

Exercises of Chapter 2: Galois cohomology and cohomological dimension

Exercise 1. The following properties of the cohomology of profinite groups were stated in class. Prove them.

1. Let G be a profinite group and let A be a G -module. There is a canonical isomorphism:

$$H^i(G, A) \cong \varinjlim_U H^i(G/U, A^U),$$

where U describes the set of open normal subgroups of G .

- 2.. Let H be a closed normal subgroup of a profinite group G and let A be a G -module. The sequence:

$$0 \rightarrow H^1(G/H, A^H) \xrightarrow{\text{Inf}} H^1(G, A) \xrightarrow{\text{Res}} H^1(H, A)$$

is exact.

3. Let G be a profinite group and let A be a uniquely divisible G -module. For $i \geq 1$, the group $H^i(G, A)$ is trivial.

Exercise 2. Let H be a closed normal subgroup of a profinite group G and let A be a G -module. Prove that the inflation-restriction sequence can be extended into a 5-term exact sequence:

$$0 \rightarrow H^1(G/H, A^H) \xrightarrow{\text{Inf}} H^1(G, A) \xrightarrow{\text{Res}} H^1(H, A)^{G/H} \rightarrow H^2(G/H, A^H) \xrightarrow{\text{Inf}} H^2(G, A).$$

Exercise 3. Let G be an abelian profinite group. Assume that G/nG is finite for each $n > 0$.

1. Prove that nG is an open subgroup of G for each $n > 0$.
2. Let A be a finite G -module. Prove that $H^1(G, A)$ is finite.

Exercise 4. Let K be a perfect field of characteristic $p > 0$. Prove that, for any finite Galois module M of order a power of p and any $n \geq 2$, the group $H^n(K, M)$ vanishes.

Exercise 5.

1. Let m be a positive integer. For each $n > 0$, compute the following groups:

$$H^n(\mathbb{R}, \mathbb{C}^\times), \quad H^n(\mathbb{R}, \mathbb{Z}/2^m\mathbb{Z}), \quad H^n(\mathbb{R}, \mu_{2^m}).$$

2. What is the cohomological dimension of \mathbb{R} ? And of \mathbb{Q} ?

Exercise 6.

1. Let K be a field of characteristic 0 with cohomological dimension at most 1. Prove that, for any finite extension L of K , the Brauer group of L is trivial. Also prove that for any tower of finite extensions $M/L/K$, the norm $N_{M/L} : M^\times \rightarrow L^\times$ is surjective.
2. Let K be a field with cohomological dimension d and let A be a divisible Galois module over K . Prove that $H^n(K, A) = 0$ for each $n \geq d + 1$.
3. Let K be a field of characteristic 0. Prove that K has cohomological dimension at most d if, and only if, for any finite extension L of K , the group of $H^{d+1}(L, \bar{L}^\times)$ is trivial.

Exercise 7. Let K be a field with cohomological dimension d and let $\text{scd}(K)$ be the smallest integer n such that, for each $r \geq n + 1$, for each Galois module M over K , the group $H^r(K, M)$ vanishes.

1. Prove that

$$d \leq \text{scd}(K) \leq d + 1.$$

2. Prove that $\text{scd}(K) = d$ if, and only if, for any finite extension L of K , we have

$$H^{d+1}(L, \mathbb{Z}) = 0.$$

Exercise 8.

1. Let K be a field with cohomological dimension $d < +\infty$ and let L be a finite extension of K . Prove that L has cohomological dimension d .
2. Let L/K be a (not necessarily finite) Galois extension of fields. Let $\text{cd}(L/K)$ be the smallest integer n such that, for each finite $\text{Gal}(L/K)$ -module M and each $r \geq n + 1$, the group $H^r(\text{Gal}(L/K), M)$ vanishes. Prove that $\text{cd}(K) \leq \text{cd}(L) + \text{cd}(L/K)$.

Exercise 9. Let K be a p -adic field and let M be a Galois module over K which, as an abelian group, is of finite type. Prove that $H^1(K, M)$ is finite. Does this finiteness result still hold when K is a number field?

Exercise 10. (*Hard!*) Let M be a finite Galois module over \mathbb{R} . Consider the Galois module $M' := \text{Hom}(M, \mathbb{C}^\times)$ endowed with the action $\sigma \cdot f : x \mapsto \sigma(f(\sigma^{-1}x))$ for $f \in M'$ and $\sigma \in \text{Gal}(\mathbb{C}/\mathbb{R})$. Show that, if $i, j > 0$ are positive integers of the same parity, then the groups $H^i(\mathbb{R}, M)$ and $H^j(\mathbb{C}, M')$ are finite and dual.