FINAL EXAM 2022/2023 - ANSWERS

Solution to Question 1 : It is the subgroup spanned by the g.c.d. of 60, 80 and 110. In other words, it is $10\mathbb{Z}$.

Solution to Question 2 : Take $G = \mathbb{Z}$ and $H = 2\mathbb{Z}$. The group G has no elements with finite order other than 0, hence $\mathbb{Z}/2\mathbb{Z}$ is not isomorphic to a subgroup of G.

Solution to Question 3 : See course notes.

Solution to Question 4 :

$$\begin{aligned}
\mathbb{Z}/16\mathbb{Z} \times \mathbb{Z}/49\mathbb{Z}, &\quad \mathbb{Z}/16\mathbb{Z} \times (\mathbb{Z}/7\mathbb{Z})^2,\\
\mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/49\mathbb{Z}, &\quad \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/7\mathbb{Z})^2,\\
(\mathbb{Z}/4\mathbb{Z})^2 \times \mathbb{Z}/49\mathbb{Z}, &\quad (\mathbb{Z}/4\mathbb{Z})^2 \times (\mathbb{Z}/7\mathbb{Z})^2,\\
\mathbb{Z}/4\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}/49\mathbb{Z}, &\quad \mathbb{Z}/4\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}/7\mathbb{Z})^2,\\
(\mathbb{Z}/2\mathbb{Z})^4 \times \mathbb{Z}/49\mathbb{Z}, &\quad (\mathbb{Z}/2\mathbb{Z})^4 \times (\mathbb{Z}/7\mathbb{Z})^2.
\end{aligned}$$

Solution to Question 5 : See course notes.

Solution to Question 6 :

Action of $\mathscr{S}_3$ on ...	Faithful?	Transitive?	Number of fixed points?
the set $\{1,2,3\}$	Yes	Yes	0
the set $\{-1,1\}$ by the formula $\sigma \cdot \epsilon = \mathrm{sign}(\sigma)\epsilon$	No	Yes	0
itself by conjugation	Yes	No	1

Solution to Question 7 :

☐ $\mathbb{Z}/81\mathbb{Z}$. ☑ $\mathbb{Z}/101\mathbb{Z}$. ☐ $\mathscr{S}_3$. ☐ $\mathscr{A}_4$. ☐ D_{50}.
☑ $\mathscr{A}_{100}$. ☐ $\mathscr{A}_5 \times \mathscr{A}_5$. ☐ A 2-Sylow subgroup of $\mathscr{A}_{2022}$.
☑ An 11-Sylow subgroup of $\mathscr{A}_{13}$.

Solution to Question 8 : We have :

$$\begin{aligned}
\sigma &= (2\ 6\ 5),\\
\tau &= (1\ 4)(2\ 3).
\end{aligned}$$

Hence :

$$\mathtt{type}(\sigma) = (1,1,1,3) \neq (1,1,2,2) = \mathtt{type}(\tau),$$

and σ and τ are not conjugate.

Solution to Question 9 : Consider the signature morphism sign : $\mathscr{S}_7 \to \{-1,1\}$. It is surjective and its kernel is $\mathscr{A}_7$. Hence, by the universal property :

$$\mathscr{S}_7/\mathscr{A}_7 = \mathscr{S}_7/\ker(\mathrm{sign}) \cong \mathrm{im}(\mathrm{sign}) = \{-1,1\}.$$

Solution to Question 10 :

1. Since n is odd, $r = r^{n+1} = (r^2)^{\frac{n+1}{2}} \in H$. We deduce that H contains $\langle r,s\rangle$. But $\langle r,s\rangle = D_{2n}$. Hence $H = D_{2n}$.
2. (a) Observe that :

$$rr^2r^{-1} = r^{-1}r^2r = r^2 \in H, \quad sr^2s^{-1} = s^{-1}r^2s = r^{-2} \in H,$$
$$rsr^{-1} = sr^{-2} \in H, \quad r^{-1}sr = sr^2 \in H, \quad sss^{-1} = s^{-1}ss = s \in H.$$

   Since D_{2n} is spanned by r and s and H is spanned by r^2 and s, we deduce that H is normal in D_{2n}.

   (b) The group H contains the set $X = \{1, r^2, r^4, \dots, r^{n-2}, s, sr^2, sr^4, \dots, sr^{n-2}\}$. Moreover, $1 \in X$ and, for every $k, l \in \mathbb{Z}$:

$$r^{2k}r^{-2l} = r^{2(k-l)} \in X,$$
$$r^{2k}(sr^{2l})^{-1} = sr^{2(-k+l)} \in X,$$
$$sr^{2k}r^{-2l} = sr^{2(k-l)} \in X,$$
$$sr^{2k}(sr^{2l})^{-1} = r^{2(-k+l)} \in X.$$

   We deduce that X is a subgroup of G, and hence that $X = H$. In particular, H has order n

   (c) By Lagrange's theorem, D_{2n}/H is a group of order $\frac{2n}{n} = 2$. It is therefore isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

Solution to Question 11 :

1. Consider the reduction modulo 2 morphism $f : \mathrm{GL}_2(\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z})$. Its kernel is H. Moreover, the elements of $\mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z})$ are the matrices :

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}, \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix},$$

   which can all be lifted to $\mathrm{GL}_2(\mathbb{Z})$. We deduce that f is surjective, and hence, by the universal property, G/H is isomorphic to $\mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z})$.

2. Consider the $\mathbb{Z}/2\mathbb{Z}$-vector space $V := (\mathbb{Z}/2\mathbb{Z})^2$ and the natural action of $\mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z})$ on $V \setminus \{0\}$. This action is faithful, and hence it induces an injective morphism :

$$\phi : \mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z}) \to \mathrm{Bij}(V \setminus \{0\}) \cong \mathscr{S}_3.$$

Since both $\mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z})$ and $\mathscr{S}_3$ have order 6, we deduce that ϕ is an isomorphism. We conclude that $G/H \cong \mathscr{S}_3$.

Solution to Question 12 :

1. Observe that $14872 = 2^3 \cdot 11 \cdot 13^2$. Let n_{13} be the number of 13-Sylow subgroups of G. By the Sylow theorems :

$$n_{13} \equiv 1 \mod 13, \quad n_{13} | 2^3 \cdot 11.$$

Hence $n_{13} = 1$. The group G therefore has a unique 13-Sylow subgroup H, which has to be normal. In particular, G is not simple.

2. Let m_{11} be the number of 11-Sylow subgroups of G/H. Since G/H has $2^3 \cdot 11$ elements, the Sylow theorems show that :

$$m_{11} \equiv 1 \mod 11, \quad m_{11} | 8.$$

Hence $m_{11} = 1$. The group G/H therefore has a unique 11-Sylow subgroup K, which is necessarily normal.

Now, K is an 11-group and $(G/H)/K$ is a 2-group : we deduce that K and $(G/H)/K$ are both solvable, and hence so is G/H. But H is also solvable since it is a 13-group. Hence G is solvable.

Solution to Question 13 :

1. Let $f : \mathscr{A}_5 \to \mathrm{GL}_3(K)$ be a morphism. The kernel of f is a normal subgroup of $\mathscr{A}_5$, and hence it is either $\{1\}$ or $\mathscr{A}_5$. Since $|\mathscr{A}_5| = 60 \nmid 168 = |\mathrm{GL}_3(K)|$, the morphism f cannot be injective, so that $\ker(f) \neq \{1\}$. We deduce that $\ker(f) = \mathscr{A}_5$, and hence f is trivial.

2. (a) Take $(x_1, x_2, x_3, x_4, x_5) \in K^5$ and $\sigma, \tau \in \mathscr{A}_5$. Then :

$$\mathrm{Id} \cdot (x_1, x_2, x_3, x_4, x_5) = (x_1, x_2, x_3, x_4, x_5),$$

$$\begin{aligned}
\sigma \cdot (\tau \cdot (x_1, x_2, x_3, x_4, x_5)) &= \sigma \cdot (x_{\tau^{-1}(1)}, x_{\tau^{-1}(2)}, x_{\tau^{-1}(3)}, x_{\tau^{-1}(4)}, x_{\tau^{-1}(5)}) \\
&= (x_{\tau^{-1}\sigma^{-1}(1)}, x_{\tau^{-1}\sigma^{-1}(2)}, x_{\tau^{-1}\sigma^{-1}(3)}, x_{\tau^{-1}\sigma^{-1}(4)}, x_{\tau^{-1}\sigma^{-1}(5)}) \\
&= (x_{(\sigma\tau)^{-1}(1)}, x_{(\sigma\tau)^{-1}(2)}, x_{(\sigma\tau)^{-1}(3)}, x_{(\sigma\tau)^{-1}(4)}, x_{(\sigma\tau)^{-1}(5)}) \\
&= (\sigma\tau) \cdot (x_1, x_2, x_3, x_4, x_5).
\end{aligned}$$

The suggested formula therefore defines a group action.

(b) The zero-vector is a fixed point. In particular, the action is not transitive. The action is faithful since :

$$\sigma \cdot (1,2,3,4,5) = (\sigma^{-1}(1), \sigma^{-1}(2), \sigma^{-1}(3), \sigma^{-1}(4), \sigma^{-1}(5)) \neq (1,2,3,4,5)$$

for $\sigma \in \mathscr{A}_5 \setminus \{\mathrm{Id}\}$.

(c) Let V be the hyperplane of K^5 defined by the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 0$. Then, for any $\sigma \in \mathscr{A}_5$ and any $v = (x_1, x_2, x_3, x_4, x_5)(x_1, x_2, x_3, x_4, x_5) \in K^5$:

$$\sum_{i=1}^{5} x_{\sigma^{-1}(i)} = \sum_{i=1}^{5} x_i = 0,$$

and hence $\sigma \cdot v \in V$.

(d) Restrict the action of question 2(b) to V. It induces a morphism :

$$\phi : \mathscr{A}_5 \to \mathrm{Bij}(V).$$

But, for each $\sigma \in \mathscr{A}_5$, the bijection $\phi(\sigma)$ is linear. Hence the image of ϕ is contained in $\mathrm{GL}(V) \cong \mathrm{GL}_4(K)$. We therefore obtain a morphism :

$$\phi : \mathscr{A}_5 \to \mathrm{GL}_4(K).$$

This morphism is not trivial because the action of $\mathscr{A}_5$ on V is not trivial.

(e) The kernel of ϕ is a normal subgroup of $\mathscr{A}_5$ other than $\mathscr{A}_5$. We deduce that $\ker(\phi) = \{1\}$, and hence ϕ is injective.