

Exercises of Chapter 3 (part 1): The Brauer group

Exercise 1. Check that cyclic algebras are indeed central simple algebras.

Exercise 2. Check that the map $\delta_\infty : B(L/K) \rightarrow \text{Br}(L/K)$ constructed in class is a group homomorphism.

Exercise 3. Prove that a 4-dimensional central simple algebra is either split or a quaternion algebra.

Exercise 4. Let A be a quaternion algebra over K and fix an element $a \in K^\times \setminus (K^\times)^2$. Prove that the following assertions are equivalent:

- (i) A is isomorphic to the quaternion algebra (a, b) for some $b \in K^\times$.
- (ii) The $K(\sqrt{a})$ -algebra $A \otimes_K K(\sqrt{a})$ is split.
- (iii) A contains a commutative K -subalgebra isomorphic to $K(\sqrt{a})$.

Exercise 5. Determine the prime numbers p such that the quaternion algebra $(-1, p)$ is split over \mathbb{Q} .

Exercise 6. Let K be a field and let A be a central simple K -algebra. Explain why $\dim_K A$ is a perfect square. The integer $\sqrt{\dim_K A}$ is called the degree of A .

Exercise 7. Let K be a field and let A be a central simple algebra of degree n . Assume that A contains a K -subalgebra L which is a degree n field extension of K . Prove that A splits over L .

Exercise 8. Prove that a central simple algebra A of degree n over a field K is split if, and only if, it contains a subalgebra isomorphic to K^n .

Exercise 9. Prove that a tensor product of two division K -algebras of coprime dimensions is still a division algebra.

Exercise 10. Let K be a field and let A be a central simple K -algebra.

1. Let D be the division algebra such that $A \cong \mathcal{M}_n(D)$ for some $n \geq 0$. The index $\text{ind}(A)$ of A is defined to be the degree of D . Prove that, for each $r \leq d$:

$$\text{ind}(A^{\otimes r}) \mid \binom{\text{ind}(A)}{r}.$$

2. The period $\text{per}(A)$ of A is the order of the class of A in $\text{Br } K$. Prove that $\text{per}(A) \mid \text{ind}(A)$.
3. Prove that $\text{per}(A)$ and $\text{ind}(A)$ have the same prime factors.

Exercise 11. Let A be a central division K -algebra of degree n . Assume that one can find a finite extension L of K of degree prime to n that splits A . Prove that A is split over K .

Exercise 12. (*Brauer*) Let K be a field and let D be a central division K -algebra. Write the prime decomposition $\text{ind}(A) = p_1^{m_1} \dots p_r^{m_r}$. Prove that one can find central division K -algebras D_1, \dots, D_r such that $D \cong D_1 \otimes_K \dots \otimes_K D_r$ and $\text{ind}(D_i) = p_i^{m_i}$ for each i . Prove also that the D_i 's are uniquely determined up to isomorphism. You may find exercises 9 and 10 helpful.

Exercise 13. Let K be a perfect field.

1. Compute $H^1(K, SL_n(\overline{K}))$.
2. Prove that $H^1(K, S_n)$ is the set of isomorphism classes of n -dimensional étale K -algebras.