathrm{Res}_{K_{i_0,1}/K_{i_0}}(\tilde{y}_{i_0}) = m\mathrm{Res}_{K_{\mathcal{I}}/K_{i_0}}(\tilde{y}_{i_0}).$$

Now move on to case of arbitrary j. By Lemma 3.9, the element:

$$(mt_{i_0,j})_j \in \bigoplus_j Ш^2(K_{i_0,j},\mathbb{Z}) = Ш^2(K_{\mathcal{I}},\mathbb{Z}[K_{i_0}/K])$$

comes by restriction from an element $t_{i_0} \in Ш^2(K_{\mathcal{I}},\mathbb{Z}) = Ш^2(K_{\hat{i}_0},\mathbb{Z}[K_{i_0}/K])$. In other words:

$$(mt_{i_0,j})_j = (\mathrm{Res}_{\tilde{\sigma}_{i_0,j}:K_{\mathcal{I}}\hookrightarrow K_{i_0,j}}(t_{i_0}))_j.$$

In particular, $mt_{i_0,1} = t_{i_0}$, and hence for each j:

$$\begin{aligned}mt_{i_0,j} &= \mathrm{Res}_{\tilde{\sigma}_{i_0,j}:K_{\mathcal{I}}\hookrightarrow K_{i_0,j}}(t_{i_0})\\ &= \mathrm{Res}_{\tilde{\sigma}_{i_0,j}:K_{\mathcal{I}}\hookrightarrow K_{i_0,j}}(mt_{i_0,1})\\ &= \mathrm{Res}_{\tilde{\sigma}_{i_0,j}:K_{\mathcal{I}}\hookrightarrow K_{i_0,j}}(m\mathrm{Res}_{K_{\mathcal{I}}/K_{i_0}}(\tilde{y}_{i_0}))\\ &= m\mathrm{Res}_{\sigma_{i_0,j}:K_{i_0}\hookrightarrow K_{i_0,j}}(\tilde{y}_{i_0}).\end{aligned}$$

This finishes the proofs of equalities (13) and (12). Applying ψ_1 to (12) we deduce that:

$$\mu f_0(\alpha) = \mu f_0(\psi_0((x_\alpha)_\alpha, (y_i)_i, z, (t_{ij})_{i,j})).$$

Since f_0 is injective, we get:

$$\mu\alpha = \mu\psi_0((x_\alpha)_\alpha, (y_i)_i, z, (t_{ij})_{i,j}),$$

which finishes the proof of the lemma. □

We can now finish the proof of Proposition 3.7. As recalled at the end of section 2, the group $Ш^2(K, \mathbb{Z}[E/K])$ is divisible and hence, by Lemma 3.10:

$$(m \vee m') \cdot Ш^2(K, \hat{T}) \subseteq Ш^2(K, \hat{T})_{\mathrm{div}}.$$

In other words, the group $\overline{Ш^2(K, \hat{T})}$ is $(m \vee m')$-torsion.

On the other hand, using once again the end of section 2, the group $\overline{Ш^2(L, \hat{T})}$ vanishes. Hence, by restriction-corestriction, $\overline{Ш^2(K, \hat{T})}$ is n-torsion. We deduce that $\overline{Ш^2(K, \hat{T})}$ is $((m \vee m') \wedge n)$-torsion. □

The following lemma will often allow us to check assumptions (H2) and (H3) of Proposition 3.7:

Lemma 3.12. *Let l be a finite unramified extension of k of degree n and set $L = lK$. The restriction map $\mathrm{Res}_{L/K} : Ш^2(K, \mathbb{Z}) \to Ш^2(L, \mathbb{Z})$ is surjective and its kernel is $(i_{\mathrm{res}}(C) \wedge n)$-torsion.*

Proof. Since $Ш^2(K, \mathbb{Z}) = Ш^1(K, \mathbb{Q}/\mathbb{Z})$, it is obvious that $\ker(\mathrm{Res}_{L/K})$ is killed by $i_{\mathrm{res}}(C) \wedge n$. In order to prove the surjectivity statement, consider an integral, regular, projective model $\mathcal{C}$ of C such that its reduced special fibre C_0 is an SNC divisor. Let c be the genus of the reduction graph of $\mathcal{C}$. According to Corollary 2.9 of [Kat86], for each $n \geq 1$, we have an isomorphism:

$$Ш^3(K, \mathbb{Z}/n\mathbb{Z}(2)) \cong (\mathbb{Z}/n\mathbb{Z})^c.$$

Hence, by Poitou-Tate duality, we also have:

$$Ш^1(K, \mathbb{Z}/n\mathbb{Z}) \cong (\mathbb{Z}/n\mathbb{Z})^c,$$

so that:

$$Ш^2(K, \mathbb{Z}) \cong (\mathbb{Q}/\mathbb{Z})^c.$$

Since l/k is unramified, the scheme $\mathcal{C} \times_{\mathcal{O}_k} \mathcal{O}_l$ is a suitable regular model of $C \times_k l$ and hence $Ш^2(L, \mathbb{Z})$ is also isomorphic to $(\mathbb{Q}/\mathbb{Z})^c$. The surjectivity of $\mathrm{Res}_{L/K}$ then follows from the isomorphism $Ш^2(K, \mathbb{Z}) \cong Ш^2(L, \mathbb{Z}) \cong (\mathbb{Q}/\mathbb{Z})^c$ and the finiteness of the exponent of $\ker(\mathrm{Res}_{L/K})$. □

3.1.6 Step 5: Proof of Theorem 3.1 for smooth proper varieties

In this step, we use Poitou-Tate duality to deduce Theorem 3.1 for smooth proper varieties from the previous steps:

Theorem 3.13. *Let l/k be a finite unramified extension and set $L := lK$. Let Z be a smooth proper integral K-variety. Then the quotient:*

$$K_2(K)/\langle N_{L/K}(K_2(L)), N_2(Z/K)\rangle$$

is $\chi_K(Z,E)^2$-torsion for every coherent sheaf E on Z.

Proof. Set $n := [l : k]$ and take $x \in K_2(K)$. We want to prove that:

$$\chi_K(Z,E)^2 \cdot x \in \langle N_{L/K}(K_2(L)), N_2(Z/K)\rangle.$$

First observe that, if K' stands for the algebraic closure of K in the function field of Z, then Z has a structure of a smooth proper K'-variety and that $\chi_{K'}(Z,E) = [K' : K]^{-1}\chi_K(Z,E)$. Therefore, by restriction-corestriction, we can assume that $K = K'$, and hence that Z is geometrically integral. Moreover, by Lemma 3.3, we may and do assume that C has residual index 1.

Let now S be the (finite) set of places $v \in C^{(1)}$ such that $\partial_v x \neq 0$. Given a prime number p, since the curve C has residual index 1 and the field k is large, we can find some point $w_p \in C^{(1)} \setminus S$ such that $[k(w_p) : k]_{\mathrm{res}} \wedge p = 1$. Moreover, by Proposition 3.5, we have

$$\chi_K(Z,E) \cdot K_2(K_{w_p}) \subseteq \left\langle N_{L_{w_p}/K_{w_p}}(K_2(L_{w_p})), N_2(Z_{w_p}/K_{w_p})\right\rangle. \tag{14}$$

Before moving further, we need to prove the following lemma:

Lemma 3.14. *If $v_p(n) > v_p(\chi_K(Z,E))$, then there exists a finite extension $M^{(w_p)}$ of K_{w_p} with residue field $m^{(w_p)}$ such that $Z(M^{(w_p)}) \neq \emptyset$ and $v_p([m^{(w_p)} : k(w_p)]_{\mathrm{res}}) \leq v_p(\chi_K(Z,E))$.*

Proof. By contradiction, assume that, for each finite extension M of K_{w_p} with residue field m such that $Z(M) \neq \emptyset$, we have $v_p([m : k(w_p)]_{\mathrm{res}}) > v_p(\chi_K(Z,E))$. By applying the residue map to (14) and by denoting $l(w_p)$ the residue field of L_{w_p}, we get:

$$(K_{w_p}^\times)^{\chi_K(Z,E)} \subseteq \left\langle N_{l(w_p)/k(w_p)}(l(w_p)^\times);\ N_{m/k(w_p)}(m^\times) \mid v_p([m : k(w_p)]_{\mathrm{res}}) > v_p(\chi_K(Z,E))\right\rangle.$$

By applying the valuation w_p, we deduce that:

$$\chi_K(Z,E) \in \langle [l(w_p) : k(w_p)]_{\mathrm{res}};\ [m : k(w_p)]_{\mathrm{res}} \mid v_p([m : k(w_p)]_{\mathrm{res}}) > v_p(\chi_K(Z,E))\rangle \subseteq \mathbb{Z}. \tag{15}$$

But $v_p(n) > v_p(\chi_K(Z,E))$ and $[k(w_p) : k]_{\mathrm{res}} \wedge p = 1$, and hence

$$v_p([l(w_p) : k(w_p)]_{\mathrm{res}}) > v_p(\chi_K(Z,E)).$$

Thus, every integer in:

$$\langle [l(w_p) : k(w_p)]_{\mathrm{res}};\ [m : k(w_p)]_{\mathrm{res}} \mid v_p([m : k(w_p)]_{\mathrm{res}}) > v_p(\chi_K(Z,E))\rangle,$$

is divisible by $p^{v_p(\chi_K(Z,E))+1}$, which contradicts (15). $\square$

Let us now resume the proof of Theorem 3.13. For $v \in C^{(1)} \setminus S$, we have

$$x \in N_{L_v/K_v}(K_2(L_v)) \tag{16}$$

by Lemma 3.4. For $v \in S$, Proposition 3.5 shows that we can find $M_1^{(v)}, ..., M_{r_v}^{(v)}$ finite extensions of K_v such that $Z(M_i^{(v)}) \neq \emptyset$ for all i and:

$$\chi_K(Z,E) \cdot x \in \langle N_{L_v/K_v}(K_2(L_v)); N_{M_i^{(v)}/K_v}(K_2(M_i^{(v)})), 1 \leq i \leq r_v \rangle. \tag{17}$$

By applying Proposition 3.6 inductively, we can find, for each $v \in S$ and $1 \leq i \leq r_v$, a finite extension $K_i^{(v)}$ of K satisfying the following properties:

(i) $Z(K_i^{(v)}) \neq \emptyset$;

(ii) $K_i^{(v)}$ can be seen as a subfield of $M_i^{(v)}$;

(iii) $K_i^{(v)}$ can also be seen as a subfield of the field $M^{(w_p)}$ given by Lemma 3.14 for each prime p such that $v_p(n) > v_p(\chi_K(Z,E))$;

(iv) for each pair (v_0, i_0), the field $K_{i_0}^{(v_0)}$ is linearly disjoint to the composite field

$$L_n \cdot \prod_{(v,i) \neq (v_0,i_0)} K_i^{(v)},$$

over K, where L_n stands for the composite of all cyclic extensions of L that are locally trivial everywhere and whose degrees divide n. Note that L_n is a finite extension of L since $Ш^1(L, \mathbb{Z}/n\mathbb{Z})$ is finite.

Consider the Galois module $\hat{T}$ defined by the following exact sequence:

$$0 \to \mathbb{Z} \to \mathbb{Z}[E/K] \to \hat{T} \to 0,$$

where $E := L \times \prod_{v,i} K_i^{(v)}$. To conclude, we introduce the composite field $K_{\mathcal{I}} = \prod_{v,i} K_i^{(v)}$ and we check the assumptions (LD1), (LD2), (H1), (H2), (H3) and (H4) of Proposition 3.7 with $m = \chi_K(Z,E)$ and

$$m' = \left|\ker(\mathrm{Res}_{LK_{\mathcal{I}}/L} : Ш^2(L,\mathbb{Z}) \to Ш^2(LK_{\mathcal{I}},\mathbb{Z}))\right|.$$

(LD1) The extension L/K is obviously Galois. The fields L and $K_{\mathcal{I}}$ are linearly disjoint over K by (iv).

(LD2) This immediately follows from (iv).

(H1) By proceeding exactly in the same way as in Lemma 4 of [DW14], since we already have (LD1), one gets the injectivity of the restriction map:

$$H^2(K,\hat{T}) \to H^2(L,\hat{T}) \oplus H^2(K_{\mathcal{I}},\hat{T}),$$

and hence of:

$$Ш^2(K,\hat{T}) \to Ш^2(L,\hat{T}) \oplus Ш^2(K_{\mathcal{I}},\hat{T}).$$

(H2) Let $C_{\mathcal{I}}$ be the smooth projective k-curve with fraction field $K_{\mathcal{I}}$. On the one hand, by (iii), given a prime p such that $v_p(n) > v_p(\chi_K(Z,E))$, the field $K_{\mathcal{I}}$ can be seen as a subfield of $M^{(w_p)}$ and the inequality $v_p([m^{(w_p)}:k]_{\mathrm{res}}) \leq v_p(\chi_K(Z,E))$ holds by Lemma 3.14. We deduce that $v_p(i_{\mathrm{res}}(C_{\mathcal{I}})) \leq v_p(\chi_K(Z,E))$ for such p. On the other hand, for any other prime number p, we have $v_p(n) \leq v_p(\chi_K(Z,E))$. We deduce that $i_{\mathrm{res}}(C_{\mathcal{I}}) \wedge n$ divides $m = \chi_K(Z,E)$, and hence (H2) follows from Lemma 3.12.

(H3) This immediately follows from Lemma 3.12.

(H4) Since L/K is Galois, (H4) immediately follows from the choice of m'.

By Proposition 3.7, we deduce that the group $\overline{Ш^2(K,\hat{T})}$ is $((m \vee m') \wedge n)$-torsion. But by (iv), the fields $K_{\mathcal{I}}$ and L_n are linearly disjoint over K, and hence $m' \wedge n = 1$, so that $(m \vee m') \wedge n = m \wedge n$. The group $\overline{Ш^2(K,\hat{T})}$ is therefore m-torsion. If we set $\check{T} := \mathrm{Hom}(\hat{T},\mathbb{Z})$ and $T := \check{T} \otimes \mathbb{Z}(2)$, that is also the case of $Ш^3(K,T)$ according to Poitou-Tate duality.

Now, by Lemma 3.2, we may interpret x as an element of $H^3(K,T)$. Equations (16) and (17) together with assertion (ii) imply that $mx \in Ш^3(K,T)$, which is m-torsion. Thus $m^2x = 0 \in Ш^3(K,T)$. This amounts to

$$\begin{aligned} m^2x &\in \left\langle N_{L/K}(K_2(L));\, N_{K_i^{(v)}/K}(K_2(K_i^{(v)})),\, v \in S,\, 1 \leq i \leq r_v \right\rangle \\ &\subseteq \left\langle N_{L/K}(K_2(L)), N_2(Z/K) \right\rangle, \end{aligned}$$

the last inclusion being a consequence of (i). □

3.1.7 Step 6: Proof of Theorem 3.1

In this final step, we remove the smoothness assumption from the previous step and prove Theorem 3.1 for all proper varieties. For that purpose, we use the following variation of the dévissage technique given by Proposition 2.1 of [Wit15]:

Proposition 3.15 ([Wit15])**.** *Let K be a field and r a positive integer. Let (P) be a property of proper K-varieties. Suppose we are given, for each proper K-variety X, an integer m_X. Make the following assumptions:*

(1) For every morphism of proper K-schemes $Y \to X$, the integer m_X divides m_Y.

(2) For every proper K-scheme X satisfying (P), the integer m_X divides $\chi_K(X,\mathcal{O}_X)^r$.

(3) For every proper and integral K-scheme X, there exists a proper K-scheme Y satisfying (P) and a K-morphism $f : Y \to X$ with generic fiber Y_η such that m_X and $\chi_{K(X)}(Y_\eta,\mathcal{O}_{Y_\eta})$ are coprime.

Then for every proper K-scheme X and every coherent sheaf E on X, the integer m_X divides $\chi_K(X,E)^r$.

Proof. One can prove this result by following almost word by word the proof of Proposition 2.1 of [Wit15]. Alternatively, for each proper K-scheme X, write the prime decomposition of m_X:

$$m_X = \prod_p p^{\alpha_p},$$

and consider the integer

$$n_X := \prod_p p^{\lceil \frac{\alpha_p}{r} \rceil}.$$

One can then easily check that the correspondence $X \mapsto n_X$ satisfies assumptions (1), (2) and (3) of Proposition 2.1 of [Wit15]. We deduce that $n_X | \chi_K(X, E)$, and hence that $m_X | \chi_K(X, E)^r$, for every proper K-scheme X and every coherent sheaf E on X. □

Proof of Theorem 3.1. Given a proper K-variety Z, we denote by m_Z the exponent of the quotient

$$K_2(K)/\langle N_{L/K}(K_2(L)), N_2(Z/K)\rangle.$$

We say that Z has property (P) if it is smooth and integral. We have to check the three conditions (1), (2) and (3) of Proposition 3.15. Condition (1) is straightforward. Condition (2) follows from Theorem 3.13. Condition (3) follows from Hironaka's Theorem on resolution of singularities. □

3.2 Proof of Main Theorem A

We can now deduce Main Theorem A from Theorem 3.1.

Proof of Main Theorem A. Fix two integers $n, d \geq 1$ such that $d^2 \leq n$ and a hypersurface Z in $\mathbb{P}^n_K$ of degree d. By Lang's and Tsen's Theorems, the field $k^{\mathrm{nr}}(C)$ is C_2. Since $d^2 \leq n$, we deduce that there exists a finite unramified extension l of k such that $Z(lK) \neq \emptyset$. By Theorem 3.1, the quotient:

$$K_2(K)/\langle N_{lK/K}(K_2(lK)), N_2(Z/K)\rangle = K_2(K)/N_2(Z/K)$$

is $\chi_K(Z, \mathcal{O}_Z)^2$-torsion. But since $d \leq n$, Theorem III.5.1 of [Har77] implies that:

$$\chi_K(Z, \mathcal{O}_Z) = \chi_K(\mathbb{P}^n_K, \mathcal{O}_{\mathbb{P}^n_K}) - \chi_K(\mathbb{P}^n_K, \mathcal{O}_{\mathbb{P}^n_K}(-d)) = 1.$$

Hence $K_2(K) = N_2(Z/K)$. □

4. On the C_1^2 property for p-adic function fields

The goal of this section is to prove Main Theorem B. Contrary to Main Theorem A, for which we needed to deal with unramified extensions of k, here we will have to deal with ramified extensions of k. For that purpose, the key statement is given by the following theorem:

Theorem 4.1. *Assume that C has a rational point, let ℓ be a prime number, and fix a finite Galois totally ramified extension l/k of degree ℓ. Let $\mathcal{E}^0_{l/k}$ be the set of all finite ramified subextensions of l^{nr}/k and let $\mathcal{E}_{l/k}$ be the set of finite extensions K' of K of the form $K' = k'K$ for some $k' \in \mathcal{E}^0_l$. Then:*

$$K_2(K) = \langle N_{K'/K}(K_2(K')) \,|\, K' \in \mathcal{E}_{l/k}\rangle.$$

Note that, given any two extensions k' and k'' in $\mathcal{E}^0_{l/k}$ with $k' \subset k''$, the extension k''/k' is unramified. This observation will be often used in the sequel.

Remark 4.2. We think that the assumption that C has a rational point in Theorem 4.1 cannot be removed. To check that, we invite the reader to assume that $i_{\mathrm{ram}}(C) = \ell$. Then, given an integer $n \geq 1$, consider the set $\mathcal{E}_n^0$ whose elements are extensions of k in $\mathcal{E}_l^0$ that are contained in the composite $l_n := lk_n$, where k_n is the degree ℓ^n unramified extension of k. Define the set $\mathcal{E}_n$ of finite extensions K' of K contained in $L_n := l_n K$ that are of the form $K' = k'K$ for some $k' \in \mathcal{E}_n^0$ and consider the Galois module $\hat{T}_n$ defined by the exact sequence:

$$0 \to \mathbb{Z} \to \bigoplus_{K' \in \mathcal{E}_n} \mathbb{Z}[K'/K] \to \hat{T}_n \to 0.$$

By following the proof of Proposition 4.5, one can check that, if K_1 and K_2 are two distinct degree ℓ extensions of K in $\mathcal{E}_n$, then the Tate-Shafarevich group $Ш^2(K, \hat{T}_n)$ is the direct sum of the kernel of the map:

$$(\mathrm{Res}_{K_1/K}, \mathrm{Res}_{K_2/K}) : Ш^2(K, \hat{T}_n) \to Ш^2(K_1, \hat{T}_n) \oplus Ш^2(K_2, \hat{T}_n)$$

and of a divisible group, given by the kernel of the map:

$$\mathrm{Res}_{K_1K_2/K_1} - \mathrm{Res}_{K_1K_2/K_2} : Ш^2(K_1, \hat{T}_n) \oplus Ш^2(K_2, \hat{T}_n) \to Ш^2(K_1K_2, \hat{T}_n).$$

In particular:

$$\overline{Ш^2(K, \hat{T}_n)} \cong \ker \left(Ш^2(K, \hat{T}_n) \to Ш^2(K_1, \hat{T}_n) \oplus Ш^2(K_2, \hat{T}_n) \right).$$

The computation of this kernel is a relatively simple (but a bit technical) exercise in the cohomology of finite groups, since it is contained in the group:

$$\ker \left(H^2(K, \hat{T}_n) \to H^2(L_n, \hat{T}_n) \right) \cong H^2(\mathrm{Gal}(L_n/K), \hat{T}) \cong H^2(\mathbb{Z}/\ell\mathbb{Z} \times \mathbb{Z}/\ell^n\mathbb{Z}, \hat{T}).$$

In that way, one checks that $\overline{Ш^2(K, \hat{T}_n)}$ is an $\mathbb{F}_\ell$-vector space of dimension at least $n\ell - n - 1$. Moreover, the computation being very explicit, one can even check that the morphism $\overline{Ш^2(K, \hat{T}_{n+1})} \to \overline{Ш^2(K, \hat{T}_n)}$ induced by the natural projection $\hat{T}_{n+1} \to \hat{T}_n$ is always surjective. But then, by dualizing thanks to Poitou-Tate duality, this shows that the groups:

$$\begin{aligned} Q_n := \ker(K_2(K)/\langle N_{K'/K}(K_2(K')) \,|\, K' \in \mathcal{E}_n \rangle \\ \to \prod_{v \in C^{(1)}} K_2(K_v)/\langle N_{K' \otimes K_v/K_v}(K_2(K' \otimes K_v)) \,|\, K' \in \mathcal{E}_n \rangle) \end{aligned}$$

are all non-trivial and that the natural maps $Q_n \to Q_{n+1}$ are all injective. We deduce that the non-trivial elements of Q_1 provide non-trivial elements in the quotient:

$$K_2(K)/\langle N_{K'/K}(K_2(K')) \,|\, K' \in \bigcup_n \mathcal{E}_n \rangle = K_2(K)/\langle N_{K'/K}(K_2(K')) \,|\, K' \in \mathcal{E}_{l/k} \rangle.$$

4.1 Proof of Theorem 4.1

4.1.1 Step 1: Solving the local problem

The first step to prove Theorem 4.1 consists in settling an analogous statement over the completions of K. We start with the following lemma:

Lemma 4.3. *Let ℓ be a prime number and let l/k be a finite Galois totally ramified extension of degree ℓ. Let m/k be a totally ramified extension such that ml/m is unramified. Then there exists $k' \in \mathcal{E}^0_{l/k}$ such that $k' \subset m$.*

Proof. If ml/m is trivial, then m contains l and we are done. Therefore we may and do assume that ml/m has degree ℓ. Denote by k_ℓ the unramified extension of k with degree ℓ and set $l_\ell := l \cdot k_\ell$. The extension l_ℓ/k is Galois with Galois group $(\mathbb{Z}/\ell\mathbb{Z})^2$, and since ml/m is unramified of degree ℓ, the field l_ℓ is contained in $m'l$ for some finite subextension m' of m/k. But:

$$[m' : k] \cdot [l_\ell : k] = \ell^2[m' : k] > \ell[m' : k] = [m'l : k] = [m'l_\ell : k].$$

Hence the intersection $k' := m' \cap l_\ell$ is a degree ℓ totally ramified extension of k, and $k' \in \mathcal{E}^0_{l/k}$. □

Proposition 4.4. *Let ℓ be a prime number and let l/k be a finite Galois totally ramified extension of degree ℓ. Fix $v \in C^{(1)}$. Then:*

$$K_2(K_v) = \langle N_{K'\otimes_K K_v/K_v}(K_2(K' \otimes_K K_v)) \,|\, K' \in \mathcal{E}_{l/k}\rangle.$$

Proof. Three different cases arise:

1. the field $k(v)$ contains l;
2. the extension $lk(v)/k(v)$ is unramified of degree ℓ;
3. the extension $lk(v)/k(v)$ is totally ramified of degree ℓ.

Case 1 is trivial, since:

$$K_2(K_v) = N_{lK\otimes_K K_v/K_v}(K_2(lK \otimes_K K_v)).$$

Let us now consider case 2, and denote by $k(v)_{\mathrm{nr}}$ the maximal unramified subextension of $k(v)/k$. By Lemma 4.3, since $lk(v)_{\mathrm{nr}}/k(v)_{\mathrm{nr}}$ is a Galois totally ramified extension of degree ℓ and $k(v)/k(v)_{\mathrm{nr}}$ is a totally ramified extension such that $k(v)l/k(v)$ is unramified, there exists a finite extension m of $k(v)_{\mathrm{nr}}$ such that $m \in \mathcal{E}^0_{lk(v)_{\mathrm{nr}}/k(v)_{\mathrm{nr}}} \subset \mathcal{E}^0_{l/k}$ and $m \subset k(v)$. By setting $M := mK$, we get that $M \in \mathcal{E}_{l/k}$ and that:

$$\begin{aligned} K_2(K_v) &= N_{M\otimes_K K_v/K_v}(K_2(M \otimes_K K_v)) \\ &\subset \langle N_{K'\otimes_K K_v/K_v}(K_2(K' \otimes_K K_v)) \,|\, K' \in \mathcal{E}_{l/k}\rangle, \end{aligned}$$

as wished.

Let us finally consider case 3. To do so, fix a uniformizer π of $k(v)$, and as before, let $k(v)_{\mathrm{nr}}$ be the maximal unramified subextension of $k(v)/k$. Denote by $k(v)^{\mathrm{ram}}_\pi$ the maximal abelian totally ramified extension of $k(v)$ associated to π by Lubin-Tate theory. Since l/k is abelian, the extension $lk(v)^{\mathrm{ram}}_\pi/k(v)^{\mathrm{ram}}_\pi$ must be unramified. Hence, by Lemma 4.3, there exists a finite extension m of $k(v)_{\mathrm{nr}}$ such that $m \in \mathcal{E}^0_{lk(v)_{\mathrm{nr}}/k(v)_{\mathrm{nr}}} \subset \mathcal{E}^0_{l/k}$ and $m \subset k(v)^{\mathrm{ram}}_\pi$. We deduce from Corollary 5.12 of [Yos08] that:

$$\pi \in N_{m\otimes_{k(v)_{\mathrm{nr}}} k(v)}((m \otimes_{k(v)_{\mathrm{nr}}} k(v))^\times) \subset \langle N_{k'\otimes_k k(v)/k(v)}((k' \otimes_k k(v))^\times) \,|\, k' \in \mathcal{E}^0_{l/k}\rangle.$$

This being true for every uniformizer π of $k(v)$, we deduce that:

$$k(v)^\times \subset \langle N_{k'\otimes_k k(v)/k(v)}((k' \otimes_k k(v))^\times) \,|\, k' \in \mathcal{E}^0_{l/k}\rangle,$$

and hence, by Lemma 3.4:

$$K_2(K_v) = \langle N_{K'\otimes_K K_v/K_v}(K_2(K' \otimes_K K_v)) \,|\, K' \in \mathcal{E}_{l/k}\rangle.$$

□

4.1.2 Step 2: Computation of a Tate-Shafarevich group

The second step, which is slightly technical, consists in computing the Tate-Shafarevich groups of some finitely generated free Galois modules over K associated to the fields in $\mathcal{E}_l$. Poitou-Tate duality will then allow us to obtain a local-global principle that will let us deduce Theorem 4.1 from Proposition 4.4.

Proposition 4.5. *Assume that C has a rational point, and let ℓ be a prime number. Fix a finite Galois totally ramified extension l/k of degree ℓ. Given $K_1, ..., K_r$ in $\mathcal{E}_{l/k}$ so that the fields K_1 and K_2 are linearly disjoint over K, consider the Galois module $\hat{T}$ defined by the following exact sequence:*

$$0 \to \mathbb{Z} \to \mathbb{Z}[E/K] \to \hat{T} \to 0, \tag{18}$$

where $E := K_1 \times ... \times K_r$. Then $\text{Ш}^2(K, \hat{T})$ is divisible.

Proof. Consider the following complex:

$$\begin{array}{ccccc} \text{Ш}^2(K,\hat{T}) & \xrightarrow{f_0} & \text{Ш}^2(K_1,\hat{T}) \oplus \text{Ш}^2(K_2,\hat{T}) & \xrightarrow{\quad g_0 \quad} & \text{Ш}^2(K_1K_2,\hat{T}) \\ x & \longmapsto & (\mathrm{Res}_{K_1/K}(x), \mathrm{Res}_{K_2/K}(x)) & & \\ & & (x,y) & \longmapsto & \mathrm{Res}_{K_1K_2/K_1}(x) - \mathrm{Res}_{K_1K_2/K_2}(y). \end{array} \tag{19}$$

We start by proving the following lemma:

Lemma 4.6. *The morphism f_0 is injective.*

Proof. Let $K_{\mathcal{I}}$ be the Galois closure of the composite of all the K_i's. By inflation-restriction, there is an exact sequence:

$$0 \to H^2(K_{\mathcal{I}}/K, \hat{T}) \to H^2(K, \hat{T}) \to H^2(K_{\mathcal{I}}, \hat{T}).$$

Take $q \in C(k)$ a rational point. The previous exact sequence then induces a commutative diagram with exact lines:

$$\begin{tikzcd} 0 \arrow[r] & H^2(K_{\mathcal{I}}/K,\hat{T}) \arrow[r] \arrow[d, "\cong"] & H^2(K,\hat{T}) \arrow[r] \arrow[d] & H^2(K_{\mathcal{I}},\hat{T}) \arrow[d] \\ 0 \arrow[r] & H^2(K_{\mathcal{I},q}/K_q,\hat{T}) \arrow[r] & H^2(K_q,\hat{T}) \arrow[r] & H^2(K_{\mathcal{I},q},\hat{T}) \end{tikzcd}$$

in which the first vertical map is an isomorphism since $\mathrm{Gal}(K_{\mathcal{I}}/K) = \mathrm{Gal}(K_{\mathcal{I},q}/K_q)$. We deduce that the restriction map:

$$\ker\left(H^2(K,\hat{T}) \to H^2(K_q,\hat{T})\right) \to \ker\left(H^2(K,\hat{T}) \to H^2(K_{\mathcal{I},q},\hat{T})\right)$$

is injective. Hence so is the restriction map:

$$\mathrm{Res}_{K_{\mathcal{I}}/K} : Ш^2(K,\hat{T}) \to Ш^2(K_{\mathcal{I}},\hat{T})$$

as well as the restriction maps:

$$\mathrm{Res}_{K_1/K} : Ш^2(K,\hat{T}) \to Ш^2(K_1,\hat{T}),$$
$$\mathrm{Res}_{K_2/K} Ш^2(K,\hat{T}) \to Ш^2(K_2,\hat{T}).$$

□

Now observe that the complex (19) fits in the following commutative diagram, in which the first and second rows are obtained in the same way as the third:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] \\
Ш^2(K,\mathbb{Z}) \arrow[r] \arrow[d] & Ш^2(K_1,\mathbb{Z}) \oplus Ш^2(K_2,\mathbb{Z}) \arrow[r, two heads] \arrow[d] & Ш^2(K_1K_2,\mathbb{Z}) \arrow[d] \\
Ш^2(K,\mathbb{Z}[E/K]) \arrow[r, "f"] \arrow[d] & Ш^2(K_1,\mathbb{Z}[E/K]) \oplus Ш^2(K_2,\mathbb{Z}[E/K]) \arrow[r, "g"] \arrow[d] & Ш^2(K_1K_2,\mathbb{Z}[E/K]) \arrow[d] \\
Ш^2(K,\hat{T}) \arrow[r, hook, "f_0"] & Ш^2(K_1,\hat{T}) \oplus Ш^2(K_2,\hat{T}) \arrow[r, "g_0"] \arrow[d] & Ш^2(K_1K_2,\hat{T}) \arrow[d] \\
 & 0 & 0.
\end{tikzcd} \tag{20}$$

The second and third columns are exact since the exact sequence (18) splits over K_1, K_2 and K_1K_2. The lines are all complexes. In the first one, the second arrow is surjective since the restriction map:

$$Ш^2(K_1,\mathbb{Z}) \to Ш^2(K_1K_2,\mathbb{Z})$$

is an isomorphism by Lemma 3.12 and C has a rational point. As for the second line, we have the following lemma:

Lemma 4.7. *The second line of diagram* (20) *is exact.*

Proof. For $1 \leq \alpha \leq r$, write:

$$K_1 \otimes_K K_\alpha = \prod_\beta L_{\alpha\beta}$$
$$K_2 \otimes_K K_\alpha = \prod_\gamma M_{\alpha\gamma}$$
$$L_{\alpha\beta} \otimes_{K_\alpha} M_{\alpha\gamma} = \prod_\delta N_{\alpha\beta\gamma\delta}$$

for some fields $L_{\alpha\beta}$, $M_{\alpha\gamma}$ and $N_{\alpha\beta\gamma\delta}$. By Shapiro's Lemma, the second line of (20) can be identified with the following complex:

$$\begin{tikzcd}
\bigoplus_{\alpha} Ш^2(K_\alpha, \mathbb{Z}) \arrow[d, "f"] \\
\bigoplus_{\alpha,\beta} Ш^2(L_{\alpha\beta}, \mathbb{Z}) \oplus \bigoplus_{\alpha,\gamma} Ш^2(M_{\alpha\gamma}, \mathbb{Z}) \arrow[d, "g"] \\
\bigoplus_{\alpha,\beta,\gamma,\delta} Ш^2(N_{\alpha\beta\gamma\delta}, \mathbb{Z})
\end{tikzcd}$$

where f is given by:

$$(x_\alpha) \mapsto \left(\left(\mathrm{Res}_{L_{\alpha\beta}/K_\alpha}(x_\alpha) \right)_{\alpha\beta}, \left(\mathrm{Res}_{M_{\alpha\gamma}/K_\alpha}(x_\alpha) \right)_{\alpha\gamma} \right),$$

and g:

$$((y_{\alpha\beta})_{\alpha,\beta}, (z_{\alpha\gamma})_{\alpha,\gamma})) \mapsto \left(\mathrm{Res}_{N_{\alpha\beta\gamma\delta}/L_{\alpha\beta}}(y_{\alpha\beta}) - \mathrm{Res}_{N_{\alpha\beta\gamma\delta}/M_{\alpha\gamma}}(z_{\alpha\gamma}) \right)_{\alpha\beta\gamma\delta},$$

Fix $((y_{\alpha\beta})_{\alpha,\beta}, (z_{\alpha\gamma})_{\alpha,\gamma})) \in \ker(g)$. Then:

$$\mathrm{Res}_{N_{\alpha\beta\gamma\delta}/L_{\alpha\beta}}(y_{\alpha\beta}) = \mathrm{Res}_{N_{\alpha\beta\gamma\delta}/M_{\alpha\gamma}}(z_{\alpha\gamma})$$

for all $\alpha, \beta, \gamma, \delta$. But the restrictions $\mathrm{Res}_{L_{\alpha\beta}/K_\alpha}$, $\mathrm{Res}_{M_{\alpha\gamma}/K_\alpha}$, $\mathrm{Res}_{N_{\alpha\beta\gamma\delta}/L_{\alpha\beta}}$ and $\mathrm{Res}_{N_{\alpha\beta\gamma\delta}/M_{\alpha\gamma}}$ are all isomorphisms by Lemma 3.12 and they fit into a commutative diagram:

$$\begin{tikzcd}
Ш^2(K_\alpha, \mathbb{Z}) \arrow[r, "\mathrm{Res}_{L_{\alpha\beta}/K_\alpha}"] \arrow[d, "\mathrm{Res}_{M_{\alpha\gamma}/K_\alpha}"'] & Ш^2(L_{\alpha\beta}, \mathbb{Z}) \arrow[d, "\mathrm{Res}_{N_{\alpha\beta\gamma\delta}/L_{\alpha\beta}}"] \\
Ш^2(M_{\alpha\gamma}, \mathbb{Z}) \arrow[r, "\mathrm{Res}_{N_{\alpha\beta\gamma\delta}/M_{\alpha\gamma}}"'] & Ш^2(N_{\alpha\beta\gamma\delta}, \mathbb{Z}).
\end{tikzcd}$$

We deduce that, for each α, there exists $x_\alpha \in Ш^2(K_\alpha, \mathbb{Z})$ such that:

$$\forall \beta, \ \ \mathrm{Res}_{L_{\alpha\beta}/K_\alpha}(x_\alpha) = y_{\alpha\beta},$$
$$\forall \gamma, \ \ \mathrm{Res}_{M_{\alpha\gamma}/K_\alpha}(x_\alpha) = z_{\alpha\gamma}.$$

In other words, $((y_{\alpha\beta})_{\alpha,\beta}, (z_{\alpha\gamma})_{\alpha,\gamma})) \in \mathrm{im}(f)$. □

With all the gathered information, a simple diagram chase in (8) shows that the morphism $Ш^2(K, \mathbb{Z}[E/K]) \to Ш^2(K, \hat{T})$ is surjective. But as recalled at the end of section 2, the group $Ш^2(K, \mathbb{Z}[E/K])$ is divisible. Hence so is $Ш^2(K, \hat{T})$. □

4.1.3 Step 3: Proof of Theorem 4.1

We can finally prove Theorem 4.1 by using Poitou-Tate duality.

Proof of Th. 4.1. Take $x \in K_2(K)$. By Proposition 4.4, we have:

$$K_2(K_v) = \langle N_{K' \otimes_K K_v / K_v}(K_2(K' \otimes_K K_v)) \,|\, K' \in \mathcal{E}_{l/k} \rangle$$

for all $v \in C^{(1)}$. Hence we can find $K_1, ..., K_r \in \mathcal{E}_l^0$ such that:

$$\begin{aligned} x \in \ker(K_2(K)/\langle N_{K_i/K}(K_2(K_i)) \mid 1 \leq i \leq r\rangle \\ \to \prod_{v \in C^{(1)}} K_2(K_v)/\langle N_{K_i \otimes_K K_v/K_v}(K_2(K_i \otimes_K K_v)) \mid 1 \leq i \leq r\rangle). \quad (21) \end{aligned}$$

Moreover, up to enlarging the family $(K_i)_i$, we may and do assume that K_1 and K_2 are linearly disjoint. Consider the étale K-algebra $E := K_1 \times ... \times K_r$ and the Galois module $\hat{T}$ defined by the following exact sequence:

$$0 \to \mathbb{Z} \to \mathbb{Z}[E/K] \to \hat{T} \to 0.$$

Set $\check{T} := \mathrm{Hom}(\hat{T}, \mathbb{Z})$ and $T := \check{T} \otimes \mathbb{Z}(2)$. By Lemma 3.2, equation (21) can be rewritten as:

$$x \in Ш^3(K, T).$$

But, by Poitou-Tate duality, $Ш^3(K,T)$ is dual to $\overline{Ш^2(K, \hat{T})}$, and by Proposition 4.5, the group $Ш^2(K, \hat{T})$ is divisible. We deduce that $Ш^3(K,T) = 0$, and hence that:

$$x \in \langle N_{K_i/K}(K_2(K_i)) \mid 1 \leq i \leq r\rangle \subset \langle N_{K'/K}(K_2(K')) \,|\, K' \in \mathcal{E}_{l/k} \rangle.$$

□

4.2 Proof of Main Theorem B

By combining Theorems 3.1 and 4.1, we can now settle the following theorem, from which we will deduce Main Theorem B:

Theorem 4.8. *Let K be the function field of a smooth projective curve C defined over a p-adic field k. Let l/k be a finite Galois extension and set $L := lK$. Let Z be a proper K-variety. If $s_{l/k}$ stands for the number of (not necessarily distinct) prime factors of the ramification degree of l/k, then the quotient:*

$$K_2(K)/\langle N_{L/K}(K_2(L)), N_2(Z/K)\rangle$$

is $i_{\mathrm{ram}}(C) \cdot \chi_K(Z, E)^{2s_{l/k}+4}$-torsion for every coherent sheaf E on Z.

Proof. We first assume that C has a rational point, and we prove that

$$K_2(K)/\langle N_{L/K}(K_2(L)), N_2(Z/K)\rangle$$

is $\chi_K(Z, E)^{2s_{l/k}+2}$-torsion for every coherent sheaf E on Z by induction on $s_{l/k}$. The case $s_{l/k} = 0$ immediately follows from Theorem 3.1. We henceforth assume now that $s_{l/k} > 0$. Let l_{nr} be the maximal unramified subextension of l/k and set $L_{\mathrm{nr}} := l_{\mathrm{nr}} K$. Theorem 3.1 ensures then that the quotient:

$$K_2(K)/\langle N_{L_{\mathrm{nr}}/K}(K_2(L_{\mathrm{nr}})), N_2(Z/K)\rangle$$

is $\chi_K(Z,E)^2$-torsion. Now, the extension $l/l_{\rm nr}$ is Galois and totally ramified. Since finite extensions of local fields are solvable, we can find a Galois totally ramified extension $m/l_{\rm nr}$ contained in l and of prime degree ℓ. Set $M := mK$. By Theorem 4.1, we have:

$$K_2(L_{\rm nr}) = \langle N_{K'/L_{\rm nr}}(K_2(K')) \,|\, K' \in \mathcal{E}_{m/l_{\rm nr}} \rangle.$$

But for each $k' \in \mathcal{E}_{m/l_{\rm nr}}$, the ramification degree of lk'/k' divides that of l/k. Hence, by induction, the group:

$$K_2(K')/\langle N_{LK'/K'}(K_2(LK')), N_2(Z/K')\rangle$$

is $\chi_K(Z,E)^{2s_{l/k}}$-torsion for each $K' \in \mathcal{E}_{m/l_{\rm nr}}$. We deduce that:

$$K_2(K)/\langle N_{L/K}(K_2(L)), N_2(Z/K)\rangle$$

is $\chi_K(Z,E)^{2s_{l/k}+2}$-torsion, which finishes the induction.

We do not assume anymore that C has a rational point. Let $k_1, ..., k_r$ be finite extensions of k over which C acquires rational points and such that the g.c.d.'s of their ramification degrees is $i_{\rm ram}(C)$. For each i, let $k_{i,\rm nr}$ be the maximal unramified extension of k contained in k_i, and set $K_i := k_iK$ and $K_{i,\rm nr} := k_{i,\rm nr}K$. Theorem 3.1 ensures that the quotient:

$$K_2(K)/\langle N_{K_{i,\rm nr}/K}(K_2(K_{i,\rm nr})), N_2(Z/K)\rangle$$

is $\chi_K(Z,E)^2$-torsion. Moreover, a restriction-corestriction argument shows that the quotient:

$$K_2(K_{i,\rm nr})/N_{K_i/K_{i,\rm nr}}(K_2(K_i))$$

is $[k_i : k_{i,\rm nr}]$-torsion. Since $[k_i : k_{i,\rm nr}]$ is the ramification degree of k_i/k, we deduce that:

$$K_2(K)/\langle N_{K_1/K}(K_2(K_1)), ..., N_{K_r/K}(K_2(K_r)), N_2(Z/K)\rangle$$

is $i_{\rm ram}(C) \cdot \chi_K(Z,E)^2$-torsion. But C has rational points over all the k_i's. Hence the quotients:

$$K_2(K_i)/\langle N_{LK_i/K_i}(K_2(LK_i)), N_2(Z/K_i)\rangle$$

are all $\chi_K(Z,E)^{2s_{l/k}+2}$-torsion. We deduce that:

$$K_2(K)/\langle N_{L/K}(K_2(L)), N_2(Z/K)\rangle$$

is $i_{\rm ram}(C) \cdot \chi_K(Z,E)^{2s_{l/k}+4}$-torsion. $\square$

Main Theorem B can be immediately deduced from the following corollary:

Corollary 4.9. *Let K be the function field of a smooth projective curve C defined over a p-adic field k. Then, for each $n, d \geq 1$ and for each hypersurface Z in $\mathbb{P}^n_K$ of degree d with $d \leq n$, the quotient $K_2(K)/N_2(Z/K)$ is killed by $i_{\rm ram}(C)$.*

Proof. Let Z be a hypersurface in $\mathbb{P}^n_K$ of degree d with $d \leq n$. By Tsen's Theorem, the field $\overline{k}(C)$ is C_1. Since $d \leq n$, we deduce that there exists a finite extension l of k such that $Z(lK) \neq \emptyset$. By Theorem 4.8, the quotient:

$$K_2(K)/\langle N_{lK/K}(K_2(lK)), N_2(Z/K)\rangle = K_2(K)/N_2(Z/K)$$

is $i_{\mathrm{ram}}(C)\cdot\chi_K(Z,\mathcal{O}_Z)^{2s_{l/k}+4}$-torsion. But since $d\leq n$, Theorem III.5.1 of [Har77] implies that:

$$\chi_K(Z,\mathcal{O}_Z)=\chi_K(\mathbb{P}^n_K,\mathcal{O}_{\mathbb{P}^n_K})-\chi_K(\mathbb{P}^n_K,\mathcal{O}_{\mathbb{P}^n_K}(-d))=1.$$

Hence the quotient $K_2(K)/N_2(Z/K)$ is $i_{\mathrm{ram}}(C)$-torsion. □

Main Theorem B can now be immediately deduced from the following corollary:

Corollary 4.10. *Let K be the function field of a smooth projective curve C defined over a p-adic field k. Assume that $i_{\mathrm{ram}}(C)=1$. Then, for each $n,d\geq 1$ and for each hypersurface Z in $\mathbb{P}^n_K$ of degree d with $d\leq n$, we have $N_2(Z/K)=K_2(K)$.*

Remark 4.11. By section 9.1 of [BLR90], the assumption that $i_{\mathrm{ram}}(C)=1$ automatically holds when C has an irreducible proper flat regular model whose special fibre has multiplicity 1.

References

[Blo86] Spencer Bloch. Algebraic cycles and higher K-theory. *Adv. in Math.*, 61:267—304, 1986.

[BLR90] Siegfried Bosch, Werner Lütkebohmert and Michel Raynaud. Néron models. Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, vol. 21, Springer-Verlag, Berlin, 1990.

[CTM04] Jean-Louis Colliot-Thélène and David A. Madore. Surfaces de del Pezzo sans point rationnel sur un corps de dimension cohomologique un. *J. Inst. Math. Jussieu*, 3(1):1–16, 2004.

[DW14] Cyril Demarche and Dasheng Wei. Hasse principle and weak approximation for multinorm equations. *Israel J. Math.*, 202(1):275–293, 2014.

[ELW15] Hélène Esnault, Marc Levine and Olivier Wittenberg. Index of varieties over Henselian fields and Euler characteristic of coherent sheaves. *J. Algebraic Geom.*, 24(4):693–718, 2015.

[FJ08] Michael D. Fried and Moshe Jarden. *Field arithmetic*, 3ème éd. Ergeb. Math. Grenzgeb. (3), 11, Springer, Berlin, 2008.

[Gei04] Thomas Geisser. Motivic cohomology over Dedekind rings. *Math. Z.*, 248(4):773–794, 2004.

[GL00] Thomas Geisser and Marc Levine. The K-theory of fields in characteristic p. *Invent. Math.*, 139(3):459–493, 2000.

[GL01] Thomas Geisser and Marc Levine. The Bloch-Kato conjecture and a theorem of Suslin-Voevodsky. *J. Reine Angew. Math.*, 530:55–103, 2001.

[GS17] Philippe Gille and Tamás Szamuely. Central Simple Algebras and Galois Cohomology, Second edition. Cambridge Studies in Advanced Mathematics 165, Cambridge University Press, 2017.

[HW19] Christian Haesemeyer and Charles A. Weibel. The norm residue theorem in motivic cohomology. Annals of Mathematics Studies 200, Princeton, NJ: Princeton University Press, 2019.

[Har77] Robin Hartshorne. Algebraic geometry. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York, 1977.

[ILO14] Luc Illusie, Yves Laszlo and Fabrice Orgogozo, with the collaboration of F. Déglise, A. Moreau, V. Pilloni, M. Raynaud, J. Riou, B. Stroh, M. Temkin et W. Zheng. Travaux de Gabber sur l'uniformisation locale et la cohomologie étale des schémas quasi-excellents. (Séminaire à l'École polytechnique 2006–2008). *Astérisque*, 363-364, 2014.

[Izq16] Diego Izquierdo. Théorèmes de dualité pour les corps de fonctions sur des corps locaux supérieurs. *Math. Z.*, 284(1-2):615–642, 2016.

[Izq18] Diego Izquierdo. On a conjecture of Kato and Kuzumaki. *Algebra and Number Theory*, 12(2):429–454, 2018.

[ILA21] Diego Izquierdo and Giancarlo Lucchini Arteche. Homogeneous spaces, algebraic K-theory and cohomological dimension of fields. To appear in *Journal of the European Mathematical Society*, 2021.

[Jar91] Moshe Jarden. Intersections of local algebraic extensions of a hilbertian field. In *Generators and Relations in Groups and Geometries*, NATO ASI Series 333, 1991, 343–405.

[Kat80] Kazuya Kato. A generalization of local class field theory by using K-groups. II. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 27(3):603–683, 1980.

[Kat86] Kazuya Kato. A Hasse principle for two dimensional global fields *J. Reine Angew. Math.*, 366:142–183, 1986. With an appendix by Jean-Louis Colliot-Thélène.

[KK86] Kazuya Kato and Takako Kuzumaki. The dimension of fields and algebraic K-theory. *J. Number Theory*, 24(2):229–244, 1986.

[Mer91] Alexander S. Merkurjev. Simple algebras and quadratic forms. *Izv. Akad. Nauk SSSR Ser. Mat.*, 55(1):218–224, 1991.

[NS89] Yuri V. Nesterenko, Andrei A. Suslin. Homology of the general linear group over a local ring, and Milnor's K -theory. *Izv. Akad. Nauk SSSR Ser. Mat.* 53(1):121—146, 1989.

[Pop14] Florian Pop. Little survey on large fields — old & new. In *Valuation theory in interaction*, p. 432–463, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2014.

[Rio14] Joël Riou : *La conjecture de Bloch-Kato [d'après M. Rost et V. Voevodsky]*, Astérisque **361**, Exp. No. 1073, x, 421–463 (2014).

[Ser65] Jean-Pierre Serre. Cohomologie galoisienne. With a contribution by Jean-Louis Verdier. Lecture Notes in Mathematics, No. 5. Troisième édition, vol. 1965, Springer-Verlag, Berlin-New York, 1965.

[SJ06] Andrei A. Suslin and Seva Joukhovitski. Norm varieties. *J. Pure Appl. Algebra*, 206(1-2):245–276, 2006.

[SV00] Andrei A. Suslin and Vladimir Voevodsky. Bloch-Kato conjecture and motivic cohomology with finite coefficients. In *The arithmetic and geometry of algebraic cycles (Banff, AB, 1998)*, volume 548 of *NATO Sci. Ser. C Math. Phys. Sci.*, pages 117–189. Kluwer Acad. Publ., Dordrecht, 2000.

[Tot92] Burt Totaro. Milnor K-theory is the simplest part of algebraic K-theory. *K-Theory*, 6(2):177–189, 1992.

[Voe11] Vladimir Voevodsky. On motivic cohomology with $\mathbf{Z}/l$-coefficients. *Ann. of Math. (2)*, 174(1):401–438, 2011.

[Wit15] Olivier Wittenberg. Sur une conjecture de Kato et Kuzumaki concernant les hypersurfaces de Fano. *Duke Math. J.*, 164(11):2185–2211, 2015.

[Yos08] Teruyoshi Yoshida. Local class field theory via Lubin-Tate theory. *Annales de la Faculté des Sciences de Toulouse*, Ser. 6, 17(2):411–438, 2008.

Diego Izquierdo, CMLS, École polytechnique, F–91128 Palaiseau cedex, France.
Email address: `diego.izquierdo@polytechnique.edu`

Giancarlo Lucchini Arteche, Departamento de Matemáticas, Facultad de Ciencias, Universidad de Chile, Las Palmeras 3425, Ñuñoa, Santiago, Chile.
Email address: `luco@uchile.cl`