

Exercises of Chapter 1: C_i -fields

The last exercises are harder than the first ones.

Exercise 1. By using the Arkhipov, Karatsuba and Alemu theorem, prove that number fields are not C_i for any i .

Exercise 2. Let K be a field.

1. Let L be a finite extension of K . Assume that L is C_i for some $i \geq 0$. Is K also C_i ?
2. Prove that if the field of rational functions $K(T)$ is C_{i+1} for some $i \geq 0$, then K is C_i .

Exercise 3. Let X be a smooth integral \mathbb{R} -variety such that $X(\mathbb{R}) \neq \emptyset$. Prove that $\mathbb{R}(X)$ is not C_i for any i .

Exercise 4. Let p be a prime number and let n and d be two positive integers. Let K be a p -adic field and let $a_0, \dots, a_n \in K^\times$. Consider the projective variety X defined by the equation:

$$a_0x_0^d + \dots + a_nx_n^d = 0.$$

1. Assume that d is not divisible by p . Prove that, if $n \geq d^2$, then $X(K) \neq \emptyset$.
2. In general, prove that one can always find a positive integer $f(d)$ depending only on d such that, if $n \geq f(d)$, then $X(K) \neq \emptyset$. Can the function $d \mapsto f(d)$ be chosen to be a polynomial?

Exercise 5. Let R be the ring $\mathbb{C}[[x, y]]$ of formal power series in 2 variables with coefficients in \mathbb{C} and let K be its fraction field $\mathbb{C}((x, y))$.

1. Prove that the group R^\times is divisible.
2. Prove that any element $f \in R$ can be factored as $f = ug$ with $u \in R^\times$ and $g \in \mathbb{C}[[x]][y]$.
3. Let n and d be positive integers such that $n \geq d^2$. Let a_0, \dots, a_n be elements in K . Prove that the projective hypersurface defined by the equation $a_0x_0^d + \dots + a_nx_n^d = 0$ has a rational point.

Exercise 6. Let K be a field. Assume that there exists a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that, for any $d \geq 1$, any $n \geq f(d)$ and any $a_0, \dots, a_n \in K^\times$, the projective hypersurface given by the equation:

$$a_0x_0^d + \dots + a_nx_n^d = 0$$

has a rational point in K .

1. Let $m, d \geq 1$. Prove that there exists an integer $g(d, m)$ such that, for any $n \geq g(d, m)$ and any $a_0, \dots, a_n \in K^\times$, the set of solutions of the equation:

$$a_0x_0^d + \dots + a_nx_n^d = 0$$

in K^{n+1} contains an m -dimensional vector subspace of K^{n+1} .

2. Let m, d_1, \dots, d_h be positive integers. Prove that there exists an integer $g(d_1, \dots, d_h, m)$ such that, for any $n \geq g(d_1, \dots, d_h, m)$, the set of solutions of any system of equations:

$$f_i(\mathbf{x}) = 0, \quad 1 \leq i \leq h$$

in which each f_i is a homogeneous polynomial over K in $n+1$ variables of degree d_i contains an m -dimensional vector subspace of K^{n+1} .

Exercise 7. *To do this exercise, you need to know Witt vectors.* Let k be an algebraically closed field with characteristic $p > 0$. Let $W(k)$ be the ring of Witt vectors of k and let f be a homogeneous polynomial of degree d in n variables over $W(k)$ with $d < n$.

1. Let m be a positive integer. Consider the polynomial ring $k[X_{ij}, 1 \leq i \leq m, 0 \leq j \leq m]$. For each j , denote by \mathbf{X}_j the vector $(X_{i,j})_{1 \leq i \leq m}$. Construct $m + 1$ homogeneous polynomials $f_0(\mathbf{X}_0), f_1(\mathbf{X}_0, \mathbf{X}_1), \dots, f_m(\mathbf{X}_0, \dots, \mathbf{X}_m)$ with coefficients in k such that the equation $f = 0$ has solutions in $W(k)/p^m$ not divisible by p if, and only if, the system of equations:

$$\begin{cases} f_j(\mathbf{X}_0, \dots, \mathbf{X}_j) = 0, & 0 \leq j \leq m, \\ \mathbf{X}_0 \neq 0, \end{cases}$$

has solutions in k .

2. Let m and j be positive integers. Prove that, if all solutions of the equation $f = 0$ in $W(k)/p^m$ are divisible by p , then all solutions of $f = 0$ in $W(k)/p^{m+jd}$ are divisible by p^{j+1} .
3. Use the previous question to show that, for any $m > 0$, the equation $f = 0$ has solutions in $W(k)/p^m$ which are not divisible by p .
4. Use Greenberg's approximation theorem to show that the equation $f = 0$ has a non-zero solution in $W(k)$.
5. Prove Lang's theorem: a complete discrete valuation field with algebraically closed residue field is C_1 .

Some references for the course:

-Lectures on Forms in Many Variables, Marvin J. Greenberg, W. A. Benjamin, Inc., New York-Amsterdam, 1969.

-Central Simple Algebras and Galois Cohomology, Second Edition, Philippe Gille and Tamás Szamuely, Cambridge Studies in Advanced Mathematics, Vol. 165, 2017.

-Rational points on varieties, Bjorn Poonen, Graduate Studies in Mathematics, Vol. 186, 2017.