FINAL EXAM - PART 1

No documents allowed - 30 min - Maximum 7 points

NAME :

Question 1: `(0.75 points)` What is the subgroup of $\mathbb{Z}$ spanned by 60, 80 and 110?

Answer

Question 2: `(0.75 points)` Give an example of a group G and a normal subgroup H such that G has no subgroups isomorphic to G/H. Briefly explain your answer.

Answer

Question 3: `(0.5 point)` Give the definition of a solvable group.

Answer

Question 4: `(0.75 points)` Make the list of all the abelian groups with order 784. No justification is needed.

Answer

Question 5: `(0.5 point)` Give the definition of a faithful action.

Answer

Question 6: `(1.5 points)` Fill in the table. No justification is needed.

Action of $\mathscr{S}_3$ on ...	Faithful?	Transitive?	Number of fixed points?
the set $\{1,2,3\}$			
the set $\{-1,1\}$ by the formula $\sigma \cdot \epsilon = \mathrm{sign}(\sigma)\epsilon$			
itself by conjugation			

Question 7: `(0.75 points)` Which of the following groups are simple? Tick ALL the correct answers. No justification is needed.

☐ $\mathbb{Z}/81\mathbb{Z}$. ☐ $\mathbb{Z}/101\mathbb{Z}$. ☐ $\mathscr{S}_3$. ☐ $\mathscr{A}_4$. ☐ D_{50}.
☐ $\mathscr{A}_{100}$. ☐ $\mathscr{A}_5 \times \mathscr{A}_5$. ☐ A 2-Sylow subgroup of $\mathscr{A}_{2022}$.
☐ An 11-Sylow subgroup of $\mathscr{A}_{13}$.

Question 8: `(0.75 points)` Consider the permutations $\sigma = (1\ 2\ 3)(2\ 6\ 5)(3\ 5\ 1)$ and $\tau = (3\ 2\ 6)(6\ 4\ 1)(2\ 1\ 6)$ in $\mathscr{S}_6$. Find their respective types, and decide whether they are conjugate of each other.

Answer

Question 9: `(0.75 points)` Explain how one can use the universal property to compute the quotient $\mathscr{S}_7/\mathscr{A}_7$.

Answer

FINAL EXAM - PART 2

Course notes allowed - 90 min - Maximum 18 points

Question 10: `(5 points)` Let $n \geq 3$ be an integer. In the dihedral group D_{2n}, let r be the rotation with angle $\frac{2\pi}{n}$ and let s be some symmetry. Consider the subgroup H of D_{2n} spanned by r^2 and s.

1. Assume first that n is odd. Prove that $H = D_{2n}$.
2. Assume now that n is even.
   (a) Prove that H is a normal subgroup of D_{2n}.
   (b) Prove that H has order n.
   (c) Compute the quotient D_{2n}/H.

Question 11: `(3 points)` Consider the group $G := \mathrm{GL}_2(\mathbb{Z})$. Its elements are invertible matrices that have integer coefficients and whose inverse also has integer coefficients. Let H be the subgroup of G consisting of matrices that have odd entries on the diagonal and even entries elsewhere.

1. Prove that H is normal in G and that $G/H \cong \mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z})$.
2. Deduce that $G/H \cong \mathscr{S}_3$.
   Hint : Use a group action!

Question 12: `(3 points)` Let G be a group with order 14872.

1. Prove that G is not simple.
2. Prove that G is solvable.

Question 13: `(7 points)` We consider the field $K := \mathbb{Z}/2\mathbb{Z}$.

1. Prove that there are no non-trivial morphisms from $\mathscr{A}_5$ to $\mathrm{GL}_3(K)$.
2. (a) Prove that the formula $\sigma\cdot(x_1, x_2, x_3, x_4, x_5) = (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, x_{\sigma^{-1}(3)}, x_{\sigma^{-1}(4)}, x_{\sigma^{-1}(5)})$ defines a group action of $\mathscr{A}_5$ on K^5.
   (b) Is it transitive? Is it faithful? Are there any fixed points?
   (c) Find a 4-dimensional sub-K-vector space of K^5 such that $\sigma \cdot v \in V$ for all $\sigma \in \mathscr{A}_5$ and all $v \in V$.
   (d) Deduce from the previous question that there exists a non-trivial group homomorphism from $\mathscr{A}_5$ to $\mathrm{GL}_4(K)$.
   (e) Is it injective?