

FINAL EXAM

Important instructions

- No documents are allowed.
- Unless otherwise stated, *you should justify all your answers.*
- The exam is divided into two parts. Part I is about group theory and part II is about rings and fields. You do not need to solve all the questions to get the maximum grade.
- You can solve the exercises in the order you want. Even if you do not succeed in solving a question, you can use it in subsequent questions.
- You can only use the results that have been seen during the course. Any other result has to be proved.

Part I : Group theory

Exercise 1: (1 point)

Give an example of a group G and a normal subgroup H of G such that G is not isomorphic to $H \times G/H$.

Exercise 2: (2,5 points)

1. State the structure theorem of finite abelian groups.
2. Make the list of all abelian groups of order 400. No justification is needed.

Exercise 3: (4,5 points)

1. State the Sylow theorems.
2. Let G be a group with order 2020.
 - (a) Prove that G is not simple.
 - (b) Prove that G is solvable.

Exercise 4: (4 points)

1. Let p be a prime number and let G be a finite simple group whose order is divisible by p^2 .
 - (a) Prove that, for each $k \in \{1, 2, 3, \dots, 2p - 1\}$, every group homomorphism from G to \mathcal{S}_k is trivial.
 - (b) Deduce that any strict subgroup of G has index at least $2p$.
Hint : use a group action!
2. Conversely, for each **odd** prime p , give an example of a finite simple group G whose order is divisible by p^2 and that has a subgroup of index $2p$.

Part II : Rings and fields

Exercise 5: (2,5 points)

1. Let A be a principal ideal domain and let $a \in A \setminus \{0\}$. We have seen during the course that, if a is irreducible, then the ideal (a) is maximal. Recall the proof of this result.
2. Give an example of a ring A together with an irreducible element $a \in A$ such that (a) is not a prime ideal. No justification is needed.
3. Give an example of a ring A together with a non-zero prime ideal I that is not maximal. No justification is needed.

Exercise 6: (2,5 points)

1. State Eisenstein's criterion.
2. Set $\alpha := \sqrt[5]{1 + \sqrt[11]{43}}$. Find the minimal polynomial of α and deduce the degree of the extension $\mathbb{Q}(\alpha)/\mathbb{Q}$.

Exercise 7: (4,5 points)

Find the degrees of the splitting fields of the following polynomials over \mathbb{Q} :

$$f = X^{17} - 28, \quad g = (X^2 - 3)(X^3 - 1), \quad h = \sum_{k=0}^{1999} X^k.$$

Exercise 8: (6,5 points)

1. Let p be an **odd** prime number. Consider the polynomial $\phi = X^4 + 1 \in \mathbb{F}_p[X]$.
 - (a) Prove that ϕ has a root in \mathbb{F}_p if, and only if, $p \equiv 1 \pmod{8}$.
 - (b) Let L be a splitting field of ϕ over \mathbb{F}_p . Let ζ be a root of ϕ in L and set $\beta := \zeta + \zeta^{-1}$. Check that $\beta^2 = 2$.
 - (c) Deduce that 2 is a square in \mathbb{F}_p if, and only if, $p \equiv \pm 1 \pmod{8}$.
2. Consider the ring $A = \mathbb{Z}[\sqrt{2}]$.
 - (a) Prove that A is isomorphic to the quotient $\mathbb{Z}[X]/(X^2 - 2)$.
 - (b) For which odd prime numbers p is the ideal (p) in A prime? And maximal?