

## FINAL EXAM - PART 1

No documents allowed - 30 min - Maximum 7 points

NAME :

**Question 1:** (1 point) State Sylow's theorems.

Answer

**Question 2:** (0.75 points) Give an example of two non-isomorphic finite groups that have the same simple factors. Briefly explain your answer.

Answer

**Question 3:** (0.75 points) Make the list of all the abelian groups with order 216. No justification is needed.

Answer

**Question 4:** (1.5 points) Fill in the table. No justification is needed.

Action of $\mathcal{S}_5$ on ...	Faithful?	Transitive?	Number of fixed points?
the set $\{1, 2, 3, 4, 5\}$			
$\mathcal{S}_5 / \langle (1\ 2\ 3\ 4\ 5) \rangle$ by left multiplication			
itself by conjugation			

**Question 5:** (0.75 points) Which of the following groups are solvable? Tick ALL the correct answers. No justification is needed.

- $\mathcal{S}_3$ .        $\mathcal{S}_{21}$ .        $D_{50}$ .        $D_{16} \times \mathbb{Z}/99\mathbb{Z}$ .  
 A  $\mathbb{Z}/5\mathbb{Z}$ -vector space of dimension 100.       A 3-Sylow subgroup of  $\mathcal{S}_{1000}$ .

**Question 6:** (0.75 points) Consider the permutations  $\sigma = (3\ 4\ 5\ 6)(2\ 3\ 5\ 8)$  and  $\tau = (7\ 8\ 3\ 1)(1\ 7\ 2\ 4)$  in  $\mathcal{S}_8$ . Find their respective types, and decide whether they are conjugate of each other.

Answer

**Question 7:** (0.75 points) Explain how one can use the universal property to compute the quotient  $GL_3(\mathbb{R})/SL_3(\mathbb{R})$ .

Answer

**Question 8:** (0.75 points) Prove that the group  $GL_3(\mathbb{Z}/7\mathbb{Z})$  does not contain any subgroup of order 243.

Answer

## FINAL EXAM - PART 2

Course notes allowed - 90 min - Maximum 16 points

**Question 9:** (4 points) In the symmetric group  $\mathcal{S}_{11}$ , consider the elements :

$$\sigma := (1\ 2), \quad \tau := (1\ 2\ 4\ 3)(5\ 6\ 7\ 8)(10\ 3\ 2\ 4\ 8\ 9).$$

Set  $H := \langle \sigma, \tau \rangle$ .

1. Write  $\tau$  as a product of cycles with disjoint supports. What is the order of  $\tau$ ? Does  $\tau$  belong to the alternating group  $\mathcal{A}_{11}$ ?
2. Can you recognize the group  $H$ ? What is its index in  $\mathcal{S}_{11}$ ?
3. Prove that, for any  $\rho \in \mathcal{S}_{11} \setminus H$ , the group  $\mathcal{S}_{11}$  is spanned by  $\sigma, \tau$  and  $\rho$ .

**Question 10:** (3 points) Let  $G$  be the group of 2-by-2 upper-triangular invertible matrices with real coefficients. Let  $H$  be the subgroup of  $G$  given by matrices that have ones on the diagonal.

1. Prove that  $H$  is a normal subgroup of  $G$  and that the quotient  $G/H$  is isomorphic to  $\mathbb{R}^\times \times \mathbb{R}^\times$ .
2. Is the group  $G$  solvable?

**Question 11:** (3 points) Let  $n \geq 5$ .

1. Consider the action of  $\mathcal{A}_n$  on  $\{1, \dots, n\}$  defined by :

$$\sigma \cdot i = \sigma(i)$$

for  $\sigma \in \mathcal{A}_n$  and  $i \in \{1, \dots, n\}$ . Is this action transitive? Is it faithful?

2. Prove that every action of  $\mathcal{A}_n$  on  $\{1, \dots, n-1\}$  is trivial.  
*Recall that an action of a group  $G$  on a set  $X$  is said to be trivial if  $g \cdot x = x$  for all  $g \in G$  and  $x \in X$ .*

**Question 12:** (3 points) Let  $G$  be a group with order 6256.

1. Prove that  $G$  is not simple.
2. Prove that  $G$  is solvable.

**Question 13:** (3 points) Let  $G$  be a group and let  $H$  be some subgroup of  $G$ .

1. We set  $N_G(H) = \{g \in G \mid gHg^{-1} = H\}$ . Prove that  $N_G(H)$  is a subgroup of  $G$  and that  $H$  is a normal subgroup of  $N_G(H)$ .
2. Assume that  $|G| = p^a \cdot m$  for some prime  $p$ , some  $a \geq 1$  and some  $m$  not divisible by  $p$  and not congruent to 1 modulo  $p$ . Let  $H$  be a  $p$ -Sylow subgroup of  $G$ . By introducing a well-chosen group action, prove that  $N_G(H) \neq H$ .